Domain Representability of Certain Function Spaces

by

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Abstract: Let $C_p(X)$ be the space of all continuous real-valued functions on a space X, with the topology of pointwise convergence. In this paper we show that $C_p(X)$ is not domain representable unless X is discrete for a class of spaces that includes all pseudo-radial spaces and all generalized ordered spaces. This is a first step toward our conjecture that if X is completely regular, then $C_p(X)$ is domain representable if and only if X is discrete. In addition, we show that if X is completely regular and pseudonormal, then in the function space $C_p(X)$, Oxtoby's pseudocompleteness, strong Choquet completeness, and weak Choquet completeness are all equivalent to the statement "every countable subset of X is closed."

MR Subject Classifications: Primary = 54C35; Secondary = 54E52, 06B35, 05F30, 54F05.

Key Words and Phrases: domain representable space, function space, pointwise convergence topology, pseudo-radial space, transfinite sequence, pseudocomplete, weakly Choquet complete, strongly Choquet complete.

****draft of March 27, 2008*****

1 Introduction

A topological space is *domain representable* if it is homeomorphic to the subspace of maximal elements of a domain, topologized with the Scott topology. (See Section 2 for definitions.) A wide range of topological spaces are domain representable – for example, any completely metrizable space, the Sorgenfrey line, the Michael line, and any space of ordinals. A central problem in domain representation theory is to determine which spaces are domain representable.

Domain representability is a kind of Baire-category completeness property that lies toward the top of the hierarchy of strengthenings of the Baire space property (that any countable intersection of dense open sets must be dense). For example, every subcompact regular T_3-space [4] is domain-representable [2], every domain-representable space is strongly Choquet complete [7], and every strongly Choquet complete space is a Baire space. (See Section 4 for definitions related to completeness.)

For any space X, let $C_p(X)$ be the set of all continuous, real-valued functions on X, equipped with the pointwise convergence topology. In this paper, we investigate domain representability and strong completeness properties of $C_p(X)$.

For completely regular T_1-spaces, $C_p(X)$ is a dense subspace of the full topological product $\mathbb{R}^X$. The literature shows that while the full product space $\mathbb{R}^X$ has Baire-category completeness properties like subcompactness [4] and strong Choquet completeness, it is difficult for the subspace $C_p(X)$ to have such properties [5, 6, 9, 11]. Starting with work by Lutzer and McCoy [5] Pytkeev and van Douwen gave restrictive necessary and sufficient conditions on X for $C_p(X)$ to be a Baire space [9, 11]. Lutzer and

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McCoy characterized spaces X for which $C_p(X)$ is weakly Choquet complete [5] as being the spaces in which each countable set is closed. More recently, starting with ideas of van Mill and Lutzer, Tkachuk [6] proved that if X is completely regular then $C_p(X)$ is never subcompact, unless X is discrete (in which case $C_p(X) = \mathbb{R}^X$).

A natural question asks for a characterization of those spaces X whose function space $C_p(X)$ is domain representable. In the light of Tkachuk's result on subcompact $C_p(X)$ we make the following conjecture, which would be a strengthening of Tkachuk's theorem:

> **Conjecture**: *If X is a completely regular T_1-space, then $C_p(X)$ is domain representable if and only if X is discrete.*

In this paper, we prove a special case of our conjecture, namely

> **Main Theorem** *Suppose that, for the completely regular space X, there is a cardinal κ such that every subset $S \subseteq X$ with $|S| < \kappa$ is closed and such that X contains a transfinite sequence $\{y(\alpha) : \alpha < \kappa\}$ that converges to some point of $X - \{y(\alpha) : \alpha < \kappa\}$. Then $C_p(X)$ is not domain representable. Consequently, for a completely regular pseudo-radial T_1-space X (and hence for any generalized ordered space X), the following are equivalent:*
>
> *a) $C_p(X)$ is domain representable;*
>
> *b) X is discrete (so that $C_p(X) = \mathbb{R}^X$);*
>
> *c) $C_p(X)$ is Scott-domain representable.*

An easy consequence of our result (taking $\kappa = \omega$) is that $C_p(X)$ is not domain representable if X is first-countable and not discrete. A slightly more complicated consequence is that if $X = [0, \omega_1]$ is the generalized ordered space in which each countable ordinal is isolated and ω_1 has its usual neighborhoods, then $C_p(X)$ is not domain representable. This second example puts limits on how far Tkachuk's result about subcompact $C_p(X)$ can be generalized, because it shows that the function space $C_p(X)$ can be strongly Choquet complete when X is not discrete. See Example 4.2.

Our paper is organized as follows. Section 2 presents definitions and preliminary results. Section 3 gives the proof of the Main Theorem and Section 4 discusses the equivalence of Oxtoby's pseudocompleteness, weak Choquet completeness, and strong Choquet completeness for function spaces $C_p(X)$ where X is pseudonormal and completely regular. See Section 4 for definitions.

Throughout the paper, all spaces will be at least T_3 and $\mathbb{R}$ will denote the usual set of real numbers.

2 Definitions and preliminary results

We will think of cardinal numbers as initial ordinals, and for any set S, we let $|S|$ denote the cardinality of S. If α is an ordinal, then $|\alpha| \leq \alpha$.

Basic neighborhoods of a function $f \in C_p(X)$ have the form $O(f, S, \varepsilon) := \{g \in C_p(X) : \text{ for all } x \in S,\ g(x) \in (f(x) - \varepsilon, f(x) + \varepsilon)\}$ where $S \subseteq X$ is finite and $\varepsilon > 0$.

Given a partially ordered set $(P,\sqsubseteq)$, a non-empty set $D \subseteq P$ is *bounded* if there is some $p \in P$ with $d \sqsubseteq p$ for each $d \in D$. For a nonempty bounded subset $D \subseteq P$, $\sup(D)$ is an upper bound for D in P that is less than or equal to every upper bound for D. Note that $\sup(D)$ may, or may not, exist in P. Throughout this paper, if a poset $(P,\sqsubseteq)$ is given and D is a nonempty subset of P, we will write $\sup(D) \in P$ to mean that the supremum of D exists in P. A non-empty set $D \subseteq P$ is *directed* if, for each pair $d_1, d_2 \in D$, some $d_3 \in D$ has $d_1, d_2 \sqsubseteq d_3$. A *directed complete partial order* (dcpo) is a partially ordered set with the property that if $D \subset P$ is nonempty and directed then $\sup(D)$ exists in P. If p, q are elements of a partially ordered set, we write $p \ll q$ to mean that if a directed set D has $q \sqsubseteq \sup(D)$ then some $d \in D$ has $p \sqsubseteq d$. We define $\Uparrow(p) = \{q \in P : p \ll q\}$ and $\Downarrow(q) = \{p \in P : p \ll q\}$. A partially ordered set is *continuous* if for each $q \in P$, the set $\Downarrow(q)$ is directed and has $q = \sup(\Downarrow(q))$. A *domain* is a continuous dcpo, and a *Scott domain* is a continuous dcpo P with the additional property that if $p, q, r \in P$ and $p, q \sqsubseteq r$, then $\sup\{p, q\} \in P$.

We will need three lemmas about domains. The first is called the *Interpolation Lemma* and appears in [8].

Lemma 2.1 *Suppose $(P,\sqsubseteq)$ is a domain and $p, r \in P$ have $p \ll r$. Then for some $q \in P$, $p \ll q \ll r$.*

Lemma 2.2 *If p, q, r are points in a domain P with $p \in \Uparrow(q) \cap \Uparrow(r)$ then there is some $s \in P$ with $p \in \Uparrow(s) \subseteq \Uparrow(q) \cap \Uparrow(r)$ and $q, r \ll s \ll p$.*

Proof: We have $q, r \in \Downarrow(p)$ so that because $\Downarrow(p)$ is directed, some $s_1 \in \Downarrow(p)$ has $q, r \sqsubseteq s_1 \ll p$. Now use the Interpolation Lemma to find $s \in P$ with $s_1 \ll s \ll p$. $\square$

Lemma 2.3 *Suppose P is a domain and $E \subseteq P$ has the property that $\bigcap\{\Uparrow(e) : e \in E\} \neq \emptyset$. Then there is a set $E^* \subseteq P$ with:*

a) $E \subseteq E^*$;

b) E^* is directed;

c) for each $e \in E^*$ some $\hat{e} \in E^*$ has $e \ll \hat{e}$;

d) $|E^*| = |E| \cdot \omega$.

Proof: Fix $g \in \bigcap\{\Uparrow(e) : e \in E\}$. In this proof, we will apply the following statement recursively. Start with some $F \subseteq E$ and the fixed $g \in \bigcap\{\Uparrow(p) : p \in F\}$, Then for each $f_1, f_2 \in F$ we have $g \in \Uparrow(f_1) \cap \Uparrow(f_2)$ and we apply Lemma 2.2 to obtain $r_1(f_1, f_2) \in P$ with

$$g \in \Uparrow(r_1(f_1, f_2)) \subseteq \Uparrow(f_1) \cap \Uparrow(f_2)$$

and with $f_1, f_2 \ll r_1(f_1, f_2) \ll g$. Recursively apply Lemma 2.1 to find $r_k(f_1, f_2) \in P$ with

$$f_1, f_2 \ll r_1(f_1, f_2) \ll r_2(f_1, f_2) \ll \cdots \ll g$$

and let $R_g(F) = F \cup \{r_k(f_1, f_2) : 1 \leq k < \omega \text{ and } f_1, f_2 \in F\}$.

Now let E_1 be the set E given in the statement of Lemma 2.3. Let $E_2 = R_g(E_1)$. Note that $g \in \Uparrow(e)$ for each $e \in E_2$ and that $|E_2| = |E_1| \cdot \omega$. Recursively define $E_{k+1} = R_g(E_k)$ and then let $E^* = \bigcup\{E_k : 1 \leq k < \omega\}$. Clearly E^* is directed and has the other properties required by the lemma. $\square$

In a domain P, the collection $\{\Uparrow(p) : p \in P\}$ is a base for the *Scott topology* on the set P. The subspace $\max(P)$ consisting of all maximal elements of P has a special role to play. We say that a topological space Y is *domain representable* if and only if there is some domain P such that Y is homeomorphic to the subspace $\max(P)$, endowed with the relative Scott topology. In such a case, we often abuse notation and write $Y = \max(P)$. In particular, in Section 3 we will consider situations where $C_p(X)$ is a domain-representable function space and we will write $C_p(X) = \max(P)$ where P is some domain.

The following result must be well-known but we could not find it in the literature, nor could several domain theorists whom we consulted.

Proposition 2.4 *For any set X, the topological product $\mathbb{R}^X$ is Scott-domain representable.*

Proof: Because the proof is the natural one, we only sketch it. We will say that a function ϕ is *useful* if the domain of ϕ is the set X and for each $x \in X, \phi(x)$ is either the entire set $\mathbb{R}$ or is a closed, bounded interval $[a_x, b_x] \subseteq \mathbb{R}$, with $a_x = b_x$ being allowed. Let $B(\phi) = \Pi\{\phi(x) : x \in X\}$ and let $P := \{B(\phi) : \phi \text{ is useful}\}$. Let $\sqsubseteq$ be reverse-inclusion in the set P. Then $(P, \sqsubseteq)$ is a poset and a set $D \subseteq P$ is directed if and only if for each $B(\phi_1), B(\phi_2) \in D, B(\phi_1) \cap B(\phi_2)$ contains some $B(\phi_3) \in D$, and for any directed set $D, \sup(D) = B(\psi)$ where $\psi(x) := \bigcap\{\phi(x) : \phi \in D\}$ for each $x \in X$. One proves that $B(\phi) \ll B(\psi)$ if and only if $\psi(x) \subseteq \mathrm{Int}_{\mathbb{R}}(\phi(x))$ for each $x \in X$ and the set $Restrict(\phi) := \{x \in X : \phi(x) \neq \mathbb{R}\}$ is finite. Maximal elements of P have the form $B(\phi)$ where for all $x \in X, \phi(x)$ is a singleton. Consequently there is a natural 1-1, onto function from $\mathbb{R}^X$ to $\max(P)$ that sends $f \in \mathbb{R}^X$ to $B(\phi_f)$ given by $\phi_f(x) = \{f(x)\}$ for all $x \in X$. Because $\phi \ll \phi_f$ means that $Restrict(\phi)$ is finite, this mapping is a homeomorphism. $\square$

Let κ be a limit ordinal. A *transfinite sequence in a set* Y is a function $\sigma : [0, \kappa) \to Y$. We often identify the transfinite sequence with a listing of its points, writing $\sigma = \{y(\alpha) : \alpha < \kappa\}$. To say that the transfinite sequence $\{y(\alpha) : \alpha < \kappa\}$ *converges* to the point $z \in X$ means that for each neighborhood U of z, there is some $\beta < \kappa$ with the property that $y(\alpha) \in U$ for each $\alpha \in [\beta, \kappa)$. It is easy to show that if $\{y(\alpha) : \alpha < \kappa\}$ converges to $z \in X$ and if $L \subseteq [0, \kappa)$ is cofinal in κ, then $\{y(\alpha) : \alpha \in L\}$ also converges to z. Consequently, we may replace any transfinite sequence by a sub-sequence indexed by a regular cardinal. It is also easy to see that if $\{y(\alpha) : \alpha < \kappa\}$ is a transfinite sequence converging to $z \in X - \{y(\alpha) : \alpha < \kappa\}$, where κ is a regular infinite cardinal, then some subsequence of distinct points converges to z.

A space X is *pseudo-radial* provided a set $Y \subseteq X$ fails to be closed if and only if there is a transfinite sequence σ in Y and a point $z \in X - Y$ to which σ converges (see [1]). First-countable spaces and generalized ordered spaces are well-known examples of pseudo-radial spaces. Recall that a *generalized ordered space* is a triple $(X, <, \tau)$ where $<$ is a linear ordering of X and τ is a Hausdorff topology on X such that each point of X has a τ-neighborhood base consisting of order-convex (possibly degenerate) sets. If τ is the usual open interval topology of the linear order, then $(X, <, \tau)$ is a *linearly ordered topological space* (LOTS). Čech proved that topological space is a GO-space if and only if it can be embedded topologically into some LOTS.

3 Proof of the Main theorem

In this section, we prove the main theorem announced in the Introduction. To that end, consider a completely regular space X and a cardinal κ such that subsets of X of size smaller than κ are closed and discrete, and we suppose that X also contains a non-trivial transfinite sequence $\{y(\alpha) : \alpha < \kappa\}$ that converges to some point $z \in X - Y$, where $Y = \{y(\alpha) : \alpha < \kappa\}$. We want to show that $C_p(X)$ is not domain representable. For contradiction, suppose that $C_p(X)$ *is* domain representable, say $C_p(X) = \max(P)$ where $(P, \sqsubseteq)$ is some domain. Our proof will produce a function $h : X \to X$ that must be continuous, and yet cannot be continuous.

Claim 1: We claim that $\kappa > \omega$. If $\kappa = \omega$ then our transfinite sequence is a simple infinite sequence $\{y(n) : n < \omega\}$ that converges to $z \in X - Y$. For each $n \geq 1$, let

$$G_n := \{g \in C_p(X) : \text{ for some } i, j > n,\ |g(i) - g(j)| > 1\}.$$

Then G_n is a dense open set in $C_p(X)$ and $\bigcap\{G_n : 1 \leq n < \omega\} = \emptyset$. That is impossible because $C_p(X)$, being domain representable, is a Baire space [7]. Therefore $\kappa > \omega$ and Claim 1 is established.

We will say that a pair (λ, C) is *acceptable* if

1) λ is a limit ordinal with $\omega \leq \lambda < \kappa$;

2) $C \subseteq P$ is a directed set;

3) $|C| \leq |\lambda|$;

4) if $p \in C$ then some $p' \in C$ has $p \ll p'$;

5) if $f \in \max(P)$ has $f \in \bigcap\{\Uparrow(p) : p \in C\}$ then $f(z) = 0$ and for some $\alpha \geq \lambda$, $f(y(\alpha)) = 1$.

Let Ψ be the collection of all acceptable pairs, and partially order Ψ by the rule that $(\lambda_1, C_1) \preceq (\lambda_2, C_2)$ if and only if either $(\lambda_1, C_1) = (\lambda_2, C_2)$ or else $\lambda_1 < \lambda_2$ and $C_1 \subseteq C_2$.

Claim 2: $\Psi \neq \emptyset$.

To prove Claim 2 we will exhibit an acceptable pair (ω, C). First note that, in the light of Claim 1, the point $y(\omega)$ belongs to the transfinite sequence. Because X is completely regular, there is some $f_0 \in C_p(X)$ with $f_0(y(\omega)) = 1$ and $f_0(z) = 0$. Let $S_0 = \{y(\omega), z\}$ and let $\varepsilon_0 = 1$. The basic neighborhood $O(f_0, S_0, \varepsilon_0)$ in $C_p(X)$ is a relatively open subset of $\max(P)$ so there is some $p_0 \in P$ with $f_0 \in \Uparrow(p_0) \cap \max(P) \subseteq O(f_0, S_0, \varepsilon_0)$. Then there is a finite set S_1 and some $\varepsilon_1 \in (0, \frac{1}{2})$ with $f_0 \in O(f_0, S_1, \varepsilon_1) \subseteq \Uparrow(p_0) \cap \max(P)$. Necessarily $S_0 \subseteq S_1$. Because $f_0 \in O(f_0, S_1, \varepsilon_1)$ we may find $p_1 \in P$ with $f_0 \in \Uparrow(p_1) \cap \max(P) \subseteq O(f_0, S_1, \varepsilon_1) \subseteq \Uparrow(p_0)$. It follows from Lemma 2.2 that we may assume $p_0 \ll p_1$. Continue this process recursively to obtain finite sets $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k$ and positive numbers $\varepsilon_k < 2^{-k}$ and elements $p_0 \ll p_1 \ll \cdots \ll p_k \in P$ with

$$f_0 \in \Uparrow(p_{k+1}) \cap \max(P) \subseteq O(f_0, S_{k+1}, \varepsilon_{k+1}) \subseteq \Uparrow(p_k) \text{for each } k < \omega.$$

Let $C = \{p_k : k < \omega\}$. It is clear that (ω, C) satisfies the first four parts of the definition of an acceptable pair. To verify the fifth, suppose $f \in \max(P) = C_p(X)$ has $f \in \bigcap\{\Uparrow(p_k) : k < \omega\}$. Then $f \in \Uparrow(p_k) \cap \max(P) \subseteq O(f_0, S_k, \varepsilon_k)$ for each k so that $z \in S_0 \subseteq S_k$ yields $|f(z) - f_0(z)| < \varepsilon_k < 2^{-k}$ showing that $f(z) = f_0(z) = 0$. Similarly, $f(y(\omega)) = f_0(y(\omega)) = 1$. Hence $(\omega, C) \in \Psi$ and Claim 1 holds.

Because $(\Psi, \preceq)$ is a nonempty poset, Zorn's Lemma provides a maximal chain $\Phi \subseteq \Psi$. Let π_i be projection onto the i^{th} coordinate for $i = 1, 2$. Then $\pi_1[\Phi] = \{\lambda :$ some element of Φ has first coordinate $\lambda\}$

which is a subset of $[0,\kappa)$. Let $\mu = \sup(\pi_1[\Phi])$. Then $\mu \leq \kappa$. Because Φ is a chain, each $\lambda \in \pi_1[\Phi]$ is the first coordinate of exactly one member of Φ, and we denote that member by (λ, C_λ). Consequently, $|\Phi| \leq |\mu| \leq \mu \leq \kappa$. Also, let $D = \bigcup\{C_\lambda : (\lambda, C_\lambda) \in \Phi\}$. Then D is a directed subset of P, so $\sup(D) \in P$. Choose any maximal element $g_0 \in \max(P) = C_p(X)$ with $\sup(D) \sqsubseteq g_0$. Note that for each $d \in D$, some $\hat{d} \in D$ has $d \ll \hat{d} \sqsubseteq \sup(D) \sqsubseteq g_0$, so that $g_0 \in \bigcap\{\Uparrow(d) : d \in D\}$.

Claim 3: We claim that $\mu = \kappa$. If not, then $\mu < \kappa$ so that $|\mu| \leq \mu < \kappa = |\kappa|$. From above $g_0 \in \Uparrow(d) \cap \max(P)$ for each $d \in D$. That yields a finite set $T_d \subseteq X$ and a positive δ_d with $g_0 \in O(g_0, T_d, \delta_d) \subseteq \Uparrow(d) \cap \max(P)$ for each $d \in D$. We may assume that $z \in T_d$ for all $d \in D$. Let $T = \bigcup\{T_d : d \in D\}$. Being a union of at most $|\mu|$ many finite sets, $|T| \leq |\mu| < \kappa$ so that T is closed and discrete.

Because κ is a regular uncountable cardinal and $\mu < \kappa$, we know that $\mu + \omega < \kappa$. We know that $|T| < \kappa$ so we may choose some $\gamma \in [\mu + \omega, \kappa)$ with $y(\gamma) \notin T$. Because $y(\gamma) \notin T$, complete regularity of X gives a continuous function $g_1 \in C_p(X)$ with $g_1(x) = g_0(x)$ for each $x \in T$ and $g_1(y(\gamma)) = 1$. Because $g_1(x) = g_0(x)$ for all $x \in T$, we know that $g_1 \in O(g_0, T_d, \delta_d) \subseteq \Uparrow(d)$ for each $d \in D$.

Let $R_0 = \{y(\gamma), z\}$ and $\eta_0 = 1$. Consider the relatively open set $O(g_1, R_0, \eta_0)$. We may find $q_0 \in P$ with $g_1 \in \Uparrow(q_0) \cap \max(P) \subseteq O(g_1, R_0, \eta_0)$. Following the pattern in Claim 1, we recursively find finite sets R_k, positive numbers $\eta_k < 2^{-k}$, and points $q_k \in P$ with $q_0 \ll q_1 \ll \cdots \ll q_k$, $R_0 \subseteq R_1 \subseteq \cdots \subseteq R_k$ and $g_1 \in \Uparrow(q_k) \cap \max(P) \subseteq O(g_1, R_k, \eta_k) \subseteq \Uparrow(q_{k-1})$ whenever $k \geq 1$. We note that any $g \in \max(P) \cap \bigcap\{\Uparrow(q_k) : k < \omega\}$ has $g(z) = g_0(z) = 0$ and $g(y(\gamma)) = g_0(y(\gamma)) = 1$. Let $E = D \cup \{q_k : k < \omega\}$ and note that $g_1 \in \bigcap\{\Uparrow(e) : e \in E\}$. Now apply Lemma 2.3 to find a directed set $E^* \subseteq P$ containing E, with $|E^*| = |E| = |D| = |\mu| = |\mu + \omega|$. Therefore $(\mu + \omega, E^*) \in \Psi$ and $(\mu + \omega, E^*)$ is strictly above every $(\lambda, C_\lambda) \in \Phi$, contradicting the fact that Φ is a maximal chain in $(\Psi, \preceq)$. Therefore, Claim 3 is established.

At this stage of the proof, we have constructed the directed set $D = \bigcup\{C_\lambda : (\lambda, C_\lambda) \in \Phi\}$ and we know that $\{\lambda < \kappa :$ for some $C, (\lambda, C) \in \Phi\}$ is cofinal in $[0,\kappa)$. Then $\sup(D) \in P$ because P is a domain. Choose any $h \in \max(P) = C_p(X)$ that has $\sup(D) \sqsubseteq h$. Then $h \in \bigcap\{\Uparrow(d) : d \in D\} \subseteq \bigcap\{\Uparrow(d) : d \in C_\lambda\}$ for each $\lambda \in \pi_1[\Phi]$. Because $h \in \bigcap\{\Uparrow(d) : d \in D\}$ we know that $h(z) = 0$. Because h is continuous and the transfinite sequence $\{y(\alpha) : \alpha < \kappa\}$ converges to z, there is some $\beta < \kappa$ such that $h(y(\alpha)) \in (-\frac{1}{2}, \frac{1}{2})$ for all $\alpha \in [\beta, \kappa)$. Because $\{\lambda : (\lambda, C_\lambda) \in \Phi\}$ is a cofinal subset of $[0,\kappa)$, for some $(\lambda, C_\lambda) \in \Phi$ we have $\beta < \lambda < \kappa$. Using the fifth property of the acceptable pair (λ, C_λ), because $h \in \bigcap\{\Uparrow(d) : d \in D\} \subseteq \bigcap\{\Uparrow(d) : d \in C_\lambda\}$ we know that for some $\delta \geq \lambda$ we have $h(y(\delta)) = 1$. That contradiction completes the proof that, for the types of spaces considered in the Main Theorem, $C_p(X)$ cannot be domain representable unless X is discrete.

Now consider any pseudo-radial space X. If X is not discrete, then there is a minimal cardinal κ such that some set of cardinality κ is not closed. Let M be such a non-closed set. Then there is some $z \in X - M$ and some transfinite sequence $\{y(\alpha) : \alpha < \lambda\}$ in M that converges to z. As noted above, we may assume that λ is a regular cardinal. Consequently we may assume that $y(\alpha) \neq y(\beta)$ whenever $\alpha < \beta < \lambda$. Because $|M| = \kappa$ we conclude that $\lambda \leq \kappa$. Because sets of cardinality $< \kappa$ are closed, we know that $\kappa \leq \lambda$, so $\kappa = \lambda$. Now we have exactly the situation described in the first part of the proof so we know that $C_p(X)$ cannot be domain representable. Hence (a) implies (b) in the Main Theorem. That (b) implies (c) follows from Proposition 2.4, and trivially (c) implies (a).$\square$

4 Other strong completeness conditions in $C_p(X)$

Starting in the 1950s, several strong completeness conditions were studied in an attempt to understand products of Baire spaces. Oxtoby [10] called a space X *pseudocomplete* if there is a sequence $\langle \mathcal{P}_n \rangle$ of pseudobases (= π-bases) such that $\bigcap\{P_n : n \geq 1\} \neq \emptyset$ whenever $P_n \in \mathcal{P}_n$ with $\text{cl}(P_{n+1}) \subseteq P_n$. (Remember that all spaces in this paper are at least regular.) Choquet [3] introduced two topological games. In the first, now called the *weak Choquet game*, Players 1 and 2 alternate specifying non-empty open sets $U_1, U_2, U_3, \cdots$ having $U_{n+1} \subseteq U_n$ for each n. The second game, called the *strong Choquet game*, is a non-symmetric version of the first: Player 1 specifies a pair (x_1, U_1) where U_1 is open and $x_1 \in U_1$. Then Player 2 chooses an open set U_2 with $x_1 \in U_2 \subseteq U_1$. In general, Player 1 specifies a pair (x_{2n+1}, U_{2n+1}) with U_{2n+1} open and $x_{2n+1} \in U_{2n+1} \subseteq U_{2n}$. Then Player 2 responds by choosing an open set U_{2n+2} with $x_{2n+1} \in U_{2n+2} \subseteq U_{2n+1}$. In both Choquet games, Player 2 wins the game if $\bigcap\{U_n : n \geq 1\} \neq \emptyset$, and the question is whether Player 2 has a *winning strategy*, i.e., a strategy for choosing responses that leads to a win for Player 2, no matter what Player 1 does. If Player 2 has a winning strategy for the weak Choquet game (resp. the strong Choquet game), then the space X is said to be *weakly Choquet complete* (resp. *strongly Choquet complete)*. The winning strategy in either of the Choquet games is allowed to depend upon the entire history of the game up to the point where Player 2 must choose the next open set. In some references, Player 2 is said to have "perfect information." But it might happen in the strong Choquet game (resp., weak Choquet game) that Player 2 can determine U_{2n+2} knowing only the pair (x_{2n+1}, U_{2n+1}) (resp., knowing only the set U_{2n+1}) and in that case the strategy used by Player 2 is called a *stationary strategy*.

In general, weak and strong Choquet completeness are distinct concepts. For example, a metric space that is not complete and has a dense set of isolated points will be weakly Choquet complete, but not strongly Choquet complete. However, in function spaces $C_p(X)$ the situation is quite different, as our next result shows. The hypothesis of the next proposition includes a property called *pseudonormaility*. To say that a space X is *pseudonormal* means that two disjoint closed sets, one of which is countable, can be separated by open sets and, as pointed out in Lemma 8.3 of [5], in a completely regular pseudonormal space, any continuous function defined on a countable, closed, discrete subspace can be extended continuously over the entire space.

Proposition 4.1 *Suppose X is a completely regular pseudonormal space. Then the following are equivalent:*

- *a)* $C_p(X)$ *is strongly Choquet complete and Player 2 has a stationary strategy in the strong Choquet game;*
- *b)* $C_p(X)$ *is strongly Choquet complete;*
- *c)* $C_p(X)$ *is weakly Choquet complete;*
- *d)* $C_p(X)$ *is pseudocomplete;*
- *e)* $C_p(X)$ *has nonempty intersection with every nonvoid* G_δ*-subset of the product space* R^X*;*
- *f) every countable subset of X is closed.*

Proof: The equivalence of statements c), d), e), and f) was established in Theorem 8.4 of [5]. Obviously a) implies b).

To see that b) always implies c), let σ be a winning strategy for Player 2 in the strong Choquet game, and suppose that Player 1 opens the weak Choquet game by specifying a nonempty open set U_1. Player 2 picks any point $x_1 \in U_1$ and then uses σ to determine the response to $U_2 = \sigma(x_1, U_1)$. If Player 1 responds to U_2 by specifying the nonempty open set U_3, then Player 2 chooses any point $x_3 \in U_3$ and uses strategy σ to choose $U_4 = \sigma((x_1, U_1), U_2, (x_3, U_3))$. Continuing in this fashion guarantees a win for Player 2 in the weak Choquet game.

To prove that f) implies a) we define a stationary winning strategy for Player 2 in the strong Choquet game in $C_p(X)$. In response to any pair (g, U) proposed by Player 1 at any stage of the game, Player 2 should find a finite set S and a positive ε so that the basic open set $O(g, S, 2\varepsilon) \subseteq U$ and then Player 2 should respond with $\sigma(g, U) = O(g, S, \varepsilon)$. To see the role of the number 2 in this strategy, consider three consecutive moves in the game, say $U_{2k} = O(f, S, \delta)$ followed by Player 1's response (g, U), followed by Player 2's response $U_{2k+2} = O(g, T, \varepsilon)$. Because $O(f, S, \delta) \supseteq U \supseteq O(g, T, 2\varepsilon)$ we know that $S \subseteq T$ and for any $x \in S$ we have $(f(x) - \delta, f(x) + \delta) \supseteq (g(x) - 2\varepsilon, g(x) + 2\varepsilon) \supseteq [g(x) - \varepsilon, g(x) + \varepsilon]$. Now suppose that Player 2 uses the strategy σ to choose the even-numbered terms in the sequence $(f_1, U_1), U_2, (f_3, U_3), U_4, \cdots$. Then $U_{2k+2} = O(g_{2k+1}, S_{2k+2}, \varepsilon_{2k+2})$ for some finite set S_{2k+2} with $S_{2k} \subseteq S_{2k+2}$ for each k. Let $T = \bigcup\{S_{2k} : k \geq 1\}$. Then T is countable and for each $x \in S_{2k} \subseteq T$ there is some real number $h(x) \in \bigcap\{(g_{2j-1}(x) - \varepsilon_{2j}, g_{2j-1}(x) + \varepsilon_{2j}) : k \leq j < \omega\}$. This defines a function $h : T \to \mathbb{R}$. Because T is countable, T is closed and discrete in X so that $h : T \to \mathbb{R}$ is continuous. Because X is completely regular and pseudonormal, the function h has a continuous extension $\hat{h} \in C_p(X)$. Then $\hat{h} \in \bigcap\{U_k : k \geq 1\}$ as required to show that the second player's strategy σ is a winning strategy in the strong Choquet game on $C_p(X)$. □

Combining our results in Sections 3 and 4 gives a simple example showing that our conjecture in Section 1 cannot be extended to include the situation where $C_p(X)$ is strongly Choquet complete but not domain representable.

Example 4.2 *There is a GO-space X such that $C_p(X)$ is strongly Choquet complete and yet $C_p(X)$ is not domain representable.*

Proof: Let $X = [0, \omega_1]$ where each countable ordinal is isolated and ω_1 has its usual neighborhoods. Then X is a GO-space so that by our Main Theorem, $C_p(X)$ is not domain representable. However by Proposition 4.1, $C_p(X)$ is strongly Choquet complete (and Player 2 has a stationary winning strategy). □

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