CHAPTER 3

Recent Developments in the Topology of Ordered Spaces

Harold R. Bennett
Texas Tech University, Lubbock, TX 79409, U.S.A.
E-mail: graddir@math.ttu.edu

David J. Lutzer
College of William and Mary, Williamsburg, VA 23187-8795, U.S.A.
E-mail: lutzer@math.wm.edu

Contents

RECENT PROGRESS IN GENERAL TOPOLOGY II
Edited by Miroslav Hušek and Jan van Mill

1. Introduction

The last ten years have seen substantial progress in understanding the topology of linearly ordered spaces and their subspaces, the generalized ordered spaces. Some old problems have been solved, and ordered space constructions have been used to solve several problems that were posed in more general settings.

This paper surveys progress in several parts of ordered space theory. Chapter 2 mentions some recent results concerning orderability. Chapter 3 focuses on perfect ordered spaces. Chapter 4 deals with special base properties in ordered spaces. Chapter 5 investigates the role of diagonal and off-diagonal properties in metrization. Chapter 6 discusses Dugundji extension theory in ordered spaces. Chapter 7 briefly mentions Mary Ellen Rudin's recent solution of Nikiel's problem and its consequences for Hahn-Mazurkiewicz theory. Chapter 8 samples recent work on the structure of the Banach space $C(K)$ where K is a compact LOTS and Chapter 9 summarizes recent work on products of ordered spaces.

Recall that a *generalized ordered space* (GO-space) is a Hausdorff space X equipped with a linear order and having a base of order-convex sets. In case the topology of X coincides with the open interval topology of the given linear order, we say that X is a *linearly ordered topological space* (or LOTS). Čech showed that the class of GO-spaces is the same as the class of spaces that can be topologically embedded in some LOTS. (See LUTZER [1971].)

Throughout this paper, we will adopt the convention that all spaces are at least regular and T_1. Of course, the ordered spaces that we will consider have stronger separation – each GO-space is monotonically normal (HEATH, LUTZER and ZENOR [1973]) and hence hereditarily collectionwise normal. We reserve the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{R}$ for the usual sets of positive integers, all integers, rational numbers, irrational numbers, and real numbers, respectively.

For other related surveys, see TODORČEVIĆ [1984], MAYER and OVERSTEEGEN [1992], and LUTZER [1980].

2. Orderability

The classical orderability problem asks for topological characterizations of spaces whose topology can be given by the usual open interval topology of some linear ordering of the ground set. The survey paper PURISCH [1998] shows that this problem has a long history, going back to the early topological characterizations of the unit interval. The orderability problem for zero-dimensional metric spaces was solved by HERRLICH [1965] and later PURISCH [1977] gave necessary and sufficient conditions for orderability of any metric space. The general orderability problem was solved by VAN DALEN and WATTEL [1973] who proved:

2.1. THEOREM. *A T_1 space X is orderable if and only if X has a subbase $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ where each $\mathcal{S}_i$ is linearly ordered by inclusion and has the property that if $T \in \mathcal{S}_i$ has $T = \bigcap\{S \in \mathcal{S}_i : T \subset S \text{ and } T \neq S\}$, then T also has $T = \bigcup\{S \in \mathcal{S}_i : S \subset T \text{ and} S \neq T\}$.*

More recent work by VAN MILL and WATTEL [1984] significantly sharpened a selection-theoretic orderability theorem that MICHAEL [1951] proved for compact connected spaces.

For any space X, let 2^X be the collection of all closed non-empty subsets of X, topologized using the Vietoris topology , and let $X(2) = \{T \in 2^X : |T| = 2\}$ be topologized as a subspace of 2^X. VAN MILL and WATTEL [1984] showed:

2.2. THEOREM. *For a compact Hausdorff space X, the following are equivalent:*

a) X is orderable;

b) there is a continuous function $f : 2^X \to X$ having $f(S) \in S$ for each $S \in 2^X$;

c) there is a continuous function $g : X(2) \to X$ having $g(T) \in T$ for each $T \in X(2)$.

Functions such as the ones described in parts b) and c) of the previous theorem are called *continuous selections* and *continuous weak selections*, respectively. Theorem 2.2 has been the basis for further research. For example, FUJII and NOGURA [1999] have used selections to characterize compact spaces of ordinals as follows:

2.3. THEOREM. *A compact Hausdorff space is homeomorphic to a compact ordinal space if and only if there is a continuous selection $f : 2^X \to X$ with the property that $f(C)$ is an isolated point of C for each $C \in 2^X$.*

Recently ARTICO, MARCONI, PELANT, ROTTER and TKACHENKO [200?] have proved the following two results:

2.4. THEOREM. *Let X be a space with a continuous weak selection. If X^2 is pseudocompact, then X is countably compact and is a GO-space. In particular, if X is a pseudocompact k-space with a continuous weak selection, then X is a countably compact GO-space, and if X is a countably compact Tychonoff space with a continuous selection, then X is a GO-space.*

2.5. THEOREM. *For a completely regular space X, the following are equivalent:*

a) βX is orderable;

b) X is a pseudocompact GO-space;

c) X is countably compact and has a continuous weak selection;

d) X^2 is pseudocompact and X has a continuous weak selection.

3. Perfect ordered spaces

Recall that a topological space X is *perfect* if each closed subset of X is a G_δ-subset of X. Among GO-spaces, to be perfect is a very strong property.

3.1. THEOREM (LUTZER [1971]). *Any perfect GO-space is hereditarily paracompact and first countable.*

Determining whether or not a given GO-space is perfect is often one of the crucial steps in metrization problems for GO-spaces. (See 4.14 and 5.6 for examples.) The literature contains many generalizations of the property "X is perfect," and it is often useful to know which ones are equivalent to being perfect (in a GO-space). Here is a sample.

3.2. THEOREM (BENNETT, HOSOBUCHI and LUTZER [1999]). *The following properties of a GO-space X are equivalent:*

(a) *X is perfect;*

(b) *each relatively discrete subspace of X is σ-closed-discrete* (FABER [1974])*;*

(c) *for each open set $U \subseteq X$ there are open sets V_n having* $\mathrm{cl}(V_n) \subseteq U$ *and* $U \subseteq \mathrm{cl}(\bigcup\{V_n : n \geq 1\})$ (REED [1971])*;*

(d) *each open subset $U \subseteq X$ contains a dense subset S that is an F_σ-subset of X* (KOČINAC [1986])*;*

(e) *every closed nowhere dense subset of X is a G_δ-subset of X;*

(f) *every regularly closed subset of X is a G_δ-subset of X* (BENNETT *and* LUTZER [1984]).

Before continuing with the main theme of this section, let us pause to mention one interesting generalization of "X is perfect" that does *not* belong in the list given in 3.2. KOČINAC [1983] defined that a topological space is *weakly perfect* if each closed set $F \subseteq X$ contains a set S that is dense in F and is a G_δ-subset of X. It is easy to see that not all GO-spaces are weakly perfect: consider the lexicographically ordered set $X = \mathbb{R} \times [0,1]$ with its usual open interval topology. It is also easy to see that some GO-spaces are weakly perfect but not perfect: consider the usual space of countable ordinals. (In the light of 3.1, the fact that the space of countable ordinals is weakly perfect shows how wide is the gap between weakly perfect and perfect.)

Finding compact spaces that are weakly perfect but not perfect is more of a challenge. The first examples were given by KOČINAC [1983] and involved set theory. A family of ZFC examples was given by HEATH [1989]. In subsequent work BENNETT, HOSOBUCHI and LUTZER [2000] extended Heath's examples and linked the weakly perfect property to certain ideas in classical descriptive set theory.

To see that linkage, recall that a subset $A \subseteq [0,1]$ is *perfectly meager* if for any closed dense-in-itself set $C \subseteq [0,1]$, the set $C \cap A$ is a first category subset of C, where C carries its relative topology. Uncountable perfectly meager sets exist in ZFC (see MILLER [1984]). Next, for any dense subset $A \subseteq [0,1]$, let

$$X(A) = ([0,1] \times \{0\}) \cup (A \times \{-1,1\})$$

with the lexicographic order and usual open interval topology. Then $X(A)$ is always a compact, first-countable LOTS, and we have

3.3. PROPOSITION. *Let $A \subseteq [0,1]$ be dense. Then the following are equivalent:*

(a) *A is a perfectly meager subset of $[0,1]$;*

(b) *$X(A)$ is weakly perfect;*

(c) *$X(A)$ is hereditarily weakly perfect.*

If A is uncountable, then $X(A)$ is not perfect.

Notice that $X(A)$ is weakly perfect if and only if it is hereditarily weakly perfect. That leads to an open question.

3.4. QUESTION. Suppose X is an arbitrary weakly perfect GO-space. Must X be *hereditarily* weakly perfect?

A corollary of 3.3 is a new internal characterization of perfectly meager subsets of $[0, 1]$, namely that a subset $A \subseteq [0, 1]$ is perfectly meager if and only if for each $B \subseteq A$ there is a countable set $C \subseteq B$ that is dense in B and is a G_δ-subset in the relative topology of A. (See GRUENHADE and LUTZER [2000] for an easier proof.) In addition, various subspaces of $X(A)$ answer questions posed by HEATH [1989] concerning the existence of weakly perfect, non-metrizable, quasi-developable spaces.

As it happens, there is an interesting property that (for GO-spaces) is even better than being perfect, namely the existence of a σ-closed-discrete dense set. Any GO-space with a σ-closed-discrete dense subset must be perfect, but the converse is consistently false: any Souslin space (a non-separable LOTS with countable cellularity) would be a counterexample. In the 1970s, Maurice (see VAN WOUWE [1979]) posed the first of three major problems for perfect GO-spaces, namely:

3.5. QUESTION. Is there a ZFC example of a perfect GO-space that does not have a σ-closed-discrete dense subset?

Although recent work has shed considerable light on what it would take to solve 3.5, Maurice's question remains open today.

A second problem concerning perfect ordered spaces was posed by Heath. PONOMAREV [1967] and BENNETT [1968] independently proved that if there is a Souslin space, then there is a Souslin space with a point-countable base. After Bennett constructed in ZFC a LOTS, now called the "Big Bush" (see 4.5 below for details), that has a point-countable base but not a σ-point-finite base (see BENNETT [1968] [1971]), Heath asked:

3.6. QUESTION. Is there a ZFC example of a perfect GO-space that has a point-countable base and is not metrizable?

Heath's question also remains open. It is linked with Maurice's question 3.5 by

3.7. PROPOSITION (BENNETT and LUTZER [1984]). *If a GO-space has a point-countable base and a σ-closed-discrete dense subset, then it is metrizable.*

Thus, Heath's question boils down to "In ZFC, is there a perfect GO-space with a point-countable base that does not have a σ-closed-discrete dense subset?"

A third question concerning perfect GO-spaces was posed by NYIKOS [1976]. Recall that a topological space is *non-Archimedean* if it has a base that is a tree with respect to inclusion. Nyikos asked:

3.8. QUESTION. In ZFC, is there an example of a perfect non-Archimedean space that is not metrizable?

Question 3.8 is a question about ordered spaces because, as proved in PURISCH [1983], every perfect non-Archimedean space is a LOTS under some ordering. Nyikos' question remains open.

Although the questions of Maurice, Heath, and Nyikos remain open, an important paper by QIAO and TALL [200?] linked them to a generalized Souslin problem by proving:

3.9. THEOREM. *The following statements are equivalent:*

(a) *there is a perfect LOTS that does not have a σ-closed-discrete dense subset;*

(b) *there is a perfectly normal, non-Archimedean space that is not metrizable;*

(c) *there is a LOTS X in which every disjoint collection of convex open sets is σ-discrete, and yet X does not have a σ-closed-discrete dense subset;*

(d) *there is a dense-it-itself LOTS Y that does not have a σ-closed-discrete dense set, and yet each nowhere dense subspace of Y does have a σ-closed-discrete dense subset (in its relative topology).*

The key to the proof of 3.9 is the following lemma of Qiao and Tall:

3.10. LEMMA. *All first-countable GO-spaces contain dense non-Archimedean subspaces.*

In addition, QIAO and TALL [200?] also showed how Heath's question is related to the others by proving:

3.11. THEOREM. *The following statements are equivalent:*

(a) *there is a perfectly normal, non-metrizable, non-Archimedean space having a point-countable base;*

(b) *there is a perfect LOTS that has a point-countable base but does not have a σ-closed-discrete dense subset;*

(c) *there is a LOTS X with a point-countable base and having the property that every pairwise disjoint collection of convex open sets is σ-discrete, and yet X does not have a σ-closed-discrete dense subset;*

(d) *there is a dense-in-itself LOTS Y with a point-countable base that does not have a σ-closed-discrete dense subset, and yet every nowhere dense subspace of Y has a σ-closed-discrete dense subset for its relative topology.*

A family of recent results due to BENNETT, HEATH and LUTZER [2000], BENNETT, LUTZER and PURISCH [1999], and BENNETT and LUTZER [200?a] show how to recognize when a GO-space has a σ-closed-discrete dense set, in terms of relations to more familiar types of spaces. These characterizations may be of use to researchers working on the ZFC questions posed by Maurice, Heath, Nyikos, and Lutzer.

3.12. THEOREM. *The following properties of a perfect GO-space X are equivalent:*

(a) *X has a σ-closed-discrete dense subset;*

(b) *X has a dense metrizable subspace;*

(c) *there is a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X such that for each $p \in X$, the set $\bigcap\{St(p, \mathcal{G}_n) : n \geq 1\}$ has at most two points;*

(d) *there is a sequence $\langle \mathcal{G}_n \rangle$ of open covers of X such that for each $p \in X$, the set $\bigcap\{St(p, \mathcal{G}_n) : n \geq 1\}$ is a separable subspace of X;*

(e) *X is the union of two subspaces, each having a G_δ-diagonal in its relative topology;*

(f) *X is the union of countably many subspaces, each having a G_δ-diagonal in its relative topology;*

(g) *there is a metrizable GO-space Y and a continuous $f : X \to Y$ with $|f^{-1}[y]| \leq 2$ for each $y \in Y$;*

(h) *there is a topological space Z with a G_δ-diagonal and a continuous function $g : X \to Z$ such that $g^{-1}[z]$ is a separable subspace of X for each $z \in Z$;*

(i) *X has a dense subspace $E = \bigcup\{E_n : n \geq 1\}$ where each E_n is a metrizable subspace of X;*

(j) *X is metrically fibered in the sense of Tkachuk* [1994], *i.e., there is a continuous function $f : X \to M$ where M is metrizable and $f^{-1}(m)$ is a metrizable subspace of X for each $m \in M$.*

As a final part of this chapter, we consider the embedding problem for perfect GO-spaces. It is older than the questions of Maurice, Heath, and Nyikos and turns out to be related to them because of work by W-X Shi, as we will explain later.

It has been known since the work of Čech that GO-spaces are precisely the subspaces of linearly ordered topological spaces. Furthermore, there is a canonical construction that produces, for any GO-space X, a LOTS X^* that contains X as a closed subspace and is, in some sense, the smallest LOTS with this property (LUTZER [1971]). Given a GO-space $(X, \tau, <)$, let λ be the usual order topology of the order $<$. Then $\lambda \subseteq \tau$. Let $R = \{p \in X : [p, \rightarrow[\in \tau - \lambda\}$ and $L = \{q \in X :]\leftarrow, q] \in \tau - \lambda\}$. Define X^* to be the following lexicographically ordered subset of $X \times \mathbb{Z}$:

$$X^* = (X \times \{0\}) \cup \{(p, n) : p \in R \text{ and } n \leq 0\} \cup \{(q, n) : q \in L \text{ and } n \geq 0\}.$$

Clearly X is homeomorphic to the subspace $X \times \{0\}$ of X^*. It often happens that the GO-space X has a topological property P if and only if the LOTS X^* has property P. For example:

3.13. PROPOSITION (LUTZER [1971]). *Let P be one of the following properties: paracompact, metrizable, Lindelöf, quasi-developable, or has a point-countable base. Then the GO-space X has property P if and only if its LOTS extension X^* has P.*

However, being perfect is one important topological property that cannot be added to the list in 3.13. If S is the Sorgenfrey line, then S is certainly perfect, but a Baire category argument shows that S^* is not perfect. Of course, there is a perfect LOTS that contains S as a subspace, namely the lexicographic product space $\mathbb{R} \times \{0, 1\}$. Examples of that type led to

3.14. QUESTION. Is it true that any perfect GO-space can be topologically embedded in some perfect LOTS? (LUTZER [1971] BENNETT and LUTZER [1990]).

Many related questions have been solved. For example, we know that there are perfect GO-spaces $(X, \tau, <)$ that cannot be embedded:

(a) as a closed subset, or as a G_δ-subset, of any perfect LOTS (LUTZER [1971]);

(b) as a dense subspace of a perfect LOTS whose order extends the given order $<$ of X (SHI [1997] and MIWA and KEMOTO [1993]);

(c) as a dense subspace of a perfect LOTS (SHI, MIWA and GAO [1995]) (with no restriction on the order of the extension).

However, the general question 3.14 remains open today.

We have come to understand that the perfect embedding question 3.14 is closely related to the questions of Maurice, Heath, and Nyikos because SHI [1999b] has proved:

3.15. THEOREM. *If X is a perfect GO-space with a σ-closed-discrete dense set, then there is a perfect LOTS Y that contains X and also has a σ-closed-discrete dense set.*

Shi's theorem links questions 3.14 and 3.5: if there is a model $\mathcal{M}$ of set theory in which every perfect GO-space has a σ-closed-discrete dense set, then in $\mathcal{M}$ every perfect GO-space can be embedded in a perfect LOTS.

It might be tempting to think that one could start with a model of set theory that contains a Souslin space S and modify S by isolating some points and making others "one sided" in the Sorgenfrey sense, in the hope of getting a consistent example of a perfect GO-space that cannot be embedded in a perfect LOTS. That approach cannot work because SHI, MIWA and GAO [1996] showed that if one starts with a perfect LOTS Y and constructs a perfect GO-space X by modifying it as described above, then X will embed in some perfect LOTS Z.

4. Base axioms related to metrizability

As seen in Section 3, topological properties that are distinct in general spaces can coalesce among GO-spaces. The next three theorems illustrate this phenomenon for metrizable spaces, spaces with a σ-disjoint base, and spaces with a point-countable base.

4.1. THEOREM (LUTZER [1971]). *For any GO-space X, the following are equivalent:*

(a) *X is metrizable;*

(b) *X has a σ-locally-countable base;*

(c) *X is developable;*

(d) *X is semi-stratifiable.*

4.2. THEOREM (LUTZER [1971], BENNETT [1971]). *For any GO-space X, the following are equivalent:*

(a) *X has a σ-disjoint base;*

(b) *X has a σ-point-finite base;*

(c) *X is quasi-developable.*

4.3. THEOREM (GRUENHAGE [1992]). *For any GO-space X, the following are equivalent:*

(a) *X has a point-countable base;*

(b) *X has the Collins-Roscoe property "open (G)", i.e., for each $x \in X$ there is a countable collection $\mathcal{B}(x)$ of open neighborhoods of x such that if a sequence x_n converges to x, then the collection $\bigcup\{\mathcal{B}(x_n) : n \geq 1\}$ contains a neighborhood base at the point x.*

Much of recent research on base properties in ordered spaces has focused on two major themes.

- Examine the gaps between the clusters of properties characterized in 4.1, 4.2, and 4.3, i.e., between the properties of metrizability, quasi-developability, and the existence of a point-countable base in GO-spaces;
- Expand the lists of equivalent properties in each of 4.1, 4.2, and 4.3.

We begin by considering the gap between the metrizability cluster described in (4.1) and the properties equivalent to quasi-developability given in (4.2). First of all, there really *is* a gap. The Michael line M is a GO-space example of a quasi-developable space that is not metrizable and we have

4.4. EXAMPLE. The lexicographically ordered space

$$M^* = (\mathbb{R} \times \{0\}) \cup (\mathbb{P} \times \mathbb{Z})$$

is a quasi-developable LOTS that is non-metrizable.

Second, we know exactly what must be added to the properties in 4.2 in order to produce metrizability. One answer comes from the proposition that a quasi-developable GO-space is metrizable if and only if it is perfect. Another is provided by the theorem that a GO-space is metrizable if and only if it is a quasi-developable p-space (in the sense of Arhangel'skii) (BENNETT [1968] [1971]).

The gap between quasi-developability of a GO-space X and the existence of a point-countable base for X is harder to study. Finding *any* ZFC example of a GO-space with a point-countable base and not a σ-point-finite base is complicated.

4.5. EXAMPLE (BENNETT [1971]). The "Big Bush" is a LOTS in ZFC that has a point-countable base but not a σ-disjoint base. One way to describe this space is to view it as a certain lexicographically ordered set. Let X be the set of all functions $f : [0, \omega_1[\to \mathbb{R}$ with the property that for some $\alpha_f < \omega_1$,

$$f(\beta) \in \mathbb{P} \text{ for each } \beta \in [0, \alpha_f[\qquad \text{and}$$
$$f(\beta) = f(\alpha_f) \in \mathbb{Q} \text{ for each } \beta \in [\alpha_f, \omega_1[.$$

Endow X with the usual open interval topology of the lexicographic ordering. Then basic neighborhoods of a point $f \in X$ will have the form

$$B(f, n) = \{g \in X : \text{if } \beta < \alpha_f \text{ then } g(\beta) = f(\beta) \text{ and } |g(\alpha_f) - f(\alpha_f)| < 1/n\}.$$

As it happens, the Big Bush is not perfect. So far, as noted in Section 3, obtaining perfect examples of non-metrizable GO-spaces with point-countable bases requires additional set theoretic axioms. If one is willing to accept consistent examples, then one has

4.6. EXAMPLE (BENNETT [1968], PONOMAREV [1967]). If there is a Souslin space (i.e., a non-separable GO-space with countable cellularity) then there is a Souslin space with a point-countable base. Such a Souslin space example is also hereditarily Lindelöf.

As noted in Section 3, Examples 4.5 and 4.6 led Heath to ask whether there is a ZFC example of a perfect non-metrizable LOTS with a point-countable base.

Among GO-spaces, what exactly is the difference between having a point-countable base and the cluster of stronger properties listed in 4.2? In other words, exactly what must one add to the existence of a point-countable base in a GO-space in order to obtain a σ-disjoint base for the space? The answer is a strange property called "Property III" (BENNETT and LUTZER ([1996a], BENNETT, LUTZER and PURISCH [1999]).

4.7. DEFINITION. A topological space X has *Property III* if for each $n \geq 1$ there are subsets U_n and D_n of X satisfying:

(a) each U_n is open in X and D_n is a relatively closed subset of U_n;

(b) each D_n is discrete-in-itself;

(c) if p is a point of an open set G, then for some $n \geq 1$, $p \in U_n$ and $G \cap D_n \neq \emptyset$.

Property III is very weak when considered in the category of all topological spaces. Most generalized metric spaces have Property III (e.g. all semi-stratifiable spaces have Property III) and every topological space X is a closed subset of a topological space $B(X)$ that has Property III. Furthermore, if X is a "nice" space (e.g., regular, normal, has a point-countable base, etc.) then $B(X)$ is nice in the same way. See Section 6 of BENNETT and LUTZER [1996a] for details.

It is clear that if X is hereditarily Lindelöf and has Property III, then X must be separable. Consequently, none of the Souslin space examples in 4.6 can have Property III.

Property III turns out to be exactly what one must add to the existence of a point-countable base (in a GO-space) in order to get quasi-developability.

4.8. THEOREM (BENNETT and LUTZER [1996a, 1996b]). *Let X be a GO-space. Then the following assertions are equivalent:*

(a) *X is quasi-developable;*

(b) *X has a σ-disjoint base;*

(c) *X has a σ-point-finite base;*

(d) *X has a point-countable base and has Property III;*

(e) *X has a $\delta\theta$-base* (AULL [1974]) *and has Property III;*

(f) *X has a point-countable base and a σ-minimal base (see below in this section);*

(g) *X has a point-countable base and has a quasi-G_δ-diagonal* (BENNETT and LUTZER [1996a]).

Theorem 4.8 suggests that a deeper investigation of Property III in GO-spaces is warranted. As a start, we can show that Property III is hereditary among GO-spaces (BENNETT and LUTZER [1996a]) and that Property III implies hereditary paracompactness for a GO-space. It is easy to construct GO-spaces with Property III that are not first-countable. What is more surprising is that there are GO-spaces having Property III that are not first-countable at any point, a fact that shows how far Property III is from the types of properties considered in 4.1, 4.2, and 4.3.

4.9. EXAMPLE. (BENNETT and LUTZER [1996a]) There is a LOTS with Property III that is not first-countable at any point. Let X be the set of sequences $\langle x_i \rangle$ with $x_i \in [0, \omega_1]$ such that for some integer $n = n(\langle x_i \rangle)$ we have $x_i < \omega_1$ for each $i \leq n$ and $x_j = \omega_1$ for all $j > n$. Equip X with the usual open interval topology of the lexicographic order.

In recent years, an idea introduced by HEATH and LINDGREN [1976] has become the basis for understanding the fine structure of the gap between metrizability and quasi-developability in GO-spaces. Heath and Lindgren defined that a collection C is *weakly uniform* if given any $x \neq y$ in X, the collection $\{C \in \mathcal{C} : \{x, y\} \subseteq C\}$ is finite, and they studied topological spaces that have a *weakly uniform base* (WUB) for their topology. They showed that any space with a WUB must have a G_δ-diagonal and concluded:

4.10. PROPOSITION. *Any LOTS with a weakly uniform base must be metrizable.*

They also noted that the Michael line is an example of a non-metrizable GO-space with a WUB, so that 4.10 cannot be extended to be a metrization theorem for GO-spaces in general. However, there is an interesting structure theorem for GO-spaces that have weakly uniform bases, namely:

4.11. THEOREM (BENNETT and LUTZER [1998b]). *For any GO-space $(X, \tau, <)$, the following are equivalent:*

(a) *X has a WUB;*

(b) *X has a G_δ-diagonal and is quasi-developable.*

□ Outline of Proof: The proof of (a) ⇒ (b) that appears in BENNETT and LUTZER [1998b] can be simplified. Suppose that X has a WUB. HEATH and LINDGREN [1976] showed that X must have a G_δ-diagonal. According to a theorem of Przymusinski (see ALSTER [1975]) there is a metrizable topology $\mu \subseteq \tau$ such that $(X, \mu, <)$ is also a GO-space. Consequently there is a set $D = \bigcup\{D_n : n \geq 1\}$ such that each D_n is a closed discrete subset of (X, μ), and hence also of (X, τ), with the property that whenever $a < b$ and $]a, b[\, \neq \emptyset$, then $D \cap \,]a, b[\, \neq \emptyset$. Let $U_0 = \{p \in X : \{p\} \in \tau\}$ and for each $n \geq 1$ let $U_n = X$. The sets U_n and D_n for $n \geq 0$ witness the fact that X has Property III. It is easy to show that any GO-space with a WUB must have a point-countable base, and now the proof that (d) ⇒ (a) in 4.8 shows that X is quasi-developable. Conversely, suppose that the GO-space X has a G_δ-diagonal and is quasi-developable. According to 4.2, X has a σ-disjoint base $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \geq 1\}$. Because X has a G_δ-diagonal, the weaker metric topology μ in Przymusinski's theorem cited above yields a sequence $\langle \mathcal{G}_n \rangle$ of point-finite open covers of X such that $\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$ and $\bigcap\{St(p, \mathcal{G}_m) : n \geq 1\} = \{p\}$ for each $p \in X$. Let $\mathcal{H}_n = \{B \cap G : B \in \mathcal{B}_n,\ G \in \mathcal{G}_n\}$. Each $\mathcal{H}_n$ is point finite and

$\mathcal{H} = \bigcup\{\mathcal{H}_n : n \geq 1\}$ is a base for X. If p belongs to infinitely many distinct members H_k of $\mathcal{H}$, then point-finiteness of each $\mathcal{H}_n$ forces $H_k \in \mathcal{H}_{n_k}$ for infinitely many distinct n_k so that $\bigcap\{H_k : k \geq 1\} \subseteq \bigcap\{St(p, \mathcal{G}_{n_k} : k \geq 1\} = \{p\}$. Thus, $\mathcal{H}$ is a WUB for the space X. □

The class of GO-spaces with weakly uniform bases lies *strictly* between the class of quasi-developable GO-spaces and the class of metrizable GO-spaces. For example, the Michael line M is non-metrizable and has a weakly uniform base (because it is a quasi-developable GO-space with a G_δ-diagonal). To obtain an example of a quasi-developable GO-space that does not have a weakly uniform base, we use the LOTS extension of the Michael line.

4.12. EXAMPLE. The LOTS M^* described in Example 4.4 is a quasi-developable LOTS that has no weakly uniform base in the light of 4.11 because (being non-metrizable) it cannot have a G_δ-diagonal (see 5.1 (a')).

A recent paper by BALOGH, DAVIS, JUST, SHELAH and SZEPTYCKI [200?] introduced a property that is substantially weaker than having a WUB. A base $\mathcal{B}$ for a space X is a $<\omega$-WUB if, given any infinite set $S \subseteq X$ there is a finite set $F \subseteq S$ such that $\{B \in \mathcal{B} : F \subset B\}$ is finite. This generalizes the notion of an n-WUB, by which we mean a base $\mathcal{B}$ for X with the property that any set with n elements is contained in at most finitely many members of $\mathcal{B}$. Clearly, a weak uniform base is a 2-WUB, and any n-WUB for a space is a $<\omega$-WUB. Examples in BENNETT and LUTZER [1998b] show that none of these implications can be reversed. In addition, we have:

4.13. PROPOSITION. *Any GO-space with a $<\omega$-WUB is quasi-developable.*

We do not know how to characterize GO-spaces that have $<\omega$-WUBs.

4.14. QUESTION. For a GO-space X, find a topological property that solves the equation

X is quasi-developable + (?) if and only if X has a $<\omega$-WUB.

[Added in Proof: The property needed in (?) of (4.14) is: there is a sequence $\mathcal{G}_n$ of open covers of X such that for any infinite subset $S \subseteq X$, there is a finite set $F \subseteq S$, a point $p \in F$, and an integer n having $F \not\subseteq \mathrm{St}(p, \mathcal{G}_n)$. See BENNETT and LUTZER [200?b].]

Theorem 4.11 allows us to extend Theorem 4.1, expanding the cluster of generally distinct topological properties that are equivalent to metrizability in GO-spaces. BENNETT and LUTZER [1998b] shows:

4.15. THEOREM. *For any GO-space X, the following are equivalent:*

(a) *X is metrizable;*

(b) *X has a σ-locally countable base;*

(c) *X is developable;*

(d) *X is semi-stratifiable;*

(e) *X has an "open-in-finite" base, i.e., a base $\mathcal{B}$ with the property that $\{B \in \mathcal{B} : U \subseteq B\}$ is finite for every non-empty open set U;*

(f) *X has a sharp base in the sense of* ARHANGELSKII, JUST, REZNICHENKO *and* SZEPTYCKI [2000], *i.e., a base $\mathcal{B}$ with the property that if $\langle B_n \rangle$ is a sequence of distinct members of $\mathcal{B}$ each containing the point p, then the collection $\{\bigcap\{B_j : j \leq n\} : n \geq 1\}$ is a local base at p.*

□ The following argument is shorter than the proof of (4.15) given in BENNETT and LUTZER [1998b]. Every metrizable space satisfies both (e) and (f). To prove that (e) ⇒ (a), suppose (e) holds. It is easy to check that X is first-countable. Therefore, X will be quasi-developable if X is the union of countably many quasi-developable subspaces. We may assume that members of the base $\mathcal{B}$ are convex. Let J be the set of all points of X that have either an immediate predecessor or an immediate successor in the given ordering of X. Let J_0 be the set of relatively isolated points of J. Let J_1 be the set of points of $J - J_0$ that have an immediate predecessor in X and let J_2 be the points of $J - J_0$ that have an immediate successor. Clearly J_0 is a quasi-developable subspace of X. One checks that $\{B \cap J_1 : B \in \mathcal{B}\}$ is a WUB for J_1 and concludes from 4.11 that J_1 is also quasi-developable. Analogously, so is J_2. Let $Y = X - J$ and verify that $\{B \cap Y : B \in \mathcal{B}\}$ is a WUB for Y. Because $X = J_0 \cup J_1 \cup J_2 \cup Y$, we see that X is quasi-developable. To complete the proof we show that X is perfect. According to Faber's theorem (see 3.2 (b), above), it is enough to show that every relatively discrete subspace D of X is σ-closed-discrete. Find a collection $\{U(d) : d \in D\}$ of pairwise disjoint open convex sets with $d \in U(d)$ for each $d \in D$. Each set $D_k = \{d \in D : U(d)$ is contained in at most k members of $\mathcal{B}\}$ is closed and discrete, and $D = \bigcup\{D_k : k \geq 1\}$, as required. Thus we have e) ⇒ c) ⇒ a).

To prove that f) ⇒ a), show that any sharp base for X is weakly uniform and once again apply Faber's characterization of perfect GO-spaces to conclude that X is quasi-developable and perfect, whence metrizable. □

For many years, it appeared that one might be able to add another equivalent condition to the list in 4.2. Aull introduced the study of σ-minimal bases in AULL [1974]. A collection $\mathcal{C}$ of subsets of X is *minimal* or irreducible if each $C \in \mathcal{C}$ contains a point $x(C)$ that is not in any other member of $\mathcal{C}$, and a collection that is the union of countably many minimal collections is called σ-minimal. Clearly, any σ-disjoint base is σ-minimal. The first example showing that the converse is not true among GO-spaces appears in BENNETT and BERNEY [1977]:

4.16. EXAMPLE. The lexicographic square $X = [0, 1] \times [0, 1]$ is a compact, non-metrizable LOTS that has a σ-minimal base for its topology, but not a σ-disjoint base. Furthermore, its closed subspace $Y = [0, 1] \times \{0, 1\}$ does not have a σ-minimal base.

Some consequences of the existence of a σ-minimal base for a GO-space are known:

4.17. PROPOSITION (BENNETT and LUTZER [1977]). *Any GO-space with a σ-minimal base is hereditarily paracompact.*

The proof of 4.17 uses stationary set techniques, but is more complicated than usual. Suppose X is a GO-space with a σ-minimal base, and suppose X is not hereditarily paracompact. Then there is a stationary subset S of some uncountable regular cardinal that embeds in X (ENGELKING and LUTZER [1976]). The usual next step would be to say that

S inherits a σ-minimal base, but (as 4.16 shows) that might not be the case. Nevertheless, a more complicated proof still works.

Examples led researchers to suspect that either a GO-space with a σ-minimal base would be quasi-developable, or else that it would contain a certain pathological type of subspace (BENNETT and LUTZER [1990]). This led us to pose two questions:

(a) Is it true that a GO-space X must be quasi-developable provided every subspace of X has a σ-minimal base for its relative topology?

(b) Is it true that a compact LOTS X must be metrizable provided every subspace of X has a σ-minimal base for its relative topology?

Both questions have been answered recently. The first question was answered in BENNETT and LUTZER [1998a] when it was discovered that every subspace of a certain non-metrizable perfect space $E(Y, X)$ has a σ-minimal base for its relative topology. (See 5.5 for a description of $E(Y, X)$.) The second, and harder, question was answered negatively by SHI [1999a] who used a branch space of an Aronszajn tree to construct a non-metrizable compact LOTS X such that every subspace of X has a σ-minimal base for its relative topology.

5. Diagonal and off-diagonal conditions in GO-spaces

There are some striking parallels between the metrization theory for compact Hausdorff spaces and for LOTS. The most basic is:

5.1. THEOREM.

(a) *If X is a compact Hausdorff space having a G_δ-diagonal, then X is metrizable* (ŠNEIDER [1945]).

(a') *Any LOTS with a G_δ-diagonal is metrizable* (LUTZER [1971]).

(b) *A paracompact space that has a G_δ-diagonal and can be p-embedded in a compact Hausdorff space must be metrizable* (BORGES [1966] OKUYAMA [1964]).

(b') *A paracompact GO-space that has a G_δ-diagonal and can be p-embedded in a LOTS must be metrizable* (LUTZER [1971]).

The parallels in 5.1 are not accidental: see LUTZER [1972b] where the following is proved.

5.2. THEOREM. *Suppose X is a p-embedded subspace of a compact Hausdorff space or a p-embedded subspace of a LOTS. Then there is a sequence $\langle \mathcal{B}_n \rangle$ of open bases for X with the property that a collection $\mathcal{L}$ is a local base at a point $p \in X$ whenever*

a) $\bigcap \mathcal{L} = \{p\} = \bigcap\{\mathrm{cl}_X(L) : L \in \mathcal{L}\}$*;*
b) *$\mathcal{L}$ is a filter base;*
c) *the set $\{n : \mathcal{L} \cap \mathcal{B}_n \neq \emptyset\}$ is infinite.*

Furthermore, any completely regular space with such a sequence of bases and a G_δ-diagonal has a base of countable order so that a space is metrizable if and only if it has a sequence of bases as described above, has a G_δ-diagonal, and is paracompact.

One theme in ordered space research has been to explore how far the parallelism suggested by Theorem 5.1 extends, and that is the focus of this chapter.

One generalization of the notion of a G_δ-diagonal is Hušek's small diagonal property (HUŠEK [1976]). A space X has a *small diagonal* if, given any uncountable subset $T \subseteq X^2 - \Delta(X)$, there is an open set $U \subseteq X^2$ such that $\Delta(X) \subseteq U$ and $|T - U| > \omega$. Some of the metrization theory for spaces with small diagonals is known.

5.3. THEOREM. *Let CH denote the Continuum Hypothesis.*

(a) *Assume CH. Then any compact Hausdorff space with a small diagonal is metrizable* (JUHÁSZ and SZENTMIKLÓSSY [1992])*;*

(b) *In ZFC, any Lindelöf LOTS that has a small diagonal is metrizable* (*van Douwen and Lutzer, announced in* HUŠEK [1977] *and proved in* BENNETT *and* LUTZER [1997b]).

The symmetry suggested by 5.1 is broken, to some degree, in 5.3: if there were a strict parallelism between (a) and (b) in 5.3, then one would not need the Lindelöf hypothesis in 5.3 (b). There is no way to get around that problem because of the following example from BENNETT and LUTZER [1997a].

5.4. EXAMPLE. There is a LOTS with a small diagonal that is not paracompact (and hence not metrizable). The space in question is $S = \{\alpha < \omega_3 : \mathrm{cf}(\alpha) = \omega_2\}$. A result of PURISCH [1977] shows that there is some re-ordering of S under which S is a LOTS. Because S is a stationary subset of ω_3, S is not paracompact, and it is not hard to verify that S has a small diagonal. (This space is Example 6.2 in BENNETT and LUTZER [1997b].)

The history of metrization theory has shown that metrization theorems originally discovered for compact Hausdorff spaces can often be generalized, in the presence of paracompactness, to the progressively larger classes of locally compact Hausdorff spaces, Čech-complete spaces, and finally to the class of p-spaces introduced by Arhangel'skii. The results in Theorem 5.1 (a) and (b) are probably the best known examples. It is natural to wonder whether the same process of generalization would be possible for the Juhász-Szentmiklóssy result in 5.3 (a). The answer is "No" as is shown by:

5.5. EXAMPLE (BENNETT and LUTZER [1998a]). In ZFC there is a paracompact, perfect, Čech-complete LOTS that has a σ-closed-discrete dense subset, weight ω_1, and a small diagonal and yet is not metrizable.

□ The construction of the example in (5.5) begins with a remarkable metric space due to A.H. Stone in STONE [1963]. Stone's metric space is a certain subset $X \subseteq D^\omega$ where D is an uncountable discrete space of cardinality ω_1. The space X has the following properties:

(a) $|X| = \omega_1$;

(b) X is not the union of countably many relatively discrete subspaces;

(c) if S is any countable subset of X, then $\mathrm{cl}_X(S)$ is also countable.

Let Y be the closure of X in D^ω. Then Y is Čech-complete and has weight ω_1 because $w(D^\omega) = \omega_1$. A theorem of HERRLICH [1965] (see also Problem 6.3.2 in ENGELKING [1989]) shows that, with respect to some ordering, Y is a LOTS. Using that ordering, lexicographically order the set $E(Y,X) = (Y \times \{0\}) \cup (X \times \{0,1\})$. In the open interval topology of the lexicographic ordering, $E(Y,X)$ is a LOTS with a σ-closed-discrete dense subset (whence $E(Y,X)$ is perfect, paracompact, and first-countable), and the natural projection from $E(Y,X)$ onto Y is a perfect mapping. Hence $E(Y,X)$ is Čech-complete. Because the construction of $E(Y,X)$ involves splitting ω_1 points in a LOTS whose weight is ω_1, we know that $w(E(Y,X)) = \omega_1$. The special properties of X yield that the space $E(Y,X)$ is non-metrizable and yet has a small diagonal. □

In addition to showing the limits on possible generalizations of the Juhász-Szentmiklóssy result in 5.3, the space $E(Y,X)$ also answers negatively a question posed by Arhangel'skii and Bella who proved in ARHANGEL'SKII and BELLA [1992] that, assuming CH, any Lindelöf p-space with weight ω_1 and a small diagonal must be metrizable. Then they asked whether, under CH or in ZFC, the Lindelöf hypothesis could be weakened to paracompactness. The space $E(Y,X)$ is a ZFC counterexample to that question. The space $E(Y,X)$ also provides a solution to an old problem about σ-minimal bases, as noted at the end of Section 4, above.

The following question is related to 5.3 (b) and remains open:

5.6. QUESTION. Suppose X is a Lindelöf GO-space with a small diagonal that can be p-embedded in some LOTS. Must X be metrizable?

In the light of Proposition 3.4 of BENNETT and LUTZER [1997a], to prove metrizability of an X with the properties in 5.6, it will be necessary and sufficient to prove that X is perfect.

As it happens, one can prove in ZFC that a countably compact GO-space with a small diagonal must be metrizable, and that contrasts with the situation in more general spaces. GRUENHAGE [200?] and PAVLOV [200?] have proved

5.7. PROPOSITION. *The assertion that any countably compact completely regular space with a small diagonal must be metrizable is consistent with, and independent of, ZFC + CH.*

The metrization results mentioned so far in this section all involve properties of the diagonal. The following theorem of GRUENHAGE [1984] focused attention on the off-diagonal subspace $X^2 - \Delta$ of X^2.

5.8. THEOREM. *A compact Hausdorff space X is metrizable if and only if X is paracompact off of the diagonal* (*i.e., $X^2 - \Delta$ is paracompact in its relative topology*).

Gruenhage's theorem was generalized in GRUENHAGE and PELANT [1988] to yield a G_δ-diagonal for members of the class of paracompact Σ-spaces that are paracompact off of the diagonal, and led to further investigations of off-diagonal properties by Kombarov and Stepanova who proved:

5.9. THEOREM (KOMBAROV [1989]). *A paracompact Σ-space X has a G_δ-diagonal if and only if there is a rectangular open cover of $X^2 - \Delta$ (i.e. a cover by sets of the form $G \times H$ where G and H are disjoint open subsets of X) that is locally finite in $X^2 - \Delta$.*

5.10. THEOREM (STEPANOVA [1993]). *A paracompact p-space X is metrizable if and only if there is a family of subsets of $X^2 - \Delta$ that is a σ-locally finite* (*in* $X^2 - \Delta$) *cover of $X^2 - \Delta$, where each member of the family is a co-zero set in X^2.*

Once again with (5.1) in mind, it is reasonable to ask whether there are off-diagonal metrization theorems for GO-space or LOTS that parallel the results by Gruenhage, Kombarov, and Stepanova, above. This time, the answer is "No," as shown by a single example.

5.11. EXAMPLE (BENNETT and LUTZER [1997b]). Let M^* be the LOTS extension of the Michael line described in 4.4 above. Then M^* is non-metrizable and first-countable, does not have a G_δ-diagonal, and has the properties that $X^2 - \Delta$ is paracompact and admits the kinds of rectangular and co-zero covers described in 5.9 and 5.10.

A subspace of M^* provides a consistent counterexample that answers a question of KOMBAROV [1989]. After proving 5.9 above, Kombarov asked whether a space X must have a G_δ-diagonal provided X is Lindelöf and regular, and $X^2 - \Delta$ has a countable cover by sets of the form $G \times H$, where G and H are disjoint open subsets of X. The next example provides a consistent negative answer.

5.12. EXAMPLE. Assume CH or $\mathbf{b} = \omega_1$. Then there is a Lindelöf LOTS X that does not have a G_δ-diagonal and yet admits a countable cover by sets of the form $G \times H$ where G and H are disjoint open subsets of X.

□ One uses CH or $\mathbf{b} = \omega_1$ to find an uncountable set L of real numbers that is concentrated on the set $\mathbb{Q}$ of rational numbers (i.e., if U is open in $\mathbb{R}$ and contains $\mathbb{Q}$, then $L - U$ is countable). Then the desired space X is the lexicographically ordered LOTS $(\mathbb{Q} \times \{0\}) \cup ((L - \mathbb{Q}) \times \mathbb{Z})$. □

Even though the off-diagonal conditions in 5.8, 5.9, and 5.10 do not yield metrizability or a G_δ-diagonal in GO-spaces, they do yield hereditary paracompactness. One shows that no stationary set in a regular uncountable cardinal can satisfy any of the off-diagonal conditions in 5.8, 5.9, or 5.10, and then a result in ENGELKING and LUTZER [1976] gives hereditary paracompactness in any GO-space with such off-diagonal properties. A stronger result of BALOGH and RUDIN [1992] gives hereditary paracompactness in any monotonically normal space satisfying any one of the off-diagonal conditions of Gruenhage, Kombarov, or Stepanova.

A different kind of off-diagonal property was introduced in STEPANOVA [1994]. She studied the role of a strong form of the Urysohn property in metrization theory. A space X is a *Urysohn space* if for each $(x, y) \in X^2 - \Delta$ there is a continuous, real-valued function $f_{x,y}$ such that $f_{x,y}(x) \neq f_{x,y}(y)$. If the correspondence $(x, y) \to f_{x,y}$ is continuous, where the range space $C_u(X)$ is the set of all continuous, real-valued functions on X with the topology of uniform convergence, then we say that X *has a continuous separating family*. Clearly any metric space (X, d) has a continuous separating family: define $f_{x,y}(z) = d(x, z)$. Stepanova proved:

5.13. THEOREM. *If X is a paracompact p-space, then X is metrizable if and only if X has a continuous separating family.*

The role of continuous separating families in the theory of GO-spaces is not yet clear. Using a stationary set argument, one can show that

5.14. PROPOSITION (BENNETT and LUTZER [2002]). *Any GO-space and any monotonically normal space that has a continuous separating family must be hereditarily paracompact.*

For separable GO-spaces, we understand the role of continuous separating families:

5.15. PROPOSITION. *For any separable GO-space X, and more generally for any GO-space X such that X^2 contains a dense subspace that is either Lindelöf or has countable cellularity, the following are equivalent:*

(a) *X has a G_δ-diagonal;*

(b) *X has a weaker metric topology;*

(c) *X has a continuous separating family.*

Therefore, a separable LOTS with a continuous separating family must be metrizable.

Without separability, little is known. It is easy to see that a LOTS can have a continuous separating family and yet fail to be first-countable. Consider the lexicographic product $([0, \omega_1[\times \mathbb{Z}) \cup \{(\omega_1, 0)\}$. But even if one restricts attention to first-countable GO-spaces, Stepanova's Theorem 5.13 above has no analog for GO-spaces, as can be seen from

5.16. EXAMPLE. (a) The LOTS M^* described in (4.4) above has a σ-disjoint base, is hereditarily paracompact, and has a continuous separating family, but is not metrizable. Under CH or $\mathbf{b} = \omega_1$, there is a Lindelöf LOTS that is not metrizable and yet has a continuous separating family (see 5.12 above).
(b) In ZFC there is a Lindelöf, non-metrizable LOTS that has a σ-disjoint base, is hereditarily paracompact, and has a continuous separating family.

□ For the example mentioned in (b), let $B \subseteq [0, 1]$ be a Bernstein set, i.e., a set such that for each uncountable compact set K, $K \cap B \neq \emptyset$ and $K - B \neq \emptyset$. Such sets exist in ZFC: see OXTOBY [1971]. Let $= (B \times \mathbb{Z}) \cup (C \times \{0\})$, where $C = [0, 1] - B$, and topologize X using the open interval topology of the lexicographic order. It is easy to verify that X is Lindelöf, and has a σ-disjoint base. Hence X is hereditarily paracompact. To see that X has a continuous separating family, suppose $((x, i), (y, j)) \in X^2 - \Delta$. If $x \neq y$ then define $f_{(x,i),(y,j)}(z, k) = |x - z|$. If $x = y$ then $i \neq j$ and $x \in B$ and we let $f_{(x,i),(y,j)}$ be the characteristic function of the set $\{(y, j)\}$. In either case, $f_{(x,i),(y,j)}$ is continuous and separates (x, i) and (y, j). Finally suppose that $((x_n, i_n), (y_n, j_n))$ is a sequence in $X^2 - \Delta$ that converges to $((x_0, i_0), (y_0, j_0)) \in X^2 - \Delta$. A case by case analysis, depending upon which (if any) of the points x_0, y_0 belong to the set B, shows that $\langle f_{(x_n,i_n),(y_n,j_n)} \rangle$ converges uniformly to $f_{(x_0,i_0),(y_0,j_0)}$. Thus, X has a continuous separating family. □

5.17. REMARK. Note that the spaces in 5.16 have uncountable cellularity. It is natural to ask whether separability in 5.15 could be replaced with countable cellularity, i.e., whether a LOTS must be metrizable if it has countable cellularity and has a continuous separating family. Gruenhage has shown that the answer is consistently "No" by showing that if there is a Souslin space, then there is a Souslin space with a continuous separating family. A proof will appear in BENNETT, LUTZER and RUDIN [200?].

A natural question is whether the existence of a continuous separating family in a first-countable GO-space yields special kinds of bases, e.g., a σ-disjoint base or a point-countable base. The Big Bush described in 4.5 provides the necessary counterexample (BENNETT and LUTZER [2002]).

5.18. EXAMPLE. The Big Bush has a continuous separating family and a point-countable base, but does not have a σ-disjoint base. An extension of the Big Bush described in BENNETT and LUTZER [1996b] is a first-countable LOTS that has a continuous separating family and does not have a point-countable base.

5.19. REMARK. Note that in a metric space (X, d), the continuous separating family given by $f_{x,y}(z) = d(x, z)$ really depends only on the parameter x. In a recent paper HALBEISEN and HUNGERBUHLER [200?] proved that a topological space X has a continuous separating family that depends on only one parameter if and only if the space X has a weaker metric topology, and they describe a paracompact space that has a continuous separating family but does not have a one-parameter continuous separating family. In the light of their characterization, the LOTS M^* in (4.4) is an easier example of a paracompact space that has a continuous separating family but does not admit a one-parameter continuous separating family.

The space M^* used above, is a LOTS built on the Michael line. If, instead, one begins with the Sorgenfrey line S, then one obtains the lexicographically ordered LOTS $S^* = \mathbb{R} \times \{n \in \mathbb{Z} : n \leq 0\}$ that is often a useful counterexample in GO-space theory. Whether or not S^* has a continuous separating family may be axiom-sensitive. We have:

5.20. PROPOSITION (BENNETT and LUTZER [2002]). *If there is an uncountable subspace T of the Sorgenfrey line S such that T^2 is a Lindelöf space, then S^* does not have a continuous separating family.*

When does the Sorgenfrey line have a subset T with the properties described in 5.20? MICHAEL [1971] constructed such a subset assuming CH, and BURKE and MOORE [1998] point out that such a T can exist in some models of MA plus not CH, but cannot exist given OCA or PFA. That leads to:

5.21. QUESTION. In ZFC, does S^* have a continuous separating family?

According to BENNETT and LUTZER [2002], a LOTS with a σ-closed-discrete dense subset and a continuous separating family must be metrizable, and the existence of a Souslin line yields a non-metrizable perfect LOTS with a continuous separating family. That raises a question that belongs in the Maurice-Heath-Nyikos family:

5.22. QUESTION. In ZFC, is there a non-metrizable perfect LOTS with a continuous separating family?

We emphasize that 5.22 is a question about LOTS and not a question about GO-spaces, as can be seen from the fact the Sorgenfrey line is a non-metrizable perfect GO-space that is separable and has a continuous separating family.

5.23. QUESTION. In ZFC, is there an example of a GO-space X that has a continuous separating family, but whose LOTS extension X^* does not? (The proof of 5.20 given in BENNETT and LUTZER [2002] shows that the answer is consistently negative.)

6. Dugundji extension theory

For any space X, let $C(X)$ [resp. $C^*(X)$] denote the vector space of continuous [resp. continuous and bounded] real-valued functions on X. As noted in VAN DOUWEN, LUTZER and PRZYMUSINSKI [1977], for any closed subset A of a normal space X, there is a linear function $\Phi : C(A) \to C(X)$ such that $\Phi(f)$ extends f for each $f \in C(A)$. [An analogous assertion holds for bounded functions.] Such a function Φ is called a *linear extender*.

In metric spaces, one can obtain linear extenders that are very well-behaved. DUGUNDJI [1951] proved:

6.1. THEOREM. *If A is a closed subset of a metric space X, then there is a linear extender $\Phi : C(A) \to C(X)$ such that the range of $\Phi(f)$ is contained in the convex hull of the range of f for each $f \in C(A)$.*

Later, BORGES [1966] extended this result to the much larger class of stratifiable spaces. Borrowing terminology from VAN DOUWEN [1975], we will say that the extender in 6.1 is a *ch-extender*. A weaker kind of extender is one for which the range of $\Phi(f)$ is always contained in the *closed* convex hull of the range of f, and such an extender is called a *cch-extender*.

Normal spaces, or even compact Hausdorff spaces, do not always admit cch-extenders (ARENS [1952], MICHAEL [1953]). However GO-spaces do, at least for bounded functions (HEATH and LUTZER [1974]):

6.2. PROPOSITION. *Suppose A is a closed subspace of a generalized ordered space X. Then there is a linear cch-extender from $C^*(A)$ to $C^*(X)$.*

If we consider unbounded functions, then 6.2 can fail.

6.3. EXAMPLE (HEATH and LUTZER [1974]). Let X be the Michael line and let A be the closed subset consisting of all rational numbers. Then there is no linear cch-extender from $C(A)$ to $C(X)$.

In the light of 6.3, HEATH and LUTZER [1974] asked:

6.4. QUESTION. *Suppose A is a closed subset of a perfect LOTS. Is there a linear cch-extender from $C(A)$ to $C(X)$?*

A few years later, VAN DOUWEN [1975] constructed a zero-dimensional separable GO-space having a closed subset that is not a retract and asked whether that space might be a counterexample to 6.4. Recently, GRUENHAGE, HATTORI and OHTA [1998] have proved that van Douwen's space answers 6.4 negatively. The next proposition is a special case of their Theorem 1. It settles questions of HEATH and LUTZER [1974] and of VAN DOUWEN [1975], and ties together several other results in HEATH, LUTZER and ZENOR [1975].

6.5. THEOREM. *Suppose X is a perfect GO-space and that the cardinality of X is non-measurable, and let A be a closed subspace of X. Then the following are equivalent:*

(a) *there is a continuous linear extender from $C(A)$ to $C(X)$ where both function spaces carry the topology of pointwise convergence or both carry the compact-open topology;*
(b) *there is a continuous linear extender from $C^*(A)$ to $C^*(X)$ where both function spaces carry the topology of pointwise convergence or both carry the compact-open topology;*
(c) *there is a linear cch-extender from $C(A)$ to $C(X)$;*
(d) *there is a linear ch-extender from $C(A)$ to $C(X)$;*
(e) *for each space Y, $A \times Y$ is C^*-embedded in $X \times Y$.*

If, in addition, X is zero-dimensional, then each of the above is equivalent to

(f) *A is a retract of X.*

Because van Douwen's space is separable and zero-dimensional and has a closed subspace that is not a retract, 6.5 shows that van Douwen's space is a counterexample to 6.2. In addition, GRUENHAGE, HATTORI and OHTA [1998] gave an easier example, namely:

6.6. EXAMPLE. Let X be the lexicographically ordered set $(\mathbb{Q} \times \mathbb{Z}) \cup (\mathbb{P} \times \{-1, 1\})$. With the open interval topology of that order, X is a separable (and hence perfect) zero-dimensional LOTS and its closed subspace $A = \mathbb{P} \times \{-1, 1\}$ is not a retract of X. Hence there is no linear cch-extender from $C(A)$ to $C(X)$.

The space of 6.6 gives another answer to a question raised by BORGES [1966]. It is a perfectly paracompact space that does not satisfy the Dugundji extension theorem. (VAN DOUWEN [1975] gave an earlier answer using a different example.)

Finally, GRUENHAGE, HATTORI and OHTA [1998] sharpened the results of HEATH and LUTZER [1974] for perfect GO-spaces by proving:

6.7. PROPOSITION. *Let A be a closed G_δ-subset of a GO-space X. Then there is a linear ch-extender from $C^*(A)$ to $C^*(X)$. In particular, if A is a closed subset of a perfect GO-space X, then there is a linear ch-extender from $C^*(A)$ to $C^*(X)$.*

7. Rudin's solution of Nikiel's problem, with applications to Hahn-Mazurkiewicz theory

Several authors noticed that compact monotonically normal spaces had remarkable parallels to ordered spaces, and Nikiel asked whether every compact monotonically normal space must be a continuous image of a compact LOTS. Mary Ellen Rudin published three papers that contain the most important and complicated ordered space results in recent years (RUDIN [1998a] [1998b] [200?]):

7.1. THEOREM. *Any compact monotonically normal space is the continuous image of a compact LOTS.*

Rudin's theorem has important consequences for the Hahn-Mazurkiewicz problem that asks for characterizations of topological spaces that are continuous images of some connected compact LOTS. (Compact connected LOTS are often called *arcs* . It is easy to prove that the unit interval is the unique separable arc, and consequently modern Hahn-Mazurkiewicz theory focuses on non-separable arcs.) The most basic result in this area is the original Hahn-Mazurkiewicz theorem that characterized continuous images of *separable arcs* as follows:

7.2. THEOREM. *A topological space X is a continuous image of* $[0, 1]$ *if and only if X is compact, connected, locally connected, and metrizable.*

The systematic study of images of non-separable arcs began with the work of Mardešić in the 1960s. Many theorems in this area begin with the hypothesis that a space X is the continuous image of some compact LOTS and add hypothesis that force X to be the continuous image of some connected, compact LOTS. Rudin's theorem puts such results into a more natural topological context. For example, combining Rudin's theorem with a result of Treybig and Nikiel gives:

7.3. THEOREM. *A space X is the continuous image of a compact connected LOTS if and only if X is compact, connected, locally connected, and monotonically normal.*

For further surveys of the Hahn-Mazurkiewicz problem, see the papers by TREYBIG and WARD [1981], MAYER and OVERSTEEGEN [1992], and NIKIEL, TUNCALI and TYMCHATYN [1993].

8. Applications to Banach spaces

An important problem in Banach space theory asks which Banach spaces have equivalent norms with special properties. For example, a norm $||\cdot||$ on a Banach space is *convex* if $||\frac{x+y}{2}|| < 1$ whenever $||x|| = 1 = ||y||$. A norm is *locally uniformly convex* (LUC) if whenever $||x|| = 1 = ||y_n||$ and $||x+y_n|| \to 2$, then $||x-y_n|| \to 0$, and is called a *Kadec norm* if the weak topology and the norm topology coincide on the norm's unit sphere. An often-studied type of question in Banach space theory is: does a given Banach space have an equivalent norm that is locally uniformly convex (LUC) or is a Kadec norm? It is known that the property of having an equivalent LUC norm is stronger than the property of having an equivalent Kadec norm, and that some Banach spaces have equivalent norms that are LUC or Kadec, while others do not.

Function spaces $C(X)$, where X is a compact Hausdorff space and $C(X)$ carries the sup norm, provide a wide variety of Banach spaces. When X is a compact LOTS, it is possible to study $C(X)$ in great detail, as recent results in JAYNE, NAMIOKA and ROGERS [1995] and HAYDON, JAYNE, NAMIOKA and ROGERS [2000] show. In this section, we present a sample of the results from the second of those papers.

8.1. THEOREM. *Let K be any compact LOTS. Then $C(K)$ has an equivalent Kadec norm and that norm is lower semi-continuous for the pointwise convergence topology on $C(K)$.*

Furthermore, the norm and pointwise convergence topologies coincide on the unit sphere of the Kadec norm.

Haydon, Jayne, Namioka, and Rogers then asked for which compact LOTS K would $C(K)$ have an equivalent LUC norm, a property that (as mentioned above) is stronger than having an equivalent Kadec norm. That problem was solved using the ideas of a *dyadic interval system* and a *decreasing interval function* on K. Let J be the collection of all indices $(i_1, \cdots, i_n)$ where $n \geq 1$ and $i_j \in \{0, 1\}$, together with the empty set. A *dyadic interval system* on K is a function from J to the family of all non-empty closed intervals in K where $I(0)$ and $I(1)$ are disjoint closed subintervals of the interval $I(\emptyset)$ and where $I(i_1, \cdots, i_n, 0)$ and $I(i_1, \cdots, i_n, 1)$ are disjoint closed subintervals of $I(i_1, \cdots, i_n)$. By a *decreasing interval function* on K we mean a real valued function ρ defined for each closed non-empty interval in K and having the property that if $J \subseteq I$ are closed intervals, then $\rho(I) \leq \rho(J)$.

8.2. THEOREM. *The following properties of a compact LOTS K are equivalent:*

(a) *$C(K)$ has an equivalent LUC norm that is lower semi-continuous with respect to the pointwise convergence topology on $C(K)$;*

(b) *there is an equivalent strictly convex norm on $C(K)$;*

(c) *there is a bounded decreasing interval function on K that is not constant on any dyadic interval system of K.*

8.3. EXAMPLE. (a) Let α be an ordinal and let K be the lexicographically ordered product $\{0, 1\}^\alpha$. Then $C(K)$ has an equivalent LUC norm if and only if α is countable. The same is true if we consider the lexicographic product $L = [0, 1]^\alpha$.

(b) The lexicographic product $M = [0, 1]^{\omega_1}$ has an equivalent Kadec norm, but not an equivalent LUC norm because it fails to satisfy (8.2-c). This example is considerably more simple than an earlier tree-based construction given by Haydon.

c) If there is a Souslin space, then there is a compact, connected Souslin space N, and $C(N)$ does not have an equivalent LUC norm.

In HAYDON, JAYNE, NAMIOKA and ROGERS [2000] the authors show that any connected, compact LOTS L is the continuous image of some lexicographic product $[0, 1]^\gamma$ under a continuous increasing mapping f (i.e., $x \leq y$ in $[0, 1]^\gamma$ implies $f(x) \leq f(y)$ in L), where γ is an appropriately chosen ordinal. Then they prove:

8.4. THEOREM. *Suppose that the compact LOTS K is the continuous image of a closed subset of the lexicographic product $[0, 1]^\gamma$, where γ is a countable ordinal. Then $C(K)$ has an equivalent LUC norm.*

It would be interesting to characterize those LOTS that satisfy the hypotheses of the previous theorem.

9. Products of GO-spaces

Between 1940 and 1970, simple GO-spaces proved their utility as counterexamples in product theory. Subspaces of ordinals, the Sorgenfrey line, and the Michael line became standard examples in the product theory of normality, the Lindelöf property, and paracompactness. MICHAEL [1971] showed that subspaces of the Sorgenfrey line and the Michael line can be finely tuned to generate a wide range of important examples.

That same period also saw the discovery of a positive theory for products of certain GO-spaces. Let S be the Sorgenfrey line. HEATH and MICHAEL [1971] showed that S^ω is a perfect space (i.e., closed subsets are G_δ-sets) and LUTZER [1972a] showed that S^ω is hereditarily subparacompact. VAN DOUWEN and PFEFFER [1979] showed that S^n cannot be homeomorphic to T^m for any $m, n \geq 1$, where T is the subspace of S consisting of all irrational numbers, and BURKE and LUTZER [1987] proved that S^n is homeomorphic to S^m if and only if $n = m$. That result is sharpened by BURKE and MOORE [1998] who showed that if X is an uncountable subspace of S, then no power of X can be embedded in a lower power of S. They also characterized subspaces of S that are homeomorphic to S as being those uncountable $X \subseteq S$ that are dense-in-themselves and are both an F_σ-subset and a G_δ-subset of S.

ALSTER [1975] considered the broader class of GO-spaces with G_δ-diagonals and proved, for example, that (hereditary) collectionwise normality is equivalent to (hereditary) paracompactness in finite products of GO-spaces having G_δ-diagonals. He also proved that the Continuum Hypothesis is equivalent to the assertion that $X_1 \times X_2$ is hereditarily subparacompact whenever X_1 and X_2 are Lindelöf GO-spaces with G_δ-diagonals.

More recent investigations have focussed on products of ordinal spaces and their subspaces. (By an *ordinal space* we mean a space $[0, \alpha)$ (where α is an ordinal number) with its usual order topology. Throughout this section, A and B will denote subspaces of an ordinal space.) CONOVER [1972] gave necessary and sufficient conditions for normality of the product of two ordinal spaces. Later KEMOTO and YAJIMA [1992] extended earlier work of SCOTT [1975], showing:

9.1. THEOREM. *Let A and B be subsets of ordinal spaces. Then $A \times B$ is normal if and only if $A \times B$ is orthocompact.*

It is interesting to note that Theorem 9.1 does *not* hold for subspaces of $A \times B$ even when $A = B = [0, \omega_1)$, KEMOTO [1997]. In SCOTT [1977], Scott extended the "normality = orthocompactness" theorem in a different direction, proving it for any finite product of locally compact LOTS.

Some interesting equivalences among normality-related properties of $A \times B$ have been found. Results of KEMOTO, OHTA and TAMANO [1992] and KEMOTO, NOGURA, SMITH and YAJIMA [1996] have been generalized by FLEISSNER [200?a] who proved:

9.2. THEOREM. *Let X be a subspace of the product of finitely many ordinals. The following are equivalent:*

(a) *X is normal;*

(b) *X is normal and strongly zero-dimensional;*

(c) *X is collectionwise normal;*

(d) *every open cover $\mathcal{U}$ of X has an open refinement $\{V(U) : U \in \mathcal{U}\}$ that covers X and has the property that $cl_X(V(U)) \subseteq U$ for each $U \in \mathcal{U}$.*

The hypothesis of strong zero-dimensionality in 9.2 (b) not automatic. FLEISSNER, KEMOTO and TERASAWA [200?] prove that $\mathbf{c} = 2^\omega$ is the least cardinal such that $X =$

$[0,\omega] \times [0,\mathbf{c})$ contains a subspace that is not strongly zero-dimensional; in fact X contains a strongly n-dimensional subspace (i.e., a subspace with covering dimension n) for each finite n.

Countable paracompactness is a covering property of every GO-space. In products $A \times B$ of subspaces of ordinal spaces, countable paracompactness is known to be equivalent to the property that for every locally finite closed collection $\mathcal{F}$ in $A \times B$, there is a locally finite open collection $\{U(F) : F \in \mathcal{F}\}$ with $F \subseteq U(F)$ for each $F \in \mathcal{F}$. (See KEMOTO, OHTA and TAMANO [1992].) If $A \times B$ is normal, then it is countably paracompact, but $A \times B$ can be countably paracompact without being normal (the classic example being $[0,\omega_1) \times [0,\omega_1]$). It is an open question whether every countably paracompact subspace of $[0,\omega_1)^2$ is normal. KEMOTO, SMITH and SZEPTYCKI [2000] show that the answer is consistently "yes" but the question remains open in ZFC. One of the few covering properties shared by all products $A \times B$ is hereditary countable metacompactness (KEMOTO and SMITH [1996], FLEISSNER [200?b]).

Some of the equivalences among covering properties in GO-spaces still hold in products $A \times B$ of subspaces of ordinal spaces. For example, combining results of KEMOTO and YAJIMA [1992] with work of FLEISSNER and STANLEY [2001] yields:

9.3. THEOREM. *Let $X = A \times B$ where A and B are subspaces of ordinal spaces. Then the following are equivalent:*

(a) *X is paracompact;*

(b) *X is metacompact;*

(c) *X is subparacompact;*

(d) *X is a D-space, i.e., whenever we have open sets U_x satisfying $x \in U_x$ for each $x \in X$, there is a closed discrete set $D \subseteq X$ such that $\{U_x : x \in D\}$ covers X;*

(e) *no closed subspace of X is homeomorphic to a stationary subspace of an uncountable regular cardinal.*

By way of contrast, metacompactness, paracompactness, and subparacompactness are not equivalent for subspaces of $A \times B$. For example, while metacompact subspaces of $[0,\omega_1)^2$ must be paracompact, there are metacompact subspaces of $[0,\omega_2)^2$ that are not even subparacompact (KEMOTO, TAMANO and YAJIMA [2000]).

In FLEISSNER and STANLEY [2001] Stanley extended earlier work in KEMOTO and YAJIMA [1992], proving:

9.4. THEOREM. *Let X be any subspace of a product of finitely many ordinal spaces. Then the following are equivalent:*

(a) *X is metacompact;*

(b) *X is metaLindelöf;*

(c) *X is a D-space* (*see 9.3 d*)*;*

(d) *no closed subspace of X is homeomorphic to a stationary subset of a regular uncountable cardinal.*

There is a marked difference between finite and countable products of ordinal spaces. In a recent paper, KEMOTO and SMITH [1997] have shown that the product space $([0,\omega_1))^\omega$ has a subspace that is not countably metacompact, even though every finite power of $[0,\omega_1)$ is hereditarily countably metacompact.

References

ALSTER, K.

[1975] Subparacompactness in Cartesian products of generalized ordered spaces, *Fundamenta Math.* **87**, 7-28.

ARENS, R.

[1952] Extension of functions on fully normal spaces, *Pacific J. Math.* **2**, 11–22.

ARHANGEL'SKII, A. AND A. BELLA

[1992] Few observations on topological spaces with small diagonal, *Zbornik radova Filozofkog fakulteta u Nisu* **6** (2), 211–213.

ARHANGEL'SKII, A., JUST, W., REZNICHENKO, E. AND P. SZEPTYCKI

[2000] Sharp bases and weakly uniform bases versus point-countable bases, *Topology Appl.* **100**, 39–46.

ARTICO, G., U. MARCONI, J. PELANT, L. ROTTER AND M. TKACHENKO

[200?] Selections and suborderability, preprint.

AULL, C.

[1974] Quasi-developments and $\delta\theta$-bases, *J. London Math. Soc. (2)*, **9**, 192–204.

BALOGH, Z., S. DAVIS, W. JUST, S. SHELAH AND P. SZEPTYCKI

[200?] Strongly almost disjoint sets and weakly uniform bases, to appear.

BALOGH, Z. AND M. E. RUDIN

[1992] Monotone normality, *Topology Appl.* **47**, 115–127.

BENNETT, H.

[1968] *On quasi-developable spaces*, Ph.D. dissertation, Arizona State University.

[1971] Point-countability in ordered spaces, *Proc. Amer. Math. Soc.* **28**, 598–606.

BENNETT, H. AND E. BERNEY

[1977] Spaces with σ-minimal bases, *Topology Proc.* **2**, 1–10.

BENNETT, H., R. HEATH AND D. LUTZER

[2000] GO-spaces with σ-closed-discrete dense subspaces, *Proc. Amer. Math. Soc.* **129**, 931–939.

BENNETT, H., M. HOSOBUCHI AND D. LUTZER

[1999] A note on perfect generalized ordered spaces, *Rocky Mountain J. Math.* **9**, 1195–1207.

[2000] Weakly perfect generalized ordered spaces, *Houston J. Math.* **26**, 609–627.

BENNETT, H. AND D. LUTZER
[1977] Ordered spaces with σ-minimal bases, *Topology Proc.* **2**, 371–382.
[1984] Generalized ordered spaces with capacities, *Pacific J. Math.* **122**, 11–19.
[1990] Problems in perfect ordered spaces, In *Open Problems in Topology*, ed. by J. van Mill and G.M. Reed, North Holland, Amsterdam, pp. 233–237.
[1996a] Point-countability in generalized ordered spaces, *Topology Appl.* **71**, 149–165.
[1996b] A note on property III in generalized ordered spaces, *Topology Proc.* **21**, 15–24.
[1997a] Diagonal conditions in ordered spaces, *Fundamenta Math.* **153**, 99–123.
[1997b] Off diagonal metrization theorems, *Topology Proc.* **22**, 37–58.
[1998a] A metric space of A.H. Stone and an example concerning σ-minimal bases, *Proc. Amer. Math. Soc.* **126**, 2191–2196.
[1998b] Ordered spaces with special bases, *Fundamenta Math.* **158**, 289–299.
[2002] Continuous separating families in ordered spaces and strong base conditions, *Topology Appl.* **119**, 305–314.
[200?a] Metrizably fibered generalized ordered spaces, to appear.
[200?b] Spaces with $< \omega$-weakly uniform bases, to appear.

BENNETT, H., D. LUTZER AND S. PURISCH
[1999] On dense subspaces of generalized ordered spaces, *Topology Appl.* **93**, 191–205.

BENNETT, H., D. LUTZER AND M. E. RUDIN
[200?] Trees, lines, and branch spaces, to appear.

BORGES, C.
[1966] On stratifiable spaces, *Pacific J. Math.* **17**, 1–25.

BURKE, D. AND D. LUTZER
[1987] On powers of certain lines, *Topology Appl.* **26**, 251–261.

BURKE, D. AND J. MOORE
[1998] Subspaces of the Sorgenfrey line, *Topology Appl.* **90**, 57–68.

CONOVER, R.
[1972] Normality and products of linearly ordered spaces, *Gen. Top. Appl.* **2**, 215–25.

VAN DALEN, J. AND E. WATTEL
[1973] A topological characterization of ordered spaces, *Gen. Top. Appl.* **3**, 347–354.

VAN DOUWEN, E.
[1975] *Simultaneous Extensions of Continuous Functions*, Ph.D. Thesis, Vrije Universiteit, Amsterdam.

VAN DOUWEN, E. AND W. PFEFFER
[1979] Some properties of the Sorgenfrey line and related spaces, *Pacific J. Math.* **81**, 371–377.

VAN DOUWEN, E., D. LUTZER AND T. PRZYMUSINSKI
[1977] Some extensions of the Tietze-Urysohn theorem, *Amer. Math. Monthly* **84**, 435–441.

DUGUNDJI, J.
[1951] An extension of Tietze's theorem, *Pacific J. Math.* **1**, 353–367.

ENGELKING, R.
[1989] *General Topology*, Heldermann, Berlin.

ENGELKING, R. AND D. LUTZER
[1976] Paracompactness in ordered spaces, *Fundamenta Math.* **94**, 49–58.

FABER, M.J.
[1974] *Metrizability in generalized ordered spaces*, Math. Centre Tracts no. 53, Amsterdam.

FLEISSNER, W.

[200?a] Normal subspaces of products of finitely many ordinals, to appear.

[200?b] Metacompact subspaces of products of ordinals, *Proc. Amer. Math. Soc.*, to appear.

FLEISSNER, W., N. KEMOTO AND J. TERASAWA

[200?] Strong zero dimensionality of products of ordinals, to appear.

FLEISSNER, W. AND A. STANLEY

[2001] D-spaces, *Topology Appl.* **114**, 261–271.

FUJII, S. AND T. NOGURA

[1999] Characterizations of compact ordinal spaces via continuous selections, *Topology Appl.* **91**, 65–69.

GRUENHAGE, G.

[1984] Covering properties of $X^2 - \Delta$ and compact subsets of Σ-products, *Topology Appl.* **28**, 287–304.

[1992] A note on the point-countable base question, *Topology Appl.* **44**, 157–162.

[200?] Spaces having a small diagonal, to appear.

GRUENHAGE, G., Y. HATTORI AND H. OHTA

[1998] Dugundji extenders and retracts on generalized ordered spaces, *Fundamenta Math.* **158**, 147–164.

GRUENHAGE, G. AND D. LUTZER

[2000] Baire and Volterra spaces, *Proc. Amer. Math. Soc.* **128**, 3115–3124.

GRUENHAGE, G., T. NOGURA AND S. PURISCH

[1991] Normality of $X \times \omega_1$, *Topology Appl.* **39**, 263–275.

GRUENHAGE, G. AND J. PELANT

[1988] Analytic spaces and paracompactness of $X^2 - \Delta$, *Topology Appl.* **28**, 11–18.

HALBEISEN, L. AND N. HUNGERBUHLER

[200?] On continuously Urysohn and strongly separating spaces, *Topology Appl.*, to appear.

HAYDON, R., J. JAYNE, I. NAMIOKA AND C. ROGERS

[2000] Continuous functions on totally ordered spaces that are compact in their order topologies, *J. Funct. Anal.* **178**, 23–63.

HEATH, R.

[1989] On a question of Ljubiša Kočinac, *Publ. Inst. Math. (Beograd) (N.S.)* **46**, 193–195.

HEATH, R. AND W. LINDGREN

[1976] Weakly uniform bases, *Houston J. Math.* **2**, 85–90.

HEATH, R. AND D. LUTZER

[1974] Dugundji extension theorems for linearly ordered spaces, *Pacific J. Math.* **55**, 419–425.

HEATH, R., D. LUTZER AND P. ZENOR

[1973] Monotonically normal spaces, *Trans. Amer. Math. Soc.* **178**, 481–493.

[1975] On continuous extenders, In *Studies in Topology*, ed. by N. Stravrakas and K. Allen, Academic Press, New York, pp. 203–214.

HEATH, R. AND E. MICHAEL

[1971] A property of the Sorgenfrey line, *Compositio Math.* **23**, 185–188.

HERRLICH, H.

[1965] Ordnungsfähigkeit total-discontinuierlicher Räume, *Math. Ann.* **159**, 77–80.

HUŠEK, M.

[1976] Continuous mappings on subspaces of products, in *Sympos. Math.* **17**, Academic Press, London, pp. 25–41.

[1977] Topological spaces without κ-accessible diagonals, *Comment. Math. Univ. Carolin.* **18**, 777–788.

JAYNE, J., I. NAMIOKA AND C. ROGERS

[1995] Continuous functions on compact totally ordered spaces, *J. Funct. Anal.* **134**, 261–280.

JUHÁSZ, I. AND Z. SZENTMIKLÓSSY

[1992] Convergent free sequences in compact spaces, *Proc. Amer. Math. Soc.* **116**, 1153–1160.

KEMOTO, N.

[1993] Normality in products of GO-spaces and cardinals, *Topology Proc.* **18**, 133–142.

[1997] Orthocompact subspaces in products of two ordinals, *Topology Proc.* **22**, 247–263.

KEMOTO, N., T. NOGURA, K. SMITH AND Y. YAJIMA

[1996] Normal subspaces in products of two ordinals, *Fundamenta Math.* **151**, 279–297.

KEMOTO, N., H. OHTA AND K. TAMANO

[1992] Products of spaces of ordinal numbers, *Topology Appl.* **45**, 245–260.

KEMOTO, N. AND K. SMITH

[1996] The product of two ordinals is hereditarily countably metacompact, *Topology Appl.* **74**, 91–96.

[1997] Hereditary countable metacompactness in finite and infinite product spaces of ordinals, *Topology Appl.* **77**, 57–63.

KEMOTO, N., K. SMITH AND P. SZEPTYCKI

[2000] Countable paracompactness versus normality in subspaces of ω_1^2, *Topology Appl.* **104**, 141–154.

KEMOTO, N., K. TAMANO AND Y. YAJIMA

[2000] Generalized paracompactness of subspaces in products of two ordinals, *Topology Appl.* **104**, 155–168.

KEMOTO, N, AND Y. YAJIMA

[1992] Orthocompactness in products, *Tsukuba J. Math.* **16**, 407–422.

KOČINAC, L.

[1983] An example of a new class of spaces, *Mat. Vestnik* **35**, 145–150.

[1986] Some generalizations of perfect normality, *Facta. Univ. Ser. Math. Infor.* **1**, 57–63.

KOMBAROV, A.

[1989] On rectangular covers of $X^2 - \Delta$, *Comment. Math. Univ. Carolinae* **30**, 81–83.

LUTZER, D.

[1971] *On generalized ordered spaces*, Dissertationes Math **89**.

[1972a] Another property of the Sorgenfrey line, *Compositio Math.* **24**, 359–363.

[1972b] On quasi-uniform bases, In *Proc. Univ. of Oklahoma Topology Conference 1972*, ed. by D. Kay, J. Green, L. Rubin, and L. Su, University of Oklahoma, Norman, OK.

[1980] Ordered topological spaces, In *Surveys in General Topology*, ed. by G.M. Reed, Academic Press, New York, pp.247–295.

MAYER, J. AND L. OVERSTEEGEN

[1992] Continuum Theory, In *Recent Progress in General Topology* ed. by Hušek, M. and J. van Mill, North Holland, Amsterdam, pp. 247–296.

MICHAEL, E.
[1951] Topologies on spaces of subsets, *Trans. Amer. Math. Soc.* **71**, 152–182.
[1953] Some extension theorems for continuous functions, *Pacific J. Math.* **3**, 789–806.
[1971] Paracompactness and the Lindelöf property in finite and countable Cartesian products, *Compositio Math.* **23**, 199–214.

VAN MILL, J. AND E. WATTEL
[1984] Orderability from selections: Another solution of the orderability problem, *Fundamenta Math.* **121**, 219–229.

MILLER, A.
[1984] Special subsets of the real line, In *Handbook of Set Theoretic Topology* ed. by K. Kunen and J. Vaughan, Elsevier, New York, pp. 203–223.

MIWA, T., AND N. KEMOTO
[1993] Linearly ordered extensions of GO–spaces, *Topology Appl.* **54**, 133–140.

NIKIEL, J., H. TUNCALI AND E. TYMCHATYN
[1993] Continuous images of arcs and inverse limit methods, *Memoires Amer. Math. Soc.* **104**, no. 198.

NYIKOS, P.
[1976] A survey of zero-dimensional spaces, In *Topology: Proceedings of the Memphis State University Conference*, ed. by S. Franklin and B. Thomas, Marcel Dekker, New York, pp. 87–114.

OKUYAMA, A.
[1964] On metrizability of M-spaces, *Proc. Japan Acad.* **40**, 176–179.

OXTOBY, J.
[1971] *Measure and Category*, Springer-Verlag, New York.

PAVLOV, O.
[200?] There is a perfect preimage of ω_1 with a small diagonal, preprint.

PONOMAREV, V.
[1967] Metrizability of a finally compact p-space with a point-countable base, *Soviet Math. Doklady* **8**, 765–768.

PURISCH, S.
[1977] The orderability and suborderability of metrizable spaces *Trans. Amer. Math. Soc.* **226**, 59–76.
[1983] Orderability of non-archimedean spaces, *Topology Appl.* **16**, 273–277.
[1998] Orderability of topological spaces, Topology Atlas Invited Contributions, http://at.yorku.ca/z/a/a/b/14.htm

QIAO, Y-Q. AND F. TALL
[200?] Perfectly normal non-metrizable non-archimedean spaces are generalized Souslin lines, *Proc. Amer. Math. Soc.*, to appear.

REED, G. M.
[1971] Concerning normality, metrizabilty, and the Souslin property in subspaces of Moore spaces, *Gen. Top. Appl.* **1**, 223–245.

RUDIN, M. E.
[1998a] Compact, separable linearly ordered spaces, *Topology Appl.* **82**, 397–419.
[1998b] Zero-dimensionality and monotone normality, *Topology Appl.* **85**, 319–333.
[200?] Nikiel's conjecture, to appear.

SCOTT, B.

[1975] Toward a product theory for orthocompactness, in *Studies in Topology*, ed. by N. Stavrakas and K. Allen, Academic Press, New York, pp. 517–537.

[1977] Orthocompactness and normality in finite products of locally compact LOTS, in *Set Theoretic Topology* ed. by G. Reed, Academic Press, New York, pp. 339–348.

SHI, W.

[1997] Perfect GO-spaces which have a perfect linearly ordered extension, *Topology Appl.* **81**, 23–33.

[1999a] A non-metrizable compact LOTS each subspace of which has a σ-minimal base, *Proc. Amer. Math. Soc.* **127**, 2783–2791.

[1999b] Extensions of perfect GO-spaces with σ-discrete dense sets, *Proc. Amer. Math. Soc.* **127**, 615–618.

SHI, W., T. MIWA AND Y. GAO

[1995] A perfect GO-space which cannot densely embed in any perfect orderable space, *Topology Appl.* **66**, 241–249.

[1996] Any perfect GO-space with the underlying LOTS satisfying local perfectness can embed in a perfect LOTS, *Topology Appl.* **74**, 17–24.

ŠNEIDER, V.

[1945] Continuous mappings of Souslin and Borel sets: metrization theorems, *Dokl. Akad. Nauk SSSR* **50**, 77–79.

STONE, A.

[1963] On σ-discreteness and Borel isomorphism, *Amer. J. Math.* **85**, 655–666.

STEPANOVA, E.

[1993] Extension of continuous functions and metrizability of paracompact p-spaces, *Mathematical Notes* **53**, 308–314.

[1994] On metrizability of paracompact p-spaces, *Moscow University Mathematics Bulletin* **49**, 41–43.

TKACHUK, V.

[1994] A glance at compact spaces which map “nicely” onto the metrizable ones, *Topology Proc.* **19**, 321–334.

TODORČEVIĆ, S.

[1984] Trees and linearly ordered sets, in *Handbook of Set Theoretic Topology*, ed. by K. Kunen and J. Vaughan, North-Holland, Amsterdam, pp. 235–293.

TREYBIG, L. AND L. WARD

[1981] The Hahn-Mazurkiewicz problem, in *Topology and Order Structures, Part 1*, ed. by H. Bennett and D. Lutzer, Mathematical Centre Tracts 142, Amsterdam, pp. 95–106.

VAN WOUWE, J.

[1979] *GO-spaces and generalizations of metrizability*, Mathematical Centre Tracts 104, Amsterdam.