The Gruenhage property, property *, fragmentability, and σ-isolated networks in generalized ordered spaces[1]

by

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Abstract: In this paper we examine the Gruenhage property, property * (introduced by Orihuela, Smith, and Troyanski), fragmentability, and the existence of σ-isolated networks in the context of linearly ordered topological spaces (LOTS), generalized ordered spaces (GO-spaces), and monotonically normal spaces. We show that any monotonically normal space with property * or with a σ-isolated network must be hereditarily paracompact, so that property * and the Gruenhage property are equivalent in monotonically normal spaces. (However, a fragmentable monotonically normal space may fail to be paracompact.) We show that any fragmentable GO-space must have a σ-disjoint π-base and it follows from a theorem of H.E. White that any fragmentable, first-countable GO-space has a dense metrizable subspace. We also show that any GO-space that is fragmentable and is a Baire space has a dense metrizable subspace. We show that in any compact LOTS X, metrizability is equivalent to each of the following: X is Eberlein compact; X is Talagrand compact; X is Gulko compact; X has a σ-isolated network; X is a Gruenhage space; X has property *; X is perfect and fragmentable; and the function space $C(X)^*$ has a strictly convex dual norm. We give an example of a GO-space that has property *, is fragmentable, and has a σ-isolated network and a σ-disjoint π-base but contains no dense metrizable subspace.

Key words and phrases: Gruenhage space, property *, fragmentable space, σ-isolated network, LOTS, linearly ordered topological space, GO-space, generalized ordered space, paracompactness, stationary sets, dense metrizable subspace, monotone normality, metrizability, G_δ-diagonal, σ-disjoint π-base, quasi-developable space, Sorgenfrey line, Michael line, Eberlein compact, Talagrand compact, Gulko compact, strictly convex dual norm.

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1 Introduction

Special collections of sets that separate points of a topological space are frequently used in the study of generalized metric spaces and in the study of the geometry of the Banach space $C(X)$ of continuous real-valued functions with the sup-norm, where X is compact and Hausdorff. Examples of such special collections in generalized metric space theory are G_δ-diagonals, quasi-G_δ-diagonals, and σ-discrete networks. Examples used in the study of $C(X)$ and its dual space $C(X)^*$ include T_0-separating collections in Gruenhage spaces, spaces with property *, fragmentable spaces, and spaces with σ-isolated networks. The goal of this paper is to investigate the role of these ideas in

[1]This paper is dedicated to the memory of Mary Ellen Rudin (1924-2013) and E. A. Michael (1925 - 2013).

the class of generalized ordered spaces (GO-spaces) and linearly ordered topological spaces (LOTS). (Definitions are given in Section 2.) It is known that any regular space with a σ-isolated network must be a Gruenhage space, that any Gruenhage space has property *, and that any space with property * is fragmentable [20], and we prove, for example:

> (i) any GO-space, and any monotonically normal space, that has property * or that has a σ-isolated network must be hereditarily paracompact (so that property * and the Gruenhage property are equivalent in such spaces)(see 4.4);
>
> (ii) there are GO-spaces that are first-countable and fragmentable but not paracompact and therefore do not have property * (see 7.3) and that there are GO-spaces that are Gruenhage spaces but do not have σ-isolated networks (see 7.1);
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> (iii) any fragmentable GO-space has a σ-disjoint π-base so that any first-countable fragmentable GO-space has a dense metrizable subspace (see 5.3);
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> (iv) if the GO-space X is fragmentable and a Baire space, then X has a dense metrizable subspace (see 5.7);
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> (v) any GO-space that is first-countable and has a σ-isolated network is quasi-developable and has a dense metrizable subspace (see 5.8);
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> (vi) for any compact GO-space X, the following are equivalent: X is metrizable; X is Eberlein compact; X is Talagrand compact; X is Gulko compact; X has a σ-isolated network; X is a Gruenhage space; X has property *; X is perfect and fragmentable; the function space $C(X)^*$ has a strictly convex dual norm (see 6.5);
>
> (vii) any compact connected GO-space with one of the properties in (vi) must be a singleton or must be homeomorphic to the unit interval $[0, 1]$ (see 6.6).

We conclude with a list of GO-spaces that have various point-separation properties listed in the title of the paper, and others that do not. In particular we show that

> (viii) there is a GO-space that has property * and is fragmentable, has a σ-isolated network and a σ-disjoint π-base, but is not first-countable at any point and therefore contains no dense metrizable subspace (see 7.4).

Throughout this paper, the symbols $\mathbb{R}, \mathbb{Q}$, and $\mathbb{P}$ will denote the sets of real, rational, and irrational numbers with their usual ordering, and $\mathbb{Z}$ will denote the set of all integers (positive, negative, and zero).

We wish to thank the referee for comments that significantly broadened and improved this paper. We dedicate this paper to the memory of Mary Ellen Rudin (1924-2013) and Ernest A. Michael (1925-2013) whose insights inspired so much of our work ever since we were graduate students.

2 Definitions

Recall that a space X has a *quasi-G_δ-diagonal* if there is a sequence $\langle \mathcal{G}(n) : n < \omega \rangle$ of collections of open sets such that if $x \neq y$ are points of X, then for some $n < \omega,\ x \in \mathrm{St}(x, \mathcal{G}(n)) \subseteq X - \{y\}$, where

$St(x, \mathcal{G}(n)) = \bigcup\{G \in \mathcal{G}(n) : x \in G\}$. If each of the collections $\mathcal{G}(n)$ is an open cover of X, then we say that X has a *G_δ-diagonal.* A collection $\mathcal{N}$ is a *network* for X if, given any open set U and any $p \in U$, some $N \in \mathcal{N}$ satisfies $p \in N \subseteq U$. Often, but not always, one may assume that members of the network are closed sets. We do *not* make that assumption. The network $\mathcal{N}$ is *σ-discrete* if $\mathcal{N} = \bigcup\{\mathcal{N}(k) : k < \omega\}$ where each $\mathcal{N}(k)$ is a discrete collection in X, i.e., for each $x \in X$ there is a neighborhood U of x that meets at most one member of $\mathcal{N}(k)$. Regular spaces that have σ-discrete networks are called σ-spaces and play a key role in generalized metric theory. The network $\mathcal{N}$ is *σ-isolated* if $\mathcal{N} = \bigcup\{\mathcal{N}(k) : k < \omega\}$ where for each $N \in \mathcal{N}(k)$, $N \cap \mathrm{cl}_X(\bigcup\{N' \in \mathcal{N}(k) : N' \neq N\}) = \emptyset$. (Spaces that have σ-isolated networks were originally introduced by R. Hansell under the name *descriptive spaces* [16].) We will use an equivalent version of that definition.

Lemma 2.1 *A space X has a σ-isolated network if and only if X has a network $\mathcal{N} = \bigcup\{\mathcal{N}(k) : k \geq 1\}$ where each $\mathcal{N}(k)$ is a pairwise disjoint relatively open cover of the subspace $Y(k) = \bigcup \mathcal{N}(k)$.*

Proof: Any σ-isolated network has the property described in this lemma, and any network with the property given in this lemma is σ-isolated in the sense of the original definition. □

A collection $\mathcal{S}$ is said to T_0-separate two points $x \neq y$ of a space X if there is a member $S \in \mathcal{S}$ with $x \in S \subseteq X - \{y\}$ or there is a member $S \in \mathcal{S}$ with $y \in S \subseteq X - \{x\}$. A space X is said to have *property ** [20] if there is a sequence $\langle \mathcal{V}(n) : n < \omega \rangle$ of collections of open sets such that if $x \neq y$ are any two points of X, there is some n and some $V \in \mathcal{V}(n)$ such that $V \cap \{x, y\}$ is a singleton and no member of $\mathcal{V}(n)$ contains both x and y. (Note that the collections $\mathcal{V}(n)$ need not be covers, and that $\bigcup\{\mathcal{V}(n) : n \geq 1\}$ may fail to be a cover: it may miss up to one point of X.) We will use an equivalent version of this definition, namely

Lemma 2.2 *A space X has property * if and only if there is a sequence $\langle \mathcal{V}(n) : n < \omega \rangle$ of collections of open sets such that if $x \neq y$ are points of X, then there is some n with one of the following:*

a) $x \in St(x, \mathcal{V}(n)) \subseteq X - \{y\}$ or
b) $y \in St(y, \mathcal{V}(n)) \subseteq X - \{x\}$.

In addition, for each n, $\bigcup \mathcal{V}(n)$ is a dense subset of X. □

Lemma 2.2 makes it clear that any space with a G_δ-diagonal (or a quasi G_δ-diagonal) must have property *. Property * is also a generalization of an earlier property introduced in [13] and now called the *Gruenhage space property*, namely that there is a sequence $\langle \mathcal{V}(n) : n < \omega \rangle$ of collections of open sets such that given any points $x \neq y$ in X, there is some n and some $V \in \mathcal{V}(n)$ such that $V \cap \{x, y\}$ is a singleton, and at least one of the numbers $ord(x, \mathcal{V}(n))$ and $ord(y, \mathcal{V}(n))$ is finite. (Recall that $ord(x, \mathcal{V}(n))$ is the cardinality of the set $\{V \in \mathcal{V}(n) : x \in V\}$.) In regular spaces, the relationship between having a σ-isolated network and being a Gruenhage space is a consequence of results of Raja and Smith.

Proposition 2.3 *Any regular space with a σ-isolated network is a Gruenhage space.*

Proof: Lemma 2.1 in [21]) shows that in a regular space with a σ-isolated network, there is a σ-isolated network whose members are the intersection of an open set and certain closed sets. Then Proposition 2 of [23] shows that the existence of Raja's special network is enough to make X a Gruenhage space. (A single source for these results is [25], Lemma 1 and Proposition 1.) □

A space (X, τ) is *fragmentable* if there is a metric d on the set X (not directly related to the topology τ) such that for each $\epsilon > 0$ and each nonempty $E \subseteq X$ there is some $U \in \tau$ with $U \cap E \neq \emptyset$ and the d-diameter of $(U \cap E)$ is $< \epsilon$. The relation between fragmentability and T_0-separating collections is made clear by Ribarska's internal characterization of such spaces in [22]:

Theorem 2.4 *A space X is fragmentable if and only if for each $n < \omega$ there is a well-ordered collection $\mathcal{U}(n) = \{U(n, \alpha) : \alpha \leq \lambda_n\}$ satisfying*

a) each $U(n, \alpha)$ is open in X and $U(n, \alpha) \subset U(n, \beta)$ if $\alpha < \beta \leq \lambda_n$;

b) $U(n, \lambda_n) = X$;

c) if $x \neq y$ are points of X then there is some $n < \omega$ and some $\alpha \leq \lambda_n$ such that $U(n, \alpha)$ contains exactly one of the points x and y. $\square$

The relation between Gruenhage spaces, spaces with property *, and fragmentable spaces was given in Propositions 4.1 and 4.2 in [20].

Proposition 2.5 *Any Gruenhage space has property * and any space with property * is fragmentable.* $\square$

Other important properties involve embedability in special kinds of product spaces and can be characterized using the existence of a special T_0-separating sequence. For example, from Theorem 12, p. 105, in [1] we have:

Proposition 2.6 *A space Y is Eberlein compact if and only Y is a compact Hausdorff space and there is a σ-point-finite collection $\mathcal{G} = \bigcup\{\mathcal{G}(n) : n < \omega\}$ of open F_σ-sets such that if p and q are distinct points of Y, then some $G \in \mathcal{G}$ has $p \in G \subseteq Y - \{q\}$ or $q \in G \subseteq Y - \{p\}$.* $\square$

3 Preliminary lemmas and general results

The results in this section hold in any Hausdorff space. Many are part of the folklore; we list them here for ease of reference. The first several lemmas are straightforward.

Lemma 3.1 *Let P be one of the properties "is a Gruenhage space", "has property *", or "is fragmentable". Suppose σ and τ are two Hausdorff topologies on the set X with $\sigma \subseteq \tau$. If (X, σ) has P then so does (X, τ).* $\square$

Lemma 3.2 *Let $P \in$ { "is a Gruenhage space", "has property *", "is fragmentable", "has a σ-isolated network" }. If X has P and Y is a subspace of X, then Y also has P, i.e., P is a hereditary property.* $\square$

Lemma 3.3 *Suppose $\langle \mathcal{U}(n) \rangle$ satisfies the definition of property * for the space X and that for each n we have a sequence of open collections $\langle \mathcal{V}(n, k) : k < \omega \rangle$ that refine $\mathcal{U}(n)$ and have $\bigcup\{\bigcup \mathcal{V}(n, k) : k < \omega\} = \bigcup \mathcal{U}(n)$. Then $\langle \mathcal{V}(n, k) : n, k < \omega \rangle$ also satisfies the definition of property * for the space X.* $\square$

Corollary 3.4 *Suppose X has property * and is hereditarily paracompact. Then there is a sequence $\langle \mathcal{U}(n) \rangle$ that satisfies the conclusions of Lemma 2.2 and for which each $\mathcal{U}(n)$ is a pairwise-disjoint collection whose union is dense in X.*

Proof: For each n, use hereditary paracompactness to obtain a σ-disjoint collection $\bigcup\{\mathcal{V}(n,k) : k < \omega\}$ of open sets that refines $\mathcal{U}(n)$ and covers $\bigcup \mathcal{U}(n)$. By Lemma 3.3 the family $\{\mathcal{V}(n,k) : n, k < \omega\}$ has the properties described in Lemma 2.2 and each collection $\mathcal{V}(n,k)$ is pairwise disjoint. Finally, enlarge each $\mathcal{V}(n,k)$ by adding the open set $X - \mathrm{cl}_X(\bigcup \mathcal{V}(n,k))$ to the collection $\mathcal{V}(n,k)$ to ensure that each $\bigcup \mathcal{V}(n,k)$ is dense in X. $\square$

Corollary 3.5 *Suppose that X is a weakly θ-refinable*[2] *space with a G_δ-diagonal. Then X is a Gruenhage space.* $\square$

Some hypothesis like weak θ-refinability is necessary in Corollary 3.5 because Smith has given a ZFC example of a locally compact Hausdorff space with a G_δ-diagonal that is not a Gruenhage space [24] (and consequently Smith's space is not weakly θ-refinable)[3].

Corollary 3.6 *Suppose X is a Hausdorff space that is hereditarily weakly θ-refinable. Then X has property * if and only if X is a Gruenhage space.* $\square$

Corollary 3.7 *Any quasi-developable*[4] *Hausdorff space is a Gruenhage space (and therefore has property *).*

Proof: Any quasi-developable space is hereditarily weakly θ-refinable [6] and if $\langle \mathcal{G}(n) : n < \omega \rangle$ is a quasi-development for a Hausdorff space, then $\langle \mathcal{G}(n) : n < \omega \rangle$ is the sequence needed to witnes property * for the space X. Now apply Corollary 3.6. $\square$

Corollaries 3.5 and 3.7 are independent of each other because there are GO-spaces that have G_δ-diagonals that are not quasi-developable, and vice versa. The Sorgenfrey line is of the first type, and the LOTS M^* in Example 7.1 is quasi-developable and does not have a G_δ-diagonal.

Lemma 3.8 *Suppose X is a perfect*[5] *Hausdorff space with property *. Then X has a G_δ-diagonal.*

Proof: Let $\mathcal{U}(n)$ be the open collections in the definition of property * and for each n write $\bigcup \mathcal{U}(n) = \bigcup\{F(n,k) : k < \omega\}$ where each $F(n,k)$ is closed in X. Let $\mathcal{V}(n,k) = \mathcal{U}(n) \cup \{X - F(n,k)\}$. Each $\mathcal{V}(n,k)$ is an open cover of X and for each fixed $x \in X$ we claim that $\{x\} = \bigcap\{\mathrm{St}(x, \mathcal{V}(n,k) : n, k < \omega\}$. For suppose $y \neq x$. Then either

[2] A space is weakly θ-refinable if for each open cover $\mathcal{V}$ of X there is a sequence $\langle \mathcal{W}(k) : k < \omega \rangle$ such that $\bigcup\{\mathcal{W}(k) : k < \omega\}$ covers X and refines $\mathcal{V}$, and for each $x \in X$ there is some k such that $1 \leq ord(x, \mathcal{W}(k)) < \omega$. Such spaces are also known as *submetacompact spaces*

[3] There are other well-known examples of locally compact Hausdorff spaces that have G_δ-diagonals but little more. For example, Burke [10] and Gruenhage (Example 2.17 in [15]) have each described spaces that are locally compact, Hausdorff, have G_δ-diagonals, and are not Moore spaces, but both of these examples are quasi-developable and therefore are Gruenhage spaces in the light of Corollary 3.7.

[4] A space X is quasi-developable if there is a sequence $\langle \mathcal{U}(n) : n < \omega \rangle$ of collections of open sets such that if V is open and $p \in V$, then for some n, $p \in \mathrm{St}(p, \mathcal{U}(n)) \subseteq V$.

[5] A space X is *perfect* if each open subset of X is an F_σ-set.

a) for some m, $x \in \mathrm{St}(x, \mathcal{U}(m)) \subseteq X - \{y\}$; or

b) for some n, $y \in \mathrm{St}(y, \mathcal{U}(n)) \subseteq X - \{x\}$.

In the first case, find $s < \omega$ with $x \in F(m, s)$ and consider any $V \in \mathcal{V}(m, s)$ with $x \in V$. Then $V \neq X - F(m, s)$ so that $V \in \mathcal{U}(m)$ and therefore $x \in V$ forces $V \subseteq \mathrm{St}(x, \mathcal{U}(m)) \subseteq X - \{y\}$. Therefore $y \notin \mathrm{St}(x, \mathcal{V}(m, s))$.

In the second case, find $t < \omega$ with $y \in F(n, t)$. Consider any $V \in \mathcal{V}(n, t)$ with $y \in V$. Then $V \neq X - F(n, t)$ so that $V \in \mathcal{U}(n)$. Then $y \in V$ yields $V \subseteq \mathrm{St}(y, \mathcal{U}(n)) \subseteq X - \{x\}$ showing that $x \notin \mathrm{St}(y, \mathcal{V}(n, t))$. Because $\mathcal{V}(n, t)$ is a cover of X it follows that $y \notin \mathrm{St}(x, \mathcal{V}(n, t))$.

In either case, therefore, we have found an open cover $\mathcal{V}(i, j)$ with $y \notin \mathrm{St}(x, \mathcal{V}(i, j))$ as required to show that X has a G_δ-diagonal. □

Question 3.9 *Suppose X is a perfect, fragmentable GO-space. Does X have a G_δ-diagonal?*

See Section 2 for the definition of a σ-isolated network. The existence of a σ-isolated network for X is clearly weaker than the existence of a network for X that is σ-discrete in X: the one-point compactification of an uncountable discrete set has a σ-isolated network, but not a σ-discrete network. The latter property is a well-known part of generalized metric space theory [15] and the two ideas are related as follows.

Lemma 3.10 *A regular space X has a σ-discrete network if and only if X is perfect and has a σ-isolated network. Hence any perfect space with a σ-isolated network is θ-refinable and has a G_δ-diagonal.*

Proof: If X has a σ-discrete network, then X is perfect and its network is σ-isolated. To prove the converse, suppose $\mathcal{N} = \bigcup\{\mathcal{N}(k) : k \geq 1\}$ is a σ-isolated network as described in Lemma 2.1, and let $Y(k) = \bigcup \mathcal{N}(k)$. For each $p \in Y(k)$ there is an open subset $O(p, k)$ of X such that $O(p, k)$ meets exactly one member of $\mathcal{N}(k)$. Let $U(k) = \bigcup\{O(p, k) : p \in Y(k)\}$. Because X is perfect, we may write $U(k) = \bigcup\{F(k, j) : j \geq 1\}$ where each $F(k, j)$ is closed in X. Let $\mathcal{M}(k, j) := \{N \cap F(k, j) : N \in \mathcal{N}(k)\}$. Then each $\mathcal{M}(k, j)$ is a discrete collection in X and $\bigcup\{\mathcal{M}(k, j) : k, j \geq 1\}$ is a network for X. The second assertion of the lemma follows from the fact that any regular space with a σ-discrete network is subparacompact (and hence θ-refinable) and has a G_δ-diagonal. □

The Michael line M (see Example 7.1) is an example of a GO-space that has a σ-isolated network but not a σ-discrete network (because M is not perfect).

We want to thank the referee for pointing out the following results.

Lemma 3.11 *Suppose X is fragmentable and $\mathcal{U}(n) = \{U(n, \alpha) : \alpha \leq \lambda_n\}$ are the collections given in Theorem 2.4. Let $G(n, \alpha) := U(n, \alpha) - \mathrm{cl}(\bigcup\{U(n, \beta) : \beta < \alpha\})$. Then for each $\alpha \leq \lambda_n$, $\bigcup\{G(n, \beta) : \beta \leq \alpha\}$ is a dense subset of $U(n, \alpha)$.*

Proof: For a fixed n and $\alpha \leq \lambda_n$, let V be any nonempty open set with $V \subseteq U(n, \alpha)$. Let γ be the first ordinal $\leq \alpha$ with $V \cap U(n, \gamma) \neq \emptyset$. Then V is disjoint from $\bigcup\{U(n, \delta) : \delta < \gamma\}$ so that V is disjoint from the closure of $\bigcup\{U(n, \delta) : \delta < \gamma\}$. Hence $V \cap G(n, \gamma) \neq \emptyset$, as required to prove that $\bigcup\{G(n, \beta) : \beta \leq \alpha\}$ is dense in $U(n, \alpha)$. □

Theorem 3.12 *Suppose (X, τ) is a fragmentable regular space that is a Baire space. Then X has a dense subspace that has a weaker metrizable topology.*

Proof: We have a sequence of well-ordered increasing collections of open sets $\mathcal{U}(n) = \{U(n, \alpha) : \alpha \leq \lambda_n\}$ as described in Theorem 2.4. Write $G(n, \alpha) := U(n, \alpha) - \mathrm{cl}_X(\bigcup\{U(n, \beta) : \beta < \alpha\}$ for each $\alpha \leq \lambda_n$ and $\mathcal{G}(n) := \{G(n, \alpha) : \alpha < \lambda_n\}$. Then the set $V(n) := \bigcup\{G(n, \alpha) : \alpha \leq \lambda_n\}$ is open and dense in X by Lemma 3.11. Because X is a Baire space, the set $D = \bigcap\{V(n) : n \geq 1\}$ is dense in X. For each n, the collection $\mathcal{H}(n) := \{D \cap G(n, \alpha) : \alpha \leq \lambda_n\}$ is a pairwise disjoint relatively open cover of D. Fix two points $x, y \in D$. If there is some n and some $\alpha \leq \lambda_n$ such that $x \in U(n, \alpha) \subseteq X - \{y\}$, then $x \in \mathrm{St}(x, \mathcal{H}(n)) \subseteq D - \{y\}$. If there is some n and some $\beta \leq \lambda_n$ with $y \in U(n, \beta) \subseteq X - \{x\}$ then $x \notin \mathrm{St}(y, \mathcal{H}(n))$. But then, because $\mathcal{H}(n)$ covers both x and y and is pairwise disjoint, we have $y \notin \mathrm{St}(x, \mathcal{H}(n))$. In either case, we see that $\bigcap\{\mathrm{St}(x, \mathcal{H}(m) : m \geq 1\} = \{x\}$ so that D has a G_δ-diagonal, a fact that will be needed in Corollary 5.7. To complete the proof, let $\mathcal{L}(n) = \{D \cap G_1 \cap G_2 \cap \cdots \cap G_n : G_i \in \mathcal{G}(i) \text{ for } i \leq n\}$. Each $\mathcal{L}(n)$ is a pairwise disjoint relatively open cover of D. Let σ be the topology on D for which the collection $\mathcal{L} := \bigcup\{\mathcal{L}(n) : n \geq 1\}$ is a base. Clearly $\sigma \subseteq \tau|_D$, and (D, σ) is regular. Furthermore, each $\mathcal{L}(n)$ is locally finite in (D, σ) so that (D, σ) is metrizable. Alternatively one can define a metric on D by the rule that $d(x, x) = 0$ for all $x \in D$ and if $x \neq y$ then $d(x, y) = 2^{-n}$ where n is the first integer such that x and y belong to disjoint members of $\mathcal{L}(n)$. $\square$

Proposition 3.13 *Suppose that X is a fragmentable hereditarily Lindelöf regular space. Then X has a G_δ-diagonal.*

Proof: Consider the well-ordered increasing collections $\mathcal{U}(n) = \{U(n, \alpha) : \alpha \leq \lambda_n\}$ in Theorem 2.4. We may assume that if $\alpha < \beta \leq \lambda_n$, then $U(n, \alpha)$ is a proper subset of $U(n, \beta)$. Fix any limit ordinal $\lambda \leq \lambda_n$. Applying the hereditary Lindelöf property to the collection $\{U(n, \alpha) : \alpha < \lambda\}$, we see that λ has countable cofinality. Therefore $\lambda_n < \omega_1$. Consequently, each collection $\mathcal{U}(n)$ is countable. The hereditary Lindelöf property also guarantees that X is perfect so that for each $\alpha < \lambda_n$ we may write $U(\alpha, n) = \bigcup\{F(\alpha, n, k) : k \geq 1\}$ where each $F(\alpha, n, k)$ is closed in X. Let $\mathcal{V}(\alpha, n, k) := \{U(\alpha, n), X - F(\alpha, n, k)\}$. Each $\mathcal{V}(\alpha, n, k)$ is an open cover of X and because λ_n is a countable ordinal, there are only a countable number of covers of the form $\mathcal{V}(\alpha, n, k)$ for $\alpha < \lambda_n, m \geq 1, k \geq 1$ so that the collection $\{\mathcal{V}(\alpha, n, k) : n \geq 1, \alpha < \lambda_n, k \geq 1\}$ shows that X has a G_δ-diagonal. $\square$

The following corollary is known. See Corollary 9 in [18] and page 216 in [20].

Corollary 3.14 *Any compact Hausdorff space that is fragmentable and perfect is metrizable.*

Proof: Any perfect compact Hausdorff space is regular and hereditarily Lindelöf so that Proposition 3.13 shows that X has a G_δ-diagonal. But any compact Hausdorff space with a G_δ-diagonal is metrizable, by Sneider's theorem [26]. $\square$

4 Hereditary paracompactness

Smith and Troyanski [25] used a pressing down argument to show that that the ordinal space $[0, \omega_1)$ does not have property *. Our next lemma uses the same approach and strengthens their result.

Lemma 4.1 *Suppose S is a stationary subset of a regular uncountable cardinal κ. Then, in its topology as a subspace of $[0,\kappa)$, S cannot have property *.*

Proof: For contradiction, suppose S has property *. If necessary, replace S by the subset of all of its limit points in $[0,\kappa)$. This set is also stationary, and has property * by Lemma 3.2, so we may assume that every point of S is a limit point of $[0,\kappa)$.

Suppose that $\langle \mathcal{U}(n) : n < \omega \rangle$ is a sequence of open collections for S as in Lemma 2.2. For each n, write $Y(n) = \bigcup \mathcal{U}(n)$. We may assume that all members of each $\mathcal{U}(n)$ are convex[6]. Because we have property * in S, the set $\bigcup\{Y(n) : n < \omega\}$ contains all but at most one point of S; in particular, $\bigcup\{Y(n) : n < \omega\}$ is a stationary subset of κ.

Let $N_0 := \{n < \omega : Y(n) \text{ is not stationary}\}$. Then $\bigcup\{Y(n) : n \in N_0\}$ is not stationary, so the set $N_1 = \omega - N_0$ is nonempty and the set $T := S - \bigcup\{Y(n) : n \in N_0\}$ is stationary in κ.

Fix $n \in N_1$. For each α in the stationary set $Y(n)$ there is some $U \in \mathcal{U}(n)$ with $\alpha \in U$, so there is some $f(n,\alpha) < \alpha$ with $(f(n,\alpha),\alpha] \cap S \subseteq U$. Then the Pressing Down Lemma gives some β_n with $[\beta_n,\kappa) \cap S \subseteq \mathrm{St}(\beta_n, \mathcal{U}(n))$.

Because the set T is cofinal in the uncountable cardinal κ, there are points $\beta < \gamma < \delta$ in T with $\beta_n < \beta$ for each $n \in N_1$. Because members of each $\mathcal{U}(n)$ are convex in S, it follows that

$$[\beta,\kappa) \cap S \subseteq [\beta_n,\kappa) \cap S \subseteq \mathrm{St}(\beta_n, \mathcal{U}(n))$$

for each $n \in N_1$ and because $\gamma, \delta \in [\beta,\kappa) \cap S \subseteq \mathrm{St}(\beta_n, \mathcal{U}(n))$ for each $n \in N_1$, the fact that members of $\mathcal{U}(n)$ are convex in S guarantees that $\gamma \in \mathrm{St}(\delta, \mathcal{U}(n))$ and $\delta \in \mathrm{St}(\gamma, \mathcal{U}(n))$ for each $n \in N_1$.

Because $\gamma \neq \delta$ and S has property *, there is some $m < \omega$ with either $\gamma \in \mathrm{St}(\gamma, \mathcal{U}(m)) \subseteq S - \{\delta\}$ or $\delta \in \mathrm{St}(\delta, \mathcal{U}(m)) \subseteq S - \{\gamma\}$. Consider the first case, the second being analogous. We know that $\gamma \in T = S - \bigcup\{Y(n) : n \in N_0\}$, so that $m \notin N_0$. Hence $m \in N_1$. But that contradicts the final sentence of the previous paragraph. $\Box$

Corollary 4.2 *Suppose S is a stationary subspace of a regular uncountable cardinal. Then S cannot have a σ-isolated network.*

Proof: By Proposition 2.3, any stationary set with a σ-isolated network would have property *, contrary to Lemma 4.1. Alternatively, suppose $\mathcal{N} = \bigcup\{\mathcal{N}(k) : k \geq 1\}$ is a σ-isolated network for S. Because the set $[0,\alpha] \cap S$ is open for each $\alpha \in S$, we may assume that every member of $\mathcal{N}$ is bounded. Now let $Y(k) := \bigcup \mathcal{N}(k)$. Because $S = \bigcup\{Y(k) : k \geq 1\}$ we know that some $Y(k)$ is stationary. But then $\{N \cap Y(k) : N \in \mathcal{N}(k)\}$ is a pairwise-disjoint relatively open cover of $Y(k)$ by bounded sets, and that is impossible because $Y(k)$ is stationary. $\Box$

Example 4.3 *The usual space of countable ordinals $[0,\omega_1)$ is fragmentable but not paracompact.* $\Box$

Proposition 4.4 *Suppose X is a monotonically normal space with property * or with a σ-isolated network. Then X is hereditarily paracompact. In addition, X has property * if and only if X and is a Gruenhage space.*

[6] A subset Y of a linearly ordered set $(X,<)$ is *convex* if a point b is in Y whenever there are points $a, c \in Y$ with $a < b < c$.

Proof: Balough and Rudin [3] have shown that if the monotonically normal space X is not hereditarily paracompact, then there is some uncountable regular cardinal κ and some stationary subset S of κ that embeds in X. If X has a σ-isolated network or if X has property *, then so does its subspace S, and that is impossible by Lemma 4.1 and Lemma 4.2. Now apply Corollary 3.6 to conclude that if X has property *, then X is a Gruenhage space. $\square$

Corollary 4.5 *If X is a GO-space with property * or with a σ-isolated network, then X is hereditarily paracompact. Consequently, a GO-space has property * if and only if it is a Gruenhage space.*

Proof: Any GO-space is monotonically normal [17]. Now apply Proposition 4.4. $\square$

Corollary 4.6 *Any compact LOTS with a σ-isolated network or with property * must be first countable.*

Proof: If X is a compact LOTS that is not first countable, then X contains a topological copy of the ordinal space $[0, \kappa)$ for some uncountable regular cardinal κ, and that is impossible by Lemma 4.1 and Lemma 4.2. $\square$

Corollary 4.6 is used in Theorem 6.5 below to show that any compact LOTS with property * is metrizable .

5 Dense metrizable subspaces of certain GO-spaces

Gruenhage showed in [13] that any compact Hausdorff space with what we now call the Gruenhage property must have a dense metrizable subset, and his result was generalized in Corollary 2.7 of [22] to show that a compact, Hausdorff, fragmentable space has a dense metrizable subspace. In this section, we investigate the existence of dense metrizable subspaces in GO-spaces with additional structure. We begin by proving (see 5.3) that any fragmentable GO-space (and hence any GO-space with property *) must have a *σ-disjoint π-base*, i.e., a collection $\mathcal{P} = \bigcup\{\mathcal{P}(n) : n < \omega\}$ where each $\mathcal{P}(n)$ is a pairwise disjoint collection of open sets, and where for any nonempty open set U there is some $P \in \mathcal{P}$ with $\emptyset \neq P \subseteq U$. (The collection $\mathcal{P}$ is also called a *pseudobase for X*.) It then follows from a theorem of H.E. White that a first-countable GO-space with property * must have a dense metrizable subspace (see Corollary 5.5). We also show that every GO-space that is a Baire space and is fragmentable must have a dense metrizable subset (see 5.7). Example 7.4 will show that some hypothesis in addition to property * is needed in order to guarantee that a GO-space must have a dense metrizable subspace. In 5.8 we also show that a GO-space is quasi-developable if and only if it is first-countable and has a σ-isolated network.

Let $(X, <, \tau)$ be a GO-space. We will say that a subset $J \subseteq X$ is *special* if J is a singleton open set, or if J is nonempty, convex, contains no endpoints, and has both countable cofinality and countable coinitiality, i.e., there is a strictly decreasing sequence $\langle x_n \rangle$ and a strictly increasing sequence $\langle y_n \rangle$ of points of J such that $x_1 = y_1$ and $J = \bigcup\{(x_n, y_n) : n \geq 1\}$.

Lemma 5.1 *Suppose U is a nonempty open set in a GO-space $(X, <, \tau)$. Then U contains a special open subset of X.*

Proof: Without loss of generality, suppose U is convex. Let I be the set of all τ-isolated points of X. If $U \cap I \neq \emptyset$ there is nothing to prove, so assume U contains no isolated points. Then U is infinite. Note that U might have a left endpoint p, but in that case p cannot have any immediate successor (because p cannot be isolated in X), so if necessary remove the left endpoint of U. Similarly, if necessary we may remove the right endpoint of U, so we may assume that U contains no endpoints of itself. Now let x_1 be any point of U. If the infinite set $U \cap (\leftarrow, x_1)$ contains no strictly decreasing sequence, then $U \cap (\leftarrow, x_1)$ is an infinite well-ordered set and therefore contains three consecutive points of X, contrary to $U \cap I = \emptyset$. Hence we have a strictly decreasing sequence $\langle x_n \rangle$ of points of U. Let $y_1 = x_1$. Then, because U cannot contain three consecutive points of X, the set $U \cap (y_1, \rightarrow)$ cannot be anti-well-ordered so that U must contain a strictly increasing sequence $\langle y_n \rangle$. Now let $J = \bigcup\{(x_n, y_n) : n \geq 1\}$. $\square$

Lemma 5.2 *Suppose U is a nonempty open subset of the GO-space $(X, <, \tau)$. Then there is a collection $\mathcal{V}_U$ of pairwise disjoint special open sets whose union is dense in U.*

Proof: Let $\Psi := \{\mathcal{V} : \mathcal{V}$ is a pairwise disjoint collection of special open subsets of $U\}$ and partially order Ψ by $\subseteq$. Lemma 5.1 shows that U contains a special set open J so that $\{J\} \in \Psi$ and hence $\Psi \neq \emptyset$. Clearly, the union of any nested subfamily of Ψ is a member of Ψ so that Zorn's Lemma gives a maximal member $\mathcal{V}_0$ of Ψ. If $\bigcup \mathcal{V}_0$ is not dense in U, then there is some nonempty open set $U_0 \subseteq U$ that is disjoint from $\bigcup \mathcal{V}_0$, and then Lemma 5.1 gives a special open set $J_0 \subseteq U_0$ and then the collection $\mathcal{V}_0 \cup \{J_0\}$ is a member of Ψ that is strictly larger than $\mathcal{V}_0$. Consequently the collection $\mathcal{V}_0$ has the properties required in this lemma. $\square$

Proposition 5.3 *Any fragmentable GO-space has a σ-disjoint π-base.*

Proof: Let $\mathcal{U}(n) = \{U(n, \alpha) : \alpha \leq \lambda_n\}$ be as in Theorem 2.4. Temporarily fix $n \geq 1$. Let $G(n, \alpha) = U(n, \alpha) - \mathrm{cl}(\bigcup\{U(n, \beta) : \beta < \alpha\})$. Apply Lemma 5.2 to each convex component of the open set $G(n, \alpha)$ to obtain a pairwise disjoint collection $\mathcal{C}(n, \alpha)$ of special convex open subsets of $G(n, \alpha)$ whose union is dense in $G(n, \alpha)$. For each $J \in \mathcal{C}(n, \alpha)$ with more than one point, choose co-initial and cofinal sequences $\langle x(J, k) : k \geq 1\rangle$ and $\langle y(J, k) : k \geq 1\rangle$ in the set J, as in the definition of a special set. Let

$$\mathcal{X}(n, \alpha) := \{(x(J, k+1), x(J, k)) : J \in \mathcal{C}(n, \alpha), |J| > 1, k \geq 1\}$$

and

$$\mathcal{Y}(n, \alpha) := \{(y(J, k)), y(J, k+1)) : J \in \mathcal{C}(n, \alpha), |J| > 1, k \geq 1\}$$

and let $\mathcal{D}(n) = \bigcup\{\mathcal{X}(n, \alpha) \cup \mathcal{Y}(n, \alpha) : \alpha \leq \lambda_n\}$. Each collection $\mathcal{X}(n, \alpha) \cup \mathcal{Y}(n, \alpha)$ consists of pairwise disjoint subsets of $G(n, \alpha)$ and the for a fixed $n \geq 1$, the collection $\{G(n, \alpha) : \alpha \leq \lambda_n\}$ is a pairwise disjoint collection, so that the collection

$$\mathcal{E}(n) := \bigcup\{\mathcal{X}(n, \alpha) \cup \mathcal{Y}(n, \alpha) : \alpha \leq \lambda_n\}$$

is also pairwise disjoint. Let $\mathcal{E}(0) := \{\{x\} : x \text{ is isolated in } X\}$. Therefore, if we can show that the collection $\mathcal{E} := \bigcup\{\mathcal{E}(n) : n \geq 0\}$ is a π-base for X, the proof will be complete.

To prove that $\mathcal{E}$ is a π-base for X, suppose W is any nonempty open set. We may assume W is convex. If W contains any isolated point of X, then W contains a member of $\mathcal{E}(0)$, so assume that

W contains no isolated points. Choose points $a < b < c < d$ in W in such a way that each interval $(a, b), (b, c)$, and (c, d) is infinite. As in Theorem 2.4 we may find some $m \geq 1$ and some $\alpha_0 \leq \lambda_m$ such that $U(m, \alpha_0)$ contains exactly one of the points b and c. The two cases are analogous, so we may suppose that $b \in U(m, \alpha_0) \subseteq X - \{c\}$.

From Lemma 3.11 we know that $\bigcup\{G(m, \beta) : \beta \leq \alpha_0\}$ is a dense subset of $U(m, \alpha_0)$. Because $b \in U(m, \alpha_0)$ and $b \in (a, c)$ it follows that the open set (a, c) has $(a, c) \cap (\bigcup\{G(m, \beta) : \beta \leq \alpha_0\}) \neq \emptyset$ so that we may choose $\beta_0 \leq \alpha_0$ with $(a, c) \cap G(m, \beta_0) \neq \emptyset$. Because $\bigcup \mathcal{C}(m, \beta_0)$ is a dense subset of $G(m, \beta_0)$, there is some special set $J_0 \in \mathcal{C}(m, \beta_0)$ with $J_0 \cap (a, c) \neq \emptyset$. Because no point of (a, c) can be isolated, we see that $|J_0 \cap (a, c)| > 1$. Because $\beta_0 \leq \alpha_0$ and $J_0 \subseteq U(m, \beta_0) \subseteq U(m, \alpha_0) \subseteq X - \{c\}$ we know that $c \notin J_0$. Because J_0 is convex and $J_0 \cap (a, c) \neq \emptyset$ it follows that $J_0 \subseteq (\leftarrow, c)$. Choose any point $p \in J_0 \cap (a, c)$. The sequence $\langle y(J_0, k) : k \geq 1\rangle$ is cofinal in J_0 so there is some k with $p < y(J_0, k)$. Then we have

$$a < p < y(J_0, k) < y(J_0, k + 1) \in J_0 \subseteq (\leftarrow, c)$$

so that the nonempty set $(y(J_0, k), y(J_0, k + 1)$ has $(y(J_0, k), y(J_0, k + 1)) \in \mathcal{Y}(m, \beta_0) \subseteq \mathcal{E}(m) \subseteq \mathcal{E}$ and $(y(J_0, k), y(J_0, k + 1) \subseteq (a, c) \subseteq W$. Therefore $\mathcal{E}$ is a σ-disjoint π-base for X. $\Box$

Recall the following fundamental theorem of H.E. White in [28]:

Theorem 5.4 *If X is regular and first-countable, then X has a dense metrizable subspace if and only if X has a σ-disjoint π-base.* $\Box$

Corollary 5.5 *Any first-countable fragmentable GO-space has a dense metrizable subspace.*

Proof: This corollary follows from Proposition 5.3 and H.E. White's Theorem 5.4. $\Box$

Corollary 5.6 *Any GO-space with a quasi-G_δ-diagonal* [7] *has a dense metrizable subspace.*

Proof: Let $\langle \mathcal{G}(n) : n \geq 1\rangle$ be the quasi-G_δ-diagonal sequence for the GO-space $(X, \tau, <)$. It follows from [12] that X is hereditarily paracompact, so we may assume that each $\mathcal{G}(n)$ is pairwise disjoint and that members of $\mathcal{G}(n)$ are convex. Let $\mathcal{H}(m, n) := \{G \cap G' : G \in \mathcal{G}(m), G' \in \mathcal{G}(n)\}$. Then for each $x \in X$, we have $\bigcap\{\mathrm{St}(x, \mathcal{H}(m, n)) : x \in \mathrm{St}(x, \mathcal{H}(m, n)), m \geq 1, n \geq 1\} = \{x\}$ so that X is first-countable. Let $\mathcal{H}(0, 0) := \{\{x\} : \{x\} \in \tau\}$. Then the collection $\mathcal{H} := \bigcup\{\mathcal{H}(m, n) : m, n \geq 0\}$ is a σ-disjoint π-base for X. By White's theorem [28], X has a dense metrizable subspace. $\Box$

Examples in Section 7 show that there are GO-spaces with property * that are not first-countable, so that Corollary 5.5 does not cover all possible cases. Some of the remaining cases are covered by our next result, which does not assume first-countability. In an earlier version of our paper, we gave a proof of 5.7 for GO-spaces that have property * and are Baire spaces, and the referee pointed out that a more general result is available for fragmentable GO-spaces that are Baire. Our next result strengthens 3.12 for GO-spaces.

Corollary 5.7 *Suppose X is a fragmentable GO-space that is a Baire space. Then X has a dense metrizable subspace. In particular, any GO-space that is Baire and has property * must have a dense metrizable subspace.*

[7] A space X has a *quasi-G_δ-diagonal* if there is a sequence $\langle \mathcal{G}(n) : n < \omega\rangle$ of collections of open sets such that if $a \neq b$ are points of X, then for some $n, a \in \mathrm{St}(a, \mathcal{G}(n)) \subseteq X - \{b\}$.

Proof: Proposition 3.12 shows that the fragmentable Baire space X has a dense subspace D with a G_δ-diagonal. Because D is a GO-space, it follows from Corollary 5.6 that D has a dense metrizable subspace. Therefore so does X. The remaining assertion in this corollary follows from the fact that any space with property * is fragmentable (see Proposition 4.2 of [20]). $\square$

Given Corollary 5.5 and Corollary 5.7, it is natural to ask whether *every* fragmentable GO-space, or every GO-space with property *, must have a dense metrizable subspace. Example 7.4 has property * (and therefore is fragmentable) but has no dense metrizable subset, and consequently 7.4 shows that the answer is "No".

Our next result extends a well-known characterization of quasi-developability in GO-spaces by linking it to the existence of a σ-isolated network.

Proposition 5.8 *The following properties of a GO-space $(X, \tau, <)$ are equivalent:*

a) X has a σ-disjoint base;

b) X is quasi-developable;

c) X first-countable and has a σ-isolated network.

Proof: The equivalence of (a) and (b) for LOTS is given in [5] and for GO-spaces is in [19]. Clearly a) implies c), so it remains to prove that c) implies a).

Let $\mathcal{N} = \bigcup\{\mathcal{N}(k) : k \geq 1\}$ be a σ-isolated network for the GO-space X. Let $Y(k) = \bigcup \mathcal{N}(k)$. Then $\mathcal{N}(k)$ is a pairwise disjoint relatively open cover of the GO-space $Y(k)$. Let $\mathcal{M}(k)$ be the collection of all convex components[8] of members of $\mathcal{N}(k)$ in the subspace $Y(k)$. Then, in the relative topology of $Y(k)$, each member of $\mathcal{M}(k)$ is relatively open, and the collection $\mathcal{M}(k)$ is pairwise disjoint with $Y(k) = \bigcup \mathcal{M}(k)$. For each $p \in Y(k)$ there is an open convex subset $O(p, k)$ of X that contains p and that meets exactly one member of $\mathcal{M}(k)$. Let $U(k) = \bigcup\{O(p, k) : p \in Y(k)\}$. Note that $U(k)$ is open in X and that the collection $\mathcal{M}(k)$ is relatively discrete in the subspace $U(k)$. Hereditarily collectionwise normality of the GO-space X can be applied to the collection of relative closures of members of $\mathcal{M}(k)$ in the subspace $U(k)$, so that for each $M \in \mathcal{M}(k)$ there is a relatively open set $W(M, k)$ in the subspace $U(k)$ with $M \subseteq W(M, k)$ and such that the collection $\{W(M, k) : M \in \mathcal{M}(k)\}$ is pairwise disjoint. Because $U(k)$ is open in X, each set $W(M, k)$ is also open in X.

For each $M \in \mathcal{M}(k)$ define $c(M) = \bigcup\{[a, b] : a, b \in M, a \leq b\}$. Then each $c(M)$ is convex in X and if $M \neq M'$ are in $\mathcal{M}(k)$, then $c(M) \cap c(M') = \emptyset$. Define $\mathcal{P}(k) = \{\mathrm{Int}_X(c(M)) : M \in \mathcal{M}(k)\}$. Then each $\mathcal{P}(k)$ is a pairwise disjoint collection of open sets in X.

For each $M \in \mathcal{M}(k)$, let $e(M) = c(M) - \mathrm{Int}_X(c(M))$. Then $e(M) \subseteq W(M, k)$ and $|e(M)| \leq 2$ because if $a < b < c$ are points of $e(M)$ then $b \in \mathrm{Int}_X(c(M))$ so that $b \notin e(M)$. If $|e(M)| = 1$, say $e(M) = \{p_M\}$, we have $p_M \in M \subseteq W(M, k)$. Because X is first-countable, we may choose a decreasing neighborhood base $\{V(p_M, j) : j \geq 1\}$ at p_M with $V(p_M, j) \subseteq W(M, k)$ for each $j \geq 1$. Define $\mathcal{Q}(k, j) = \{V(p_M, j) : M \in \mathcal{M}(k) \text{ and } e(M) = \{p_M\}\}$. Each $\mathcal{Q}(k, j)$ is a pairwise disjoint collection of open subsets of X.

[8] A convex component of a set $T \subseteq Z$ in a linearly ordered set $(Z, <)$ is a convex subset of Z that is maximal among convex subsets of Z that are subsets of T. In the situation considered here, a convex component in $Y(k)$ of a subset of $Y(k)$ might not be convex in X.

In case $|e(M)| = 2$ write $e(M) = \{a(M), b(M)\}$ with $a(M) < b(M)$ and choose decreasing neighborhood bases $\{V(a_M, j) : j \geq 1\}$ and $\{V(b_M, j) : j \geq 1\}$ at a_M and b_M respectively, with $V(a_M, j) \cup V(b_M, j) \subseteq W(M, k)$ for each $j \geq 1$. Define $\mathcal{R}(k, j) = \{V(a_M, j) : M \in \mathcal{M}(k)$ and $|e(M)| = 2\}$ and $\mathcal{S}(k, j) = \{V(b_M, j) : M \in \mathcal{M}(k)$ and $|e(M)| = 2\}$. Each $\mathcal{R}(k, j)$ and each $\mathcal{S}(k, j)$ is a pairwise disjoint collection of open subsets of X.

Now suppose that G is a convex open subset of X with $x \in G$. Because $\mathcal{N}$ is a σ-isolated network for X, there is some k and some $N \in \mathcal{N}(k)$ with $x \in N \subseteq G$. Hence there is some set $M \in \mathcal{M}(k)$ with $x \in M \subseteq N \subseteq G$. Then, G being convex in X, we must have $x \in M \subseteq c(M) \subseteq G$. If $x \in \mathrm{Int}_X(c(M))$ then $\mathrm{Int}_X(c(M))$ is a member of $\mathcal{P}(k)$ that contains x and is contained in G. If $x \notin \mathrm{Int}_X(c(M))$, then $x \in e(M)$ so that, depending upon whether $e(M)$ has one or two points, we obtain a member of some $\mathcal{Q}(k, j)$, or of some $\mathcal{R}(k, j) \cup \mathcal{S}(k, j)$, that contains x and is a subset of G. Therefore the σ-disjoint collection

$$\mathcal{B} := \left(\bigcup\{\mathcal{P}(k) : k \geq 1\}\right) \cup \left(\bigcup\{\mathcal{Q}(k, j) \cup \mathcal{R}(k, j) \cup \mathcal{S}(k, j) : k, j \geq 1\}\right)$$

is a base for X, as required. □

Recall that a space X is a *Blumberg space* if for each function $f : X \to \mathbb{R}$, there is a dense subspace $D \subseteq X$ such that $f|_D : D \to \mathbb{R}$ is continuous. Our next result is part of the folklore.

Corollary 5.9 *Suppose X is a GO-space with $|X| \leq 2^\omega$. If X is a Blumberg space, then X has a dense metrizable subspace.*

Proof: Let $f : X \to \mathbb{R}$ be any 1-1 function and suppose that D is a dense subspace of X such that f is continuous on D. Then D is a GO-space with a G_δ-diagonal so that Corollary 5.6 gives a dense metrizable subspace M of D. But then M is a dense metrizable subspace of X. □

One consequence of Corollary 5.9 is that no Souslin space (if one exists) can be a Blumberg space. For another example of a non-Blumberg space, see Example 7.2.

Proposition 5.10 *A GO-space X that is fragmentable and has countable cellularity has a G_δ-diagonal. Therefore, no Souslin space can be fragmentable.*

Proof: Any GO-space with countable cellularity is hereditarily Lindelöf. Now apply Proposition 3.13 to conclude that X has a G_δ-diagonal. For the assertion about Souslin spaces, see part (ii) of Example 7.2. □

6 Metrization of special LOTS and GO-spaces

Proposition 6.1 *If X is a LOTS that has property * and is perfect, then X is metrizable. If Y is a GO-space that is perfect and has a σ-isolated network, then Y is metrizable.*

Proof: According to Proposition 3.8, the space X has a G_δ-diagonal. But any LOTS with a G_δ-diagonal is metrizable [19]. According to Proposition 3.10, the GO-space Y has a σ-discrete network and therefore is metrizable [19]. □

Proposition 6.2 *Any fragmentable LOTS Y with countable cellularity (e.g., any separable fragmentable LOTS) is metrizable.*

Proof: By Proposition 5.10, Y has a G_δ-diagonal. Hence Y is metrizable [19]. $\square$

We note that the first assertion of Proposition 6.1 and Proposition 6.2 may fail if X is a GO-space rather than a LOTS, as the example of the Sorgenfrey line shows. (The Sorgenfrey line S is perfect, and Lemma 3.1 shows that S has property * because the real line has property *.)

Question 6.3 *Suppose X is a GO-space that is fragmentable and perfect. Does X have a G_δ-diagonal? Suppose Y is a LOTS that is fragmentable and perfect. Is Y metrizable?*

In an earlier version of this paper, we asked

Question 6.4 *Is it true that every compact LOTS with property * must be metrizable? Is it true that every compact, connected LOTS with at least two points and with property * must be homeomorphic to the interval $[0, 1]$ in the real line?*

We want to thank Gary Gruenhage for pointing out that existing results show that the answer is "Yes" to both parts of 6.4. We want to thank Richard Smith for pointing out the equivalence of c), d), e), h), i), and a) in our next proposition. The equivalence of a) and i) reveals the special relation between the topological properties a) through h) and the functional analytic property in i). A characterization of Eberlein compact spaces is given in 2.6. For definitions of the properties "Talagrand compact" and "Gulko compact," see [25].

Proposition 6.5 *If X is a compact LOTS, then the following are equivalent:*

a) X is metrizable

b) X is Eberlein compact

c) X is Talagrand compact

d) X is Gulko compact

e) X has a σ-isolated network

f) X is a Gruenhage space

*g) X has property **

h) X is perfect and fragmentable;

i) the function space $C(X)^*$ has a strictly convex dual norm.

Proof: The chart on page 380 of [25] shows that in any compact Hausdorff space we have a) $\Rightarrow$ b) $\Rightarrow$ c) $\Rightarrow$ d) $\Rightarrow$ e) $\Rightarrow$ f) $\Rightarrow$ g).

We next show that g) $\Rightarrow$ a). Suppose that X is a compact LOTS with property *. By Corollary 4.5, X is hereditarily paracompact. Therefore, by Corollary 3.4, there is a sequence $\langle \mathcal{U}(n) : n < \omega \rangle$ such that each $\mathcal{U}(n)$ is a pairwise disjoint collection of open sets and the sequence $\langle \mathcal{U}(n) : n < \omega \rangle$ separates points of X as described in the definition of property *. Replace each member of each

$\mathcal{U}(n)$ by the collection of its convex components; this allows us to assume that all members of each $\mathcal{U}(n)$ are convex. Because X is first-countable by Corollary 4.6, we know that each member of each $\mathcal{U}(n)$ is an open F_σ-set. Consequently, Proposition 2.6 applies to show that the compact LOTS X is Eberlein compact. But now we may invoke a result of Efimov and Čertanov [11] or a theorem from [9] which shows that any Eberlein compact LOTS is metrizable. Hence, g) $\Rightarrow$ a), showing that the first seven of the listed properties are equivalent in any compact LOTS.

Clearly, $a) \Rightarrow h)$ and the implication $h) \Rightarrow a)$ is a special cae of Proposition 3.14.

Finally, Proposition 3.2 of [20] shows that if X is compact and if $C(X)^*$ has a strictly convex dual norm, then X has property *. Therefore i) $\Rightarrow$ g) $\Rightarrow$ a). Finally note that if X is a compact metric space, then $C(X)^*$ has a strictly convex dual norm so that i) and a) are equivalent in any compact LOTS. □

Corollary 6.6 *Any compact, connected LOTS with one of the nine properties listed in Proposition 6.5 is either a single point or is homeomorphic to the usual unit interval* $[0, 1]$.

Proof: By Proposition 6.5, and such space is metrizable. But any compact, connected, metrizable LOTS with more than one point is homeomorphic to $[0, 1]$. □

Question 6.7 *Is there a first-countable, compact LOTS that is fragmentable but not metrizable?*

7 Examples

Example 7.1 *LOTS and GO-spaces that have property *:*

(i) The Sorgenfrey line S and the Michael line M have property * by Lemma 3.1 because the usual topology of $\mathbb{R}$ has property *. Because the Sorgenfrey and Michael lines are GO-spaces but not LOTS, it may be interesting to note that the following lexicographically ordered LOTS also have property *.

a) $S^* := \{(x, n) : n \in \mathbb{Z}, n \leq 0\}$

b) $M^* := \{(x, n) : \text{if } x \in \mathbb{Q} \text{ then } n = 0 \text{ and if } x \in \mathbb{P} \text{ then } n \in \mathbb{Z}\}$.

The space M^* is quasi-developable, while S^* is not. Neither S^* nor M^* has a G_δ-diagonal, and their topological sum $S^* \oplus M^*$ is a LOTS that is a Gruenhage space, has property * but is not quasi-developable, does not have a σ-isolated network, and does not have a G_δ-diagonal. The GO-space S has property * and is a Gruenhage space, but cannot have a σ-isolated network because it is perfect and does not have a σ-discrete network (see Lemma 3.10). The space M^* has a σ-isolated network, namely $\mathcal{N} = \{\{(x, k)\} : x \in \mathbb{P}, k \in \mathbb{Z}\} \cup \{((a, 0), (b, 0)) : a < b, a, b \in \mathbb{Q}\}$.

(ii) The lexicographically ordered Lindelöf space $Y = ([0, \omega_1) \times \mathbb{Q}) \cup \{(\omega_1, 0)\}$ is a LOTS with a σ-isolated network. This space is not first-countable at $(\omega_1, 0)$ and is not a Baire space because $Y = \bigcup\{C_q : q \in \mathbb{Q}\}$, where for each $q \in \mathbb{Q}$, the set $C_q = \{(\alpha, q) : \alpha < \omega_1\} \cup \{(\omega_1, 0)\}$ is closed and nowhere dense in Y. To describe a σ-isolated network for Y, fix any countable base $\mathcal{B} = \{B_n : n \geq 1\}$ of open sets for $\mathbb{Q}$ and let $\mathcal{N}(k) = \{\{\alpha\} \times B_k : \alpha < \omega_1\}$. Each $\mathcal{N}(k)$ is an isolated collection, as is the singleton collection $\mathcal{N}(0) = \{\{(\omega_1, 0)\}\}$, and $\mathcal{N} = \bigcup\{\mathcal{N}(k) : k \geq 0\}$ is a σ-isolated network for Y. It is easy to see that Y has a dense metrizable subspace, namely $M = Y - \{(\omega_1, 0)\}$. Note that Y is a Gruenhage space, has property * and is fragmentable, by 2.3 and 2.5.

Example 7.2 *LOTS and GO-spaces that do not have property * and are not fragmentable:*

(i) The lexicographically ordered double line $A = \mathbb{R} \times \{0, 1\}$ is a separable LOTS and hence has countable cellularity, so that if A cannot be fragmentable (see 6.2) and cannot have any of the nine properties listed in 6.5 (This result is due to Aviles [2].)

(ii) No Souslin space[9] can be fragmentable. For suppose the Souslin space S is fragmentable. Then Proposition 5.10 shows that S has a G_δ-diagonal and therefore has a dense metrizable subspace by Corollary 5.6. That is impossible in a non-separable space with countable cellularity. Consequently, no Souslin space can have a σ-isolated network and no Souslin space can have property *, in the light of 2.3 and 2.5.

(iii) Each space $Bush(S, T)$ (where S and T are disjoint dense subsets of $\mathbb{R}$) is a first-countable Baire space that has no dense metrizable subspace (see [8]) so that by Corollary 5.7, $Bush(S, T)$ cannot have property * and cannot be fragmentable. (Also, in the light of Corollary 5.9, no space $Bush(S, T)$ can be a Blumberg space.) No space $Bush(S, T)$ can have a σ-isolated network because $Bush(S, T)$ is first-countable but not quasi-developable [8]. Now apply Proposition 5.8.

(iv) Any η_1-set[10] with its open interval topology is a Baire LOTS that has no dense metrizable subspace, and therefore cannot have property * and cannot be fragmentable in the light of Theorem 5.7. In addition, no η_1-set can have a σ-isolated network for its open interval topology because, by 2.3, that would force the space to be a Gruenhage space and therefore fragmentable.

(v) For each bistationary subset $S \subseteq \omega_1$ Todorčevic [27] constructed a first-countable compact, connected linearly ordered topological space C_S that does not have a dense metrizable subspace. In the light of Theorem 5.7 (or Gruenhage's theorem in [13]), C_S cannot have property *, cannot be a Gruenhage space, and cannot be fragmentable (see Corollary 2.7 of [22] or Theorem 3.12 in this paper).

Example 7.3 *The ordinal space $[0, \omega_1)$ is scattered and therefore is fragmentable. However, $[0, \omega_1)$ is not paracompact so that $[0, \omega_1)$ cannot have a σ-isolated network and cannot have property *. Similarly, the ordinal space $[0, \omega_1]$ is fragmentable, compact, and non-metrizable.* □

Example 7.4 *There is a paracompact GO-space X with property * (and is, therefore, fragmentable), has a σ-disjoint π-base and a σ-isolated network, but does not contain any dense metrizable subspace.*

[9] A Souslin space is a non-separable GO-space that has countable cellularity. The existence of such a space is independent of ZFC.

[10] A linearly ordered set $(X, <)$ is an η_1-set if, given countable sets $C, D \subseteq X$ where $c < d$ for every $c \in C$ and $d \in D$, there are at least two points $x_1 < x_2$ such that $c < x_1 < x_2 < d$ for each $c \in C$ and $d \in D$.

Proof: The following example was described by the authors in [7]. Let $<$ be the usual ordering of the set $[0, \omega_1]$. Let X be the set of all sequences of ordinals of the form $x = \langle \alpha_n : 1 \leq n < \omega \rangle$ such that there is some $n = n(x) \geq 1$ with the property that

(a) $\alpha_j < \omega_1$ for all $j \leq n$ and
(b) $\alpha_j = \omega_1$ for all $j > n$.

We define the length of x, denoted $len(x)$, to be n.

Let $\prec$ be the lexicographic order on X, i.e., $\langle \alpha_i \rangle \prec \langle \beta_i \rangle$ if and only if there is some $k \geq 1$ such that $\alpha_i = \beta_i$ for $i < k$ and $\alpha_k < \beta_k$. Let λ be the LOTS topology defined on X by $\prec$. In Lemma 5.4b of [7] we showed that for each $x \in X$, the set $(\leftarrow, x)$ cannot contain a countable cofinal sequence, so that when X carries the open interval topology λ of $\prec$, the resulting LOTS $(X, \lambda, \prec)$ is nowhere first-countable.

To obtain the example needed here, we replace the LOTS topology λ by the GO-topology σ in which basic neighborhoods of a point x are intervals of the form $(y, x]$ where $y \prec x$. For the same reason that $(X, \lambda, \prec)$ is nowhere first-countable, the GO-space $(X, \sigma, \prec)$ is nowhere first-countable. In addition, if M were a dense metrizable subspace of (X, σ), then (X, σ) would be first countable at each point of M, so we see that (X, σ) does not have any dense metrizable subspace.

To show that X has a σ-isolated network, let $D(n)$ be the set of all points of X having length n. In its relative topology, each $D(n)$ is discrete so that by letting $\mathcal{N}(k) := \{\{d\} : d \in D(k)\}$ we obtain the required σ-isolated network $\bigcup\{\mathcal{N}(k) : k \geq 0\}$ for X. It follows from 2.3 and 2.5 that X is a Gruenhage space, has property *, and is fragmentable. Alternatively, to show that (X, σ) has property *, for $k \geq 1$ and $\beta < \omega_1$, let $U(k, \beta) := \{x \in X : x(k) = \beta\}$ where $x(k)$ denotes the k^{th} term of the sequence x. Then each $U(k, \beta)$ is open in (X, σ). Let $\mathcal{U}(k) := \{U(k, \beta) : \beta < \omega_1\}$. Then each $\mathcal{U}(k)$ is a pairwise disjoint collection of open sets. Suppose $x \neq y$ are points of X. Find the first integer k where $x(k) \neq y(k)$. Without loss of generality, we may assume that $x(k) < y(k)$. Then $x(k) = \beta$ for some $\beta < \omega_1$ so we have $x \in U(k, \beta)$ and $y \notin U(k, \beta)$. Because the collection $\mathcal{U}(k)$ is pairwise disjoint, we have $x \in \mathrm{St}(x, \mathcal{U}(k)) \subseteq X - \{y\}$, as required. Therefore (X, σ) has property *. Consequently X is fragmentable and has a σ-disjoint π-base by 5.3. $\square$

Remark 7.5 We note that the GO-space $(X, \sigma, \prec)$ in 7.4 has a σ-disjoint π-base in the light of Proposition 5.3, showing that first-countability cannot be omitted from the hypotheses of H.E. White's theorem in [28], even in the class of GO-spaces. In addition, Example 7.4 gives us an easy way to show that the existence of a dense metrizable subspace is not a closed-hereditary property in a LOTS. Begin with the GO-space (X, σ) of (7.4). There is a standard way to embed the GO-space (X, σ) as a closed subspace of a LOTS X^* (see [19]) by adding certain isolated points. (The spaces S^* and M^* in Example 7.1 are constructed in this way.) The set of all isolated points in X^* is a dense metrizable subset of X^* but the closed subspace X of X^* has no dense metrizable subspace. Finally, it follows from Corollary 5.9 that the GO-space $(X, \sigma, \prec)$ cannot be a Blumberg space.

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