A Decade of REU at William and Mary

by

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1 Institutional Overview and Program Goals.

The College of William and Mary is a state supported university that has always placed a major emphasis on providing high quality undergraduate education. Recent decades have seen the growth of graduate degree programs and today, in addition to its undergraduate program (with 5400 students), there are graduate programs available in 13 departments and Schools at William and Mary, involving some 2000 students.

The Department of Mathematics has 19 full-time faculty, each involved in teaching at the undergraduate level while maintaining an active research agenda. Building a bridge between faculty research and undergraduate teaching, the department has conducted NSF-funded REU programs each summer since 1990. Each of these REU programs was built around matrix analysis and its applications, one of the department's preponderant research strengths and an area in which the department is a leading research center.

The Mathematics department participates in an interdisciplinary doctoral program, cooperating with the College's Applied Science program to offer an apprentice-style program leading to a doctoral degree in applied mathematics. Currently there are four doctoral students enrolled in this graduate program, and the program has a linkage to REU, as described below.

A central objective of our REU programs is to provide talented students with experience in how mathematics is done, something that is quite different from students' typical classroom experiences. Our approach places great emphasis on the process and excitement of independent discovery by the student, working with challenging, unsolved problems, as well as on the interactions of students with faculty mentors, and of students with one another. Our hope is that this summer experience will lead students to pursue graduate work in one of the mathematical sciences.

Our REU program attempts to maximize the probability of success of the research experience for students by incorporating a number of short-term goals. For example, introductory lectures and weekly seminars are intended to increase the participants' knowledge of modern research areas as well as to whet their general mathematical curiosity. REU participants have access to, and receive special training in. mathematical software such as LaTeX, MATLAB, and MAPLE, and are encouraged to use such tools. Furthermore, students are encouraged to keep a research diary to chronicle their daily progress, their conjectures and questions, as well as ideas that didn't appear to work. By the end of the program each participant will thus have a tangible record of accomplishments that will be the basis for the participant's final report.

Our program stresses the need for communicating results to peers and to the community at large. A number of early presentations introduce students to reading, writing, and explaining mathematics. Later on, students themselves present oral interim reports on their research to the entire group of faculty and students, and then present a final oral report in the eighth week. Students typically practice these reports with their REU mentors because one of our goals is to improve students' communication skills.

2 Mathematical Theme.

Throughout the decade of our REU efforts, the overall scientific theme of the program has been “Matrix Analysis and Applications.” That theme is particularly appropriate because it encourages a combination of inductive and deductive approaches. For example, the individual research topics (samples appear below) readily lend themselves to exploration by hand or by computer and then subsequent generalization, with newly derived conjectures capable of further exploration. Significantly, only a modest undergraduate mathematics background is needed to appreciate and understand some interesting open problems formulated in terms of matrices (or graphs arising from matrices). As a result, students with varying degrees of formal mathematical training can be more easily accommodated. An added benefit is that knowledge of this area provides a solid background for further study in a number of mathematical specialties. Finally, having a common theme not only allows students to discuss problems with several faculty advisors but also encourages students to exchange ideas and assist one another, in some cases working in small teams with one or two research mentors.

A wide variety of research topics have been studied by REU participants and mentors during the decade of this project, but certain broad themes unite them. Individual projects have been open ended, and a project begun in one summer by an REU student is often continued in later summers by another, using a report or publication by the first student as a starting point for further research. The key criteria for judging possible research topics include their accessibility to well-prepared and capable undergraduates, the modularity of the topic - the extent to which it can be broken into steps of increasing depth and complexity - and the likelihood of obtaining interesting and satisfying results in an eight week summer session. By way of illustration, we list a few broad themes from which many REU problems have been drawn.

a) Linear preserver problems: The “linear preserver problem” asks for characterizations of linear operators on matrix spaces that leave invariant certain properties, relations, or subsets. Particular instances include the study of isometry problems and linear transformations that leave invariant controllable matrix pairs. In addition, many results from the matrix setting are being extended to infinite dimensional contexts and even to more general algebraic structures.

b) Determinants of matrices: Consider the set of all n x n matrices consisting of 0s and 1s, and having each row sum and each column sum equal to $k < n$. What are the possible determinant values for such matrices? Bounds are known, and complete solutions are known for special values of n and k, but the general case is open. Next consider a real symmetric matrix A with singular values $a_1, a_2, \cdots, a_n$ and a real skew symmetric matrix B with singular values $b_1, b_2, \cdots, b_n$. What are the possible determinant values of $A+B$?

c) Qualitative matrix theory: A sign pattern $A = [a_{ij}]$ is a rectangular array of signs $(+,-)$ and zeros. Let $Q(A)$ be the set of all real matrices whose i,j entry has the same sign as $a_{i,j}$. Qualitative matrix theory deals with the study of $Q(A)$ and, most often, with questions about whether there exists a $B \in Q(A)$ having a certain property P (in which case we say that “A allows P)” or whether all members of $Q(A)$ have property P (in which case “A requires P”). Classical questions of qualitative matrix theory deal with the properties P = invertibility or stability, and more recently with questions involving row and column sums, semi-positivity, structure of null vectors, and the probability of a positive determinant. Often there are corresponding questions for “patterns” (arrays that specify the positions of zero and non-zero entries in a matrix). Historically, such questions arose from biology, economics, and chemistry, and more recently from computer science and mathematics.

d) Matrix convergence questions: A set $P = \{A_i : i \in I\}$ of real $n \times n$ matrices is "point-wise convergent" provided that for each $x \in \mathbb{R}$ there is a sequence $\{p(x, j) : j > 1\}$ of elements of I such that

$$\lim_{k\to\infty} \left(\Pi_{j=k}^{1} A_{p(x,j)}\right) x = 0,$$

and P is "uniformly convergent" if a single sequence $p(j)$ can be chosen independently of x. Certain necessary and sufficient conditions for each type of convergence are known, and useful criteria for recognizing each type of convergence are available, given certain restrictions on P. Analogous criteria are needed for other sets of matrices. In particular, it is known that point-wise and uniform convergence are equivalent for classes of entry-wise non-negative matrices, and a useful test for convergence in this case would be valuable.

e) Factorizations of almost periodic matrix functions. Almost periodic matrix functions are matrices with entries of the form $\Sigma_j c_j e^{i\lambda_j x}$, where the c_j are complex numbers, the λ_js are real, and x is a real variable. The sum may be finite or infinite. Factorizations of such functions as a product of at most three almost periodic matrix functions where the left factor has all λ_js non-positive, and the middle factor has basic exponentials on the main diagonal and zeros elsewhere, arise in applications to Weiner-Hopf and convolution type equations, and in mathematical physics (e.g., inverse wave scattering). Therefore it is of interest to study factorizations for special classes of almost periodic matrix functions and, if possible, to obtain explicit formulas for the factors. It turns out that this is a challenging mathematical problem and even for many seemingly simple classes such as 2 x 2 triangular matrices, there are no satisfactory answers so far.

f) Matrix completion problems: A partial matrix is one having some entries specified while others are free to be chosen from an agreed upon set (e.g., the real numbers). A completion of a partial matrix is a choice of values for the unspecified entries that results in a conventional matrix. A matrix completion problem asks which partial matrices have a completion of a designated type (e.g., positive definite, totally positive, etc.). Often, the arrangement of specified entries plays a key role in such problems and combinatorial issues become very important.

3 Student Recruitment and Selection.

Students in our program are recruited from a national pool. In recent years, the applicant pool has ranged from 110 to 160 students, and we have chosen eight students for our program. We also recruit students from William and Mary and from nearby colleges and universities, but the overwhelming majority of our REU students come from other states. From time to time, foreign students have joined our program, with support from non-NSF sources.

Our recruitment efforts begin in the early fall and are carried out through a variety of mechanisms. Notices of the program are sent to almost 200 mathematics departments across the nation. Additional national exposure is obtained by posting notices of the program on various electronic networks (e.g., International Linear Algebra Society bulletin board, CSNET) and by providing information to relevant special interest newsletters (SIAM Activity Group on Linear Algebra Newsletter, ILAS Newsletter). To help students find us, we rely on NSF to maintain an up-to-date listing of REU programs on its web site and on organizations such as MAA to provide appropriate cross linkages from their own web sites.

Each applicant is asked to provide two letters of recommendation, a list of mathematics courses completed, including the student's final grades and texts used, a list of courses that will be completed in the spring semester before the student comes to our program, and a personal statement of interests. Students for the program are chosen by a committee of program faculty. Our selection process takes into account evidence of the applicant's ability, as evidenced by grades and letters of recommendation, and the applicant's personal motivation and the possibility of growth through the program, as evidenced by the student's personal statement and comments in letters of recommendation. We are particularly interested in students' performance in either (or both) of the first courses in modern algebra and analysis. We try to be particularly receptive to applicants from institutions that do not provide graduate programs or other research opportunities for their undergraduates. We also attempt to make sure that women have access to our programs. Over ten years, eighty students have been supported by NSF in our program - thirty women and fifty men.

To help admitted students decide whether to join our program, we ask one summer's REU students whether they would agree to be contacted by students admitted to the next summer's program. To date, all have agreed. It seems that about a fifth of admitted students make contact with the previous year's participants.

4 Research Mentorship.

Most of our REU students have worked 1-1 with advisors, with an occasional team of two students working on a problem with one or more advisors. Advisors typically meet for an hour per day with each student (or student team), and some advisors hold longer daily consultation periods.

Our experience has shown the importance of daily faculty guidance in undergraduate research. During the REU program, each student's faculty advisor poses a progressive series of problems that are of increasing difficulty and generality and that lead towards the overall goal of the student's program. This strategy allows for positive reinforcement of the student as he or she advances in incremental steps to more difficult and unknown territory.

From time to time, we have involved one of our advanced doctoral students in our REU program, as a closely supervised research mentor. REU students have responded very positively to these graduate students, probably because the age and cultural differences between REU students and the graduate students are less than the corresponding differences between REU students and the rest of us. During our 1997-8-9 grant period, with NSF approval, we expanded from one to two graduate student mentors. We invited advanced graduate students from other universities to join us. We chose graduate students who

a) were expert in mathematics related to our program theme;

b) had been involved in successful undergraduate research projects themselves.

The two outside graduate students chosen in 1997 and 1999 respectively are writing their theses at Berkeley and at M.I.T., and one was a graduate of our own REU program. During their time with us as supervised research mentors, the graduate students received training in the art of being research supervisors, and were also able to collaborate mathematically with members of our matrix and operator theory group.

5 Program Structure.

Prior to arrival on campus, the participating undergraduates receive information on the expected background in linear algebra and matrix theory (suggested sections of several textbooks are cited) and students have a sampling of typical REU project areas from previous years.

During the first week of the program, faculty present background lectures on their proposed research projects, so that each student can select an appropriate research topic and advisor. A side effect of these lectures is to introduce all students to all projects, thereby facilitating interactions between students whose projects are similar. Also during the first week of the program, we provide special training in computing tools such as MAPLE and MATLAB that are useful for research exploration in matrix analysis, and in the use of the internet and other library resources in mathematics. At an early stage of the program, students are introduced to the use of one of the standard technical text-processing languages, e.g., LaTeX. (During their time at William and Mary, REU students have access to a Pentium-equipped Windows NT laboratory, and a special departmental laboratory equipped with Linux computers. Both laboratories are equipped with MATLAB and MAPLE and are located in the mathematics department's building.)

Individual research, rather than in-class learning, is the heart of our program. Nevertheless, we do schedule seminars for our REU students. From time to time during later weeks, selected faculty members present brief talks on useful techniques for conducting research. Other seminars are presented by visitors to the department. We obtain visiting speakers at no cost to the REU program by judiciously arranging the normal trips of research visitors to the department. We run an appropriate social program involving the speaker, faculty, and REU students in conjunction with the seminar series.

Another type of talk focuses on applications of matrix analysis in other disciplines. In recent years, faculty members from economics, computer science, and physics have presented seminars on applications of linear algebra in their disciplines.

Finally, we offer a special seminar with the graduate program director from a Group I Ph.D. department to discuss what to look for in a mathematics graduate school, the application process, various kinds of financial support available to mathematics graduate students, etc. In the last three years, students have found these sessions to be very helpful. In the summer of 1999 we experimented with a virtual visit, using an internet linkage, and student response was good.

After four weeks, students present reports on their research topics and describe their preliminary findings to the entire group. In the eighth week, students present a final oral report and a final written report on their work. For many students, this report will be the basis of a journal article that (previous experience shows) students work on in the weeks after the REU program ends, collaborating with faculty mentors by e-mail.

In most years, our REU program includes a visit to one of the federal research facilities in the area, where students can be introduced to mathematical scientists and see at first hand the dynamics of on-going research programs. NASA/Langley Research Center, ICASE, the Virginia Institute of Marine Science, and the Jefferson Continuous Electronic Beam Accelerator Facility are sufficiently close for a day's excursion.

To help students feel at ease with each other and with program faculty, one day each week we schedule a fast food lunch for students and REU faculty. This is an opportunity for extremely informal discussions of small problems, progress of students and faculty on problems of current interest, and non-mathematical topics. In addition, we encourage all faculty and students to attend informal coffee and cookie breaks held from time to time during the program.

6 Program Evaluation and Follow-up.

One very important measure that we use to judge success of our REU program is the extent to which our students discover new mathematical knowledge concerning substantial unsolved problems. To make that evaluation, we rely on the judgment of our faculty research mentors, something that we trust because of our faculty members' own research records and their many years of successful experience as undergraduate research advisors. But some quantitative measures are also available, and we use them. For example, we. are interested in the percentage of REU students who eventually become co-authors of refereed mathematical papers with their faculty advisors. Since 1990, over 40% of our REU students have become co-authors with their research mentors of refereed journal articles.

Our students' REU experiences have been recognized by outside groups other than mathematical journals: for example, based upon his research in our 1995 REU program, one of our REU students won the University of Maryland's Dorfman Prize for the best research by an undergraduate in the Computer, Mathematical, and Physical Sciences. Other REU projects have become part of undergraduate theses at Harvard that have won highest honors.

To evaluate our success in encouraging students toward graduate school in mathematics, we have followed students after they completed our program. Projections from surveys of the 56 students from the 1990-96 REU programs suggest that perhaps 90% of our REU students pursued graduate study in the mathematical (or related) sciences. To obtain better response rates to our surveys, we. decided to adopt a more aggressive follow-up strategy starting in 1997. We have complete data on the 24 students supported during the 1997-8-9 grant period and can report that every one of those students is either still an undergraduate or is enrolled in graduate school, with one being a doctoral student in finance and the others being students in mathematical sciences.

In addition to the numerical data above, we survey our students at the end of their summer REU program, collecting comments about the success of various aspects of the program, and getting new ideas for improving future programs. We ask students to complete a brief anonymous questionnaire, reporting on:

a) whether, in retrospect, we sent them adequate and accurate information on the mathematical content of our program, and on housing, meals, travel, airports, etc.;

b) the adequacy of housing, office space, computer access, and library materials during our program;

c) the various seminars during the program;

d) interaction with the research mentors (Was the relationship friendly? Did they get, to spend enough time with their mentors? Were the mentors helpful in guiding research?);

e) what impact the program had on their plans for studying more mathematics and attending graduate school in mathematics.

Student responses have been very favorable. Combined with impressions gathered by faculty mentors from conversations with advisees, student responses from one summer help us plan the next.

We also maintain e-mail contact with former REU students and follow their progress. Initially tills contact is often part of a collaborative effort between student and advisor to prepare an article for submission to a journal. But the personal relationship between advisor and REU student often continues over time as students pursue their mathematical studies in graduate school and the REU program director has not been shy about contacting former REU students.

7 Publications Resulting from Previous NSF REU Support.

In the past, our REU students have often become co-authors of refereed journal articles. In the following listing, REU students are marked with an asterisk.

C. Gates*, J. Drew, C. Johnson, and C. Tart, Characterization of super-commuting matrices, Linear and Multilinear Algebra 43 (1997), 35-51.

S. Chang* and C-K Li, A special linear operator on $M_4(\mathbb{R})$, Linear and Multilinear Algebra, 30 (1993), 65-75.

S. Chang* and C-K Li, Certain isometrics on $\mathbb{R}^n$, Linear Algebra and Applications, 165 (1992), 251-261.

G.S. Cheon, S.G. Lee, C.R. Johnson, and E. Pribble*, The possible number of zeros in an orthogonal matrix, Elec. J. Lin. Alg. 5 (1999), 19-23.

A.L. Cohen*, L. Rodman, and D. Stanford, Point-wise and uniformly convergent sets of matrices, Siam J. Matrix Analysis, to appear.

S. Fallat, H.T. Hall, and C.R. Johnson, Characterizations of product inequalities for principal minors of M- and inverse M-matrices, Q.J Math (Oxford) (2) 49 (1998), 451-458.

S. Ferguson*, C. Johnson, and T. Shalom, Information requirements for determining the inverse of a persymmetric matrix, in preparation.

M. Gelfand * and I. Spitkovsky, Almost periodic factorization: applicability of the division algorithm, Advances in Math. Sci., 184 (1998), 97-109.

S. Gottleib*, C. R. Johnson, and I. Spitkovsky, Inequalities involving numerical radii, Linear Algebra and its Applications, 37 (1994), 13-24.

G. Hartless* and L. Leemis, Computational algebra applications in reliability, IEEE Transactions on Reliability, 45 (1996), 393-399.

J. Helton, D. Lam*, and H. Woerdeman, Sparsity patterns with high rank extremal semipositive definite matrices, SIAM J Matrix Anal. Appl., 15 (1994), 299-312.

C. Johnson, C. Jones*, and B. Kroschel, The Euclidean distance completion problem: cycle completability, Linear and Multilinear Algebra, 39 (1995), 195-207.

C. R. Johnson, M. K. Kerr*, and D. P. Stanford, Semi-positivity of matrices, Linear and Multilinear Algebra 37 (1994), 265-271.

C. R. Johnson and J. S. Miller*, Rank decomposition under combinatorial constraints, Linear Algebra and its Applications, 251 (1997), 97-104.

C.R. Johnson, J. Pitkin*, and D. Stanford, Line sum symmetry via the DownEig algorithm, Computational Optimization and Applications, to appear.

C. Johnson and G. Whitney*, Minimal rank completions, Linear and Multilinear Algebra 28 (1991), 271-273.

C.R. Johnson, S. Lewis*, and D. Yau*, Sign patterns that allow given line sums, Linear Algebra and Applications, to appear.

Yu. Karlovich, I. Spitkovsky, and R. Walker*, Almost periodic factorization of block triangular matrix functions revisited, Linear Algebra and Applications, 293 (1999), 199-232.

D. Keeler*, L. Rodman, and I. Spitkovsky, The numerical range of 3-by-3 matrices, Linear Algebra and its Applications, 252 (1997), 115- 139.

A-L Klaus* and C-K Li, Isometries for the vector(p,q) and induced (p,q) norms, Linear and Multilinear Algebra 38 (1995), 315-332.

C-K Li and P. Metha*, Permutation invariant norms, Linear Algebra and its Applications, 219 (1995), 93-110.

C-K Li, J. Lin*, and Rodman, L., Determinants of certain classes of zero-one matrices with equal line sums, Rocky Mountain J. Math., to appear.

C-K Li, P. Metha*, and L. Rodman, Linear operators preserving the inner and outer c-spectral radii, Linear and Multilinear Algebra 36 (1004), 195-204.

C-K Li, P. Mehta*, L. Rodman, A generalized numerical range: the range of a constrained sesquilinear form, Linear and Multilinear Algebra 34 (1994), 25-49.

C-K Li, S. Shukla*, and I. Spitkovsky, Equality of higher numerical ranges of matrices and a conjecture of Kippenhahn on hermitian pencils, Linear Algebra and its Applications, 270 (1997), 323-349.

C-K Li and W. Whitney*, Symmetric overgroups of S_n in O_n, Canadian Math. Bulletin 39 (1996), 83-94.

D. Quint*, L. Rodman, and I. Spitkovsky, New cases of almost periodic factorization of triangular matrix functions, Michigan J, Math., 45 (1998), 73-102.

A. C. M. Ran, L. Rodman, and J. E. Rubin*, Direct complements of invariant Lagrangian subspaces and minimal factorization of skew-symmetric rational matrix functions, Linear Algebra and its Applications, 180 (1993), 61-94.

I. Spitkovsky and D. Yong*, Almost periodic factorization of of certain block triangular matrix functions, Math. of Computation, to appear.

D. P. Stanford and J. Urbano*, Some convergence properties of finite matrix sets, SIAM J. Matrix Analysis and Appl. 15(1994), 1132-1140.

B. Wainberg* and H. Woerdeman, The maximum row sum non-singularity radius, Linear Algebra and its Applications, 247 (1996), 251-264.

Perhaps a dozen additional REU articles are under consideration by journals or are in preparation.

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