Perfect Images of Generalized Ordered Spaces

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Abstract. We study the class of perfect images of generalized ordered (GO) spaces, which we denote by PIGO. Mary Ellen Rudin's celebrated result characterizing compact monotonically normal spaces as the continuous images of compact linearly ordered spaces implies that every space with a monotonically normal compactification is in PIGO. But PIGO is wider: every metrizable space is in the class, but not every metrizable space has a monotonically normal compactification. On the other hand, a locally compact space is in PIGO if and only if it has a monotonically normal compactification. We answer a question of Bennett and Lutzer that asked whether a (semi)stratifiable space with a monotonically normal compactification must be metrizable by showing that any semi-stratifiable member of PIGO is metrizable. This also shows that there are monotonically normal spaces which are not in PIGO. We investigate cardinal functions in PIGO, and in particular show that if K is a compact subset of a space X in PIGO, then the character of K in X equals the pseudo-character of K in X. We show that the product of two non-discrete spaces in PIGO is not in PIGO unless both are metrizable or neither one contains a countable set with a limit point. Finally, we look at the narrower class of perfect images of linearly ordered spaces, which we denote by PILOTS. Every metrizable space is in PILOTS, and if a space in PILOTS has a G_δ-diagonal, then it must be metrizable. Thus familiar GO-spaces such as the Sorgenfrey line and Michael line are in PIGO but not in PILOTS.

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1 Introduction

Mary Ellen Rudin proved in [32] that every compact monotonically normal space is a continuous image of some compact linearly ordered topological space. An easy consequence of her theorem is that any space with a monotonically normal compactification is the perfect image of a generalized ordered (GO) space, and this began to be used as a tool by several researchers (see [9],[14],and [26]) to investigate the class MNC of spaces that have monotonically normal compactifications. This suggests that it might be useful to study the wider class of perfect images of GO-spaces, which we denote by PIGO, for its own sake. PIGO is a strictly larger class of spaces than MNC; for example, PIGO contains the class of metrizable spaces, while MNC does not. On the other hand, we will also see that PIGO is a much narrower class than the class of monotonically normal spaces in general.

One of the main results of this paper answers the main question left open in Bennett and Lutzer's study [9] of semi-stratifiable spaces with monotonically normal compactifications. They had asked whether every (semi)stratifiable space with a monotonically normal compactification must be metrizable. We give a positive answer to this question by showing that every semi-stratifiable space in PIGO is metrizable. Note that this shows that any nonmetrizable stratifiable space is an example of a monotonically normal space which is not in PIGO.

Our paper is organized as follows. Section 2 contains definitions and background results. In Section 3 we show how the class PIGO is related to other familiar classes. In Section 4 we study various cardinal functions in the class PIGO; for example, we show that character and pseudocharacter agree for spaces in PIGO (but not more generally for all monotonically normal spaces). Section 5 studies paracompactness and metrization of spaces in PIGO; in particular we prove there the aforementioned result about semi-stratifiable spaces. In Section 6 we show that the product of two nondiscrete spaces in PIGO is not in PIGO unless both are metrizable or neither contains a countable set with a limit point. Section 7 introduces a special subclass of PIGO that we call PILOTS, namely spaces that are perfect images of linearly ordered topological spaces. It follows from classical results of Herrlich and Morita that every metrizable space is in PILOTS. We show that if a space in PILOTS has a G_δ-diagonal, then it must be metrizable. It follows that familiar GO-spaces like the Sorgenfrey line and the Michael line are examples of spaces that are in PIGO but not in PILOTS. We also prove that any locally compact space in PIGO (equivalently, every locally compact space with a monotonically normal compactification) must be in PILOTS. Section 8 of our paper lists some open questions.

2 Definitions and background results

A *generalized ordered space (GO-space)* is a triple $(X, \sigma, \leq)$, where $\leq$ is a linear order on X and σ is a Hausdorff topology on X having a base of order-convex sets. If σ coincides with the open-interval topology induced by $\leq$, then X is a *linearly ordered topological space (LOTS)*.

A space X is *monotonically normal* if for each pair (H, K) of disjoint closed sets, one can assign an open set $U(H, K)$ satisfying:

(a) $H \subset U(H, K) \subset \overline{U(H, K)} \subset X \setminus K$;

(b) If $H \subset H'$ and $K \supset K'$, then $U(H, K) \subset U(H', K')$.

Metrizable spaces, and more generally stratifiable spaces, are monotonically normal, as are GO-spaces. Monotone normality is a hereditary property which is preserved under closed continuous images, and it implies collectionwise-normality [22].

The following is an equivalent condition for monotone normality of a space X:

(c) For each pair (p, U) where $p \in U \subset X$ and U is open, one can assign an open set $V(p, U)$ such that $p \in V(p, U) \subset U$, $V(p, U') \subset V(p, U)$ whenever $U' \subset U$, and $V(p, X - \{q\}) \cap V(q, X - \{p\}) = \emptyset$ if $p \neq q$.

The space X is said to be *acyclically monotonically normal* if it satisfies (c) above with the added condition that whenever $x_0, x_1, ..., x_n$ is a finite sequence of distinct points with $n \geq 2$, then $\bigcap_{i=0}^{n} V(x_i, X - \{x_{i+1}\}) = \emptyset$ (taking $x_{n+1} = x_0$). Moody and Roscoe [31] showed that the familiar examples of monotonically normal spaces (stratifiable spaces, elastic spaces, GO-spaces) are all acyclically monotonically normal, as is any closed continuous image of an acyclically monotonically normal space. They also showed that any acyclic monotonically normal space has the Kuratowski property that van Douwen called K_0.

A space X is *utterly normal* [15] if it has a *magnetic base system*, i.e., a family of neighborhood bases $\mathcal{B}(x)$ for each $x \in X$ such that, if $B_x \in \mathcal{B}(x)$, $B_y \in \mathcal{B}(y)$, and $B_x \cap B_y \neq \emptyset$, then either $x \in \overline{B_y}$ or $y \in \overline{B_x}$. (Members of $\mathcal{B}(x)$ might not be open, but must have x in their interiors.) A space X is *UNO (Utterly Normal with respect to Open sets)* if X has a magnetic base system consisting of open sets. Every utterly normal space is monotonically normal, but it is not known if the converse is true. It is shown in [14] that every space which has a monotonically normal compactification (which includes all GO-spaces) is UNO.

A space X is said to have a G_δ*-diagonal* if the diagonal $\{(x, x) : x \in X\}$ is a G_δ-set in X^2. We often make use the following well-known characterization: X has a G_δ-diagonal iff there is a sequence (called a G_δ*-diagonal sequence*) $\mathcal{G}_0, \mathcal{G}_1, \ldots$ of open covers of X such that for each $x \in X$, $\bigcap_{n\in\omega} st(x, \mathcal{G}_n) = \{x\}$ (where $st(x, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : x \in U\}$). We will also use the higher cardinal version: if μ is an infinite cardinal, then X has a G_μ-diagonal iff there are open covers $\mathcal{G}_\alpha$, $\alpha < \mu$, of X such that for each $x \in X$, $\bigcap_{\alpha<\mu} st(x, \mathcal{G}_\alpha) = \{x\}$. The *diagonal degree* of X is the least cardinal μ such that X has a G_μ-diagonal.

A T_3-space X is said to be *semi-stratifiable* if to each open set U one may assign a sequence $S_n(U)$ of closed subsets of U such that $U = \bigcup_{n\in\omega} S_n(U)$, and $U \subset V$ implies $S_n(U) \subset S_n(V)$; X is *stratifiable* if in addition $U = \bigcup_{n\in\omega} Int(S_n(U))$ for each open U.

It is well-known that every metrizable space is stratifiable, and obviously every stratifiable space is semi-stratifiable. All stratifiable spaces are monotonically normal, and X is stratifiable iff X is semi-stratifiable and monotonically normal. We will be using the nontrivial fact due to R. Heath [21] that every stratifiable space has a σ-discrete network, where $\mathcal{N}$ is a *network* for X if whenever $x \in U$, U open, then there is some $N \in \mathcal{N}$ with $x \in N \subset U$.

A surjective map $f : X \to Y$ is *perfect* if it is continuous, closed, and has compact fibers, and f is *irreducible* if there is no closed subset H of X such that the restricted map $f \restriction H$ is onto. It is well-known that every perfect map has an irreducible restriction to some closed subset of the domain. We will make frequent use of the fact that if $f : X \to Y$ is irreducible and closed, then for each nonempty open $U \subseteq X$, the set $U^* = \bigcup\{f^{-1}[y] : y \in Y \text{ and} f^{-1}[y] \subset U\}$ is a nonempty dense open subset of U and $f[U^*]$ is a nonempty open subset of Y.

The key tool in our study is a lemma from [9] which shows that for a space X in PIGO, we may assume more about the witnessing perfect map:

Lemma 2.1 *If X is a perfect image of a GO-space then there is a GO-space L and a perfect irreducible mapping $f : L \to X$ with the property that if $a < b$ in L and $f(a) = f(b)$, then there is some $c \in (a, b)$ with $f(c) \neq f(a)$.*

The latter property is equivalent to: no fiber of f contains a convex subset of L with more than one point. The following result, which was proven in [9] for the case $\mu = \omega$ (i.e., for G_δ-diagonal) but with superfluous assumptions of irreducibility and closedness of the mapping, illustrates the usefulness of this condition.

Lemma 2.2 *Suppose $f : L \to X$ is a continuous mapping of the GO-space L onto X such that no fiber of f contains a nondegenerate convex set. If X has a G_μ-diagonal, so does L.*

Proof: Let $\mathcal{G}_\alpha$, $\alpha < \mu$, be a G_μ-diagonal sequence of open covers of X. For each α, let

$$\mathcal{U}_\alpha = \{C \subset L : C \text{ is a convex component of } f^{-1}[G] \text{ for some } G \in \mathcal{G}_\alpha\}.$$

We claim that $\mathcal{U}_\alpha$, $\alpha < \mu$, is a G_μ-diagonal sequence for L.

Let $a \in L$ and suppose $b \neq a$. Without loss of generality, $a < b$. If $f(a) \neq f(b)$, there is some $\alpha < \mu$ with $f(b) \notin st(f(a), \mathcal{G}_\alpha)$. It is straightforward to check that $b \notin st(a, \mathcal{U}_\alpha)$. On the other hand, if $f(a) = f(b)$, then by the fiber condition there is some c with $a < c < b$ such that $f(c) \neq f(a)$. Then choose α such that $f(c) \notin st(f(a), \mathcal{G}_\alpha)$. Suppose $b \in st(a, \mathcal{U}_\alpha)$. Then there is a convex component C of $f^{-1}[G]$ for some $G \in \mathcal{G}_\alpha$ that contains both a and b. Then $c \in C$, whence $\{f(a), f(c)\} \subseteq G \in \mathcal{G}_\alpha$, contradicting $f(c) \notin st(f(a), \mathcal{G}_\alpha)$. □

The next lemma will be used several times; it illustrates the usefulness of the closedness and irreducibility of the mapping. Recall that a collection $\mathcal{F}$ of nonempty subsets of X is a *π-net* for X if every nonempty open subset of X contains some $F \in \mathcal{F}$.

Lemma 2.3 *Suppose the map $f : Z \to X$ is closed and irreducible, and let $\mathcal{F}$ be a π-net for X. For each $F \in \mathcal{F}$, choose a point $z_F \in f^{-1}(F)$. Then $D = \{z_F : F \in \mathcal{F}\}$ is dense in Z. Furthermore, if $\mathcal{F}$ is σ-closed-discrete in X, then D is σ-closed-discrete in Z.*

In particular, if $\mathcal{F}$ is the collection of singletons of a dense subset E of X, then Z has a dense subset of the same cardinality which is also σ-closed discrete whenever E is.

Proof: Let U be a nonempty open subset of Z. As noted above, since f is closed and irreducible, the set $U^* = \bigcup\{f^{-1}[x] : x \in X \text{ and} f^{-1}[x] \subset U\}$ is a nonempty dense open subset of U and $f[U^*]$ is a nonempty open subset of X. Pick $F \in \mathcal{F}$ with $F \subset f[U^*]$. Then $z_F \in U^* \subseteq U$, so D is dense in Z. Since f is continuous, the preimage of a closed discete collection in X is closed discrete in Z; thus $\mathcal{F}$ σ-closed-discrete implies the same for D. □

3 Relation of PIGO to other classes

We begin by listing some easily proved properties of the class PIGO:

a) Because perfect mappings preserve both monotone normality [22] and weak orthocompactness [14], and any GO-space is both monotonically normal [22] and weakly ortho-compact [34], we see that every space in PIGO is monotonically normal and weakly ortho-compact.

b) If X is a space with a monotonically normal compactification, then X is in PIGO as explained in the Introduction.

c) Any space in PIGO is a K_0-space in the sense of van Douwen, is acyclically monotonically normal, and is in the class UNO [14] because each GO-space has these properties and these three properties are preserved by perfect irreducible mappings.

As we explain below, the converses of statements a), b), and c) are false. For statement a), recall that there are non-metrizable stratifiable spaces and every stratifiable space is monotonically normal, so that a metrization theorem to be proved in Section 5 (Theorem 5.5) shows that there are monotonically normal spaces that are not in PIGO. To see that the converse of b) is false, we use the next two results.

Proposition 3.1 *Every metrizable space belongs to PIGO.*

Proof: Morita has proved that if X is a metric space then there are a discrete space D and a subspace $S \subseteq D^\omega$ and a perfect mapping $g : S \to X$ (see [17] Exercise 4.4.J).

We claim that the product topology of the space D^ω is the open interval topology of some linear ordering (we believe this is a folklore result). Let $\kappa = |D|$ and consider any linear ordering $<$ of the set D with the property that each $d \in D$ has both an immediate successor and an immediate predecessor. (One example of such an ordering is the lexicographic ordering of $[0, \kappa) \times \mathbb{Z}$ where $\mathbb{Z}$ is the usual ordered set of integers.) Now let $\prec$ be the lexicographic ordering of D^ω induced by $<$, i.e., where we have $(d_0, d_1, \cdots) \prec (e_0, e_1, \cdots)$ in D^ω provided that if n is the first integer with $d_n \neq e_n$, then $d_n < e_n$. It is easy to check that the product topology on D^ω and the open interval topology of the lexicographic ordering $\prec$ are the same. Therefore D^ω is a LOTS. But then the subspace $S \subseteq D^\omega$ is a GO-space, so that X is in PIGO. $\Box$

Remark 3.2 *In Section 7 below, we will strengthen Proposition 3.1 by showing that the subspace $S \subseteq D^\omega$ is actually a LOTS under some suitable ordering.*

Corollary 3.3 *There is a space in PIGO that does not have a monotonically normal compactification.*

Proof: Junnila, Yun, and Tomoyasu [27] have shown that the hedgehog space $J(\omega_1)$ (the metrizable hedgehog with ω_1-many spines) does not have any monotonically normal compactification. But according to Proposition 3.1 this space belongs to PIGO. $\Box$

To see that the converse of c) is false, recall that there is an acyclically monotonically normal stratifiable space that is not metrizable [21]. In light of Theorem 5.5, that space cannot be in PIGO. In addition, any countable space X with just one non-isolated point must be in UNO: for an isolated point $x \in X$, the open magnetic base is $\mathcal{B}(x) = \{\{x\}\}$ and for the unique non-isolated point p, the open magnetic base is $\mathcal{B}(p) =$ the collection of all open sets containing p. There are non-metrizable spaces of this type, e.g., $X = \omega \cup \{p\} \subseteq \beta(\omega)$, and in the light of Theorem 5.5, such a space cannot be in PIGO.

It would be interesting to know which spaces in PIGO have monotonically normal compactifications. It is easy to see that every member of PIGO that is a continuous image of the space of countable ordinals will have a monotonically normal compactification. More generally we have

Proposition 3.4 *A locally compact space X is a member of PIGO if and only if X has a monotonically normal compactification.*

Proof: For the non-trivial half of the theorem, suppose X is a locally compact member of PIGO, and let $f : L \to X$ be a perfect mapping where L is a GO-space. Then X is monotonically normal. If X is compact, there is nothing to prove, so assume X is not compact. There is a compact LOTS M such that L densely embeds in M. Let $\alpha(X)$ be the one-point-compactification of X. Then the mapping f extends in a natural way (sending all points of $M - L$ to the single point in $\alpha(X) - X$) giving a continuous map $F : M \to \alpha(X)$. But then F is a perfect mapping so that, because M is monotonically normal, so is $\alpha(X)$. $\square$

Remark 3.5 *In [33], Mary Ellen Rudin gave a locally compact monotonically normal space that has no monotonically normal compactification. This is another monotonically normal space that is not in PIGO.*

4 Cardinal functions in PIGO

A great deal is known about cardinal functions for monotonically normal spaces. A good reference for these results is [20] which summarizes results by Gartside, Ostaszewski, Moody, Williams, and Zhou. For example, using notation as in [17], for any monotonically normal space X we have

$$\tau(X) \leq c(X) = hc(X) = hL(X) \leq d(X) = hd(X) \leq c(X)^+.$$

Because every space in PIGO is monotonically normal, the above cardinal relations automatically hold for PIGO spaces. However, it is often possible to give very easy proofs of cardinal function theorems for PIGO spaces by using well-known properties of GO-spaces and perfect irreducible maps. Our next result is an example.

Proposition 4.1 *For any X in PIGO, we have*

$$(*) \quad hL(X) = c(X) \leq d(X) = hd(X).$$

Proof: For the first equality, because $c(X) \leq hL(X)$ holds for any space, it is enough to show that $hL(X) \leq c(X)$ as follows. Because X is in PIGO, we have a GO-space L and a perfect irreducible mapping $f : L \to X$ with the extra properties given in Lemma 2.1. Suppose $c(X) = \kappa$. Let $\mathcal{U}$ be any pairwise disjoint collection of nonempty open subsets of the GO-space L, and for each $U \in \mathcal{U}$ let $U^* = \bigcup\{f^{-1}[x] : f^{-1}[x] \subseteq U\}$. Then $\{f[U^*] : U \in \mathcal{U}\}$ is a pairwise disjoint collection of nonempty open sets in X, and therefore the collection $\mathcal{U}$ must have cardinality $\leq \kappa$. Therefore, $c(L) \leq \kappa$ so that in the GO-space L we have $hL(L) \leq \kappa$. But then $hL(X) \leq \kappa$.

Next we prove that $hd(X) \leq d(X)$ for X in PIGO. Suppose $d(X) = \kappa$ and let $D \subseteq X$ be a dense set with cardinality κ. By Lemma 2.3, L has a dense subset of cardinality κ. Therefore $hd(L) = d(L) \leq \kappa$. Therefore $hd(X) \leq \kappa$. $\square$

Example 4.2 *There is a non-separable space X in PIGO having countable cellularity if and only if there is a Souslin space, i.e., a GO-space that is not separable and yet has countable cellularity.*

Proof: If there is a Souslin space, then it is in PIGO. Conversely, if there is a member of PIGO that has countable cellularity but is not separable, then we use a theorem of Williams and Zhou [36] asserting that there is a non-separable monotonically normal space with countable cellularity if and only if there is a Souslin space. □

Our next result (Proposition 4.5) shows that if X is in PIGO and $x \in X$, then the pseudo-character of x in X is equal to the character of x in X. (For example, if X is in PIGO and the point of $x \in X$ is a G_δ-set in X, then X is first-countable at x.) We note that this assertion is *not* true for monotonically normal spaces in general, as our next example shows.

Example 4.3 *For each cardinal κ, there is a stratifiable (and hence monotonically normal) space Y in which every point has countable pseudo-character, and yet there is a point of Y having character at least κ.*

Proof: Let $D_1, D_2, \cdots$ be a sequence of pairwise-disjoint sets, each with cardinality κ, and let p be any point not in any D_n. Let $Y = \{p\} \cup \left(\bigcup\{D_n : n \geq 1\}\right)$. Make all points of $Y - \{p\}$ isolated, and let a set V be a neighborhood of p if $p \in V$ and for some N the set $D_n - V$ is finite for each $n \geq N$. We define a stratification for Y as follows: For any $n \geq 1$ and open set $W \subseteq Y$, if $p \in W$ then let $S_n(W) = W$, and if $p \notin W$ then let $S_n(W) = \bigcup\{D_k \cap W : k \leq n\}$.

Clearly $\{p\} = \bigcap\{W_n : n \geq 1\}$ where $W_n = \{p\} \cup \left(\bigcup\{D_k : k \geq n\}\right)$ so that each point of Y has countable pseudo-character, and it is clear that the character of Y at p is at least κ. □

Before proving that character = pseudo-character for members of PIGO, we need a lemma about compact sets in GO-spaces. We will apply the lemma to the fibers of the mapping f in Lemma 2.1.

Lemma 4.4 *Suppose L is a GO-space and C is a compact subset of L with the property that if $a < b$ are points of C, then some $x \in L - C$ has $a < x < b$. If C is the intersection of μ-many open subsets of L, then the weight of the subspace C is $\leq \mu$ and there is a collection $\mathcal{B}$ of open subsets of L having $|\mathcal{B}| \leq \mu$ and having the property that if U is open in L and $C \subseteq U$, then some $B \in \mathcal{B}$ has $C \subseteq B \subseteq U$, so that C has an outer base of cardinality $\leq \mu$.*

Proof: Write $C = \bigcap\{U(\alpha) : \alpha < \mu\}$ where each $U(\alpha)$ is an open subset of L. Let $\mathcal{U}(\alpha)$ be the collection of all convex components of the set $U(\alpha)$. Each $\mathcal{U}(\alpha)$ is an open cover of C. Suppose $a \neq b$ are points of C. We may assume $a < b$. Then some $x \in L$ has $x \in (a, b) - C$ so there is some $\alpha < \mu$ with $x \notin U(\alpha)$. Then for every member $V \in \mathcal{U}(\alpha)$ that contains a, we must have $b \notin V$, for otherwise $x \in [a, b] \subseteq V \subseteq U(\alpha)$. Therefore the collection $\{\mathcal{U}(\alpha) : \alpha < \mu\}$ of open covers of C shows that the diagonal degree of the subspace C is $\leq \mu$. Now apply Corollary 7.6 of [24] to conclude that the weight of C is at most μ.

The sets $(\leftarrow, \min(C)]$ and $[\max(C), \rightarrow)$ are called the *end jumps of* C. By an *internal jump in* C we mean a pair of points $a < b$ of C such that $[a, b] \cap C = \{a, b\}$. Consider the set of all right-hand points of all possible internal jumps of C. That subspace of C must have weight $\leq \mu$ and therefore the set J consisting of all points in internal jumps of C and the two end jumps has cardinality at most μ.

Next we will identify a subset $E \subset L$ that will be used in constructing an outer base for C. Let $x \in J$. We claim that either $(\leftarrow, x]$ is open, or there is a subset R_x of $(x, \rightarrow)$ of cardinality at most

μ with $x \in \overline{R_x}$. Suppose $(\leftarrow, x]$ is not open. For each $\alpha < \mu$, pick a point $x_\alpha \in (x, \rightarrow) \cap U(\alpha)_x$, where $U(\alpha)_x$ is the convex component of $U(\alpha)$ containing x, and let $R_x = \{x_\alpha : \alpha < \mu\}$. Then since $C = \bigcap\{U(\alpha) : \alpha < \mu\}$, it follows from the hypothesized property of C (call it $(*)$) that $x \in \overline{R_x}$. Similarly, either $[x, \rightarrow)$ is open, or there is a subset L_x of $(\leftarrow, x)$ of cardinality at most μ with $x \in \overline{L_x}$.

Now we let $E = \bigcup\{L_x \cup R_x \cup \{x\} : x \in J\}$. Let $\mathcal{B}'$ be the collection of all intervals with endpoints in E that are open in L (e.g, include $[x, y)$, $(y, x]$, and $\{x\}$ if these are open in L). It follows from the construction that $\mathcal{B}'$ contains a local base in L at every point of J. If $x \in C - J$, then by compactness x is a limit point of C from both sides, and then by $(*)$ is a limit point of J, hence of E, from both sides. Thus the set of all open intervals with endpoints in E is a local base in L at x.

Finally, we let $\mathcal{B}$ be the collection of all finite unions of members of $\mathcal{B}'$. Since E has cardinality at most μ, the same is true of $\mathcal{B}'$ and thus $\mathcal{B}$. Since $\mathcal{B}'$ contains a base in L at every point of C, it follows by compactness that if $C \subset U$ where U is open, there is some $B \in \mathcal{B}$ with $C \subset B \subset U$. So $\mathcal{B}$ is an outer base for C of cardinality $\leq \mu$. $\square$

Proposition 4.5 *Suppose $X \in PIGO$ and $x \in X$. Then the character of x in X equals the pseudo-character of x in X.*

Proof: Suppose $f : L \to X$ is a perfect mapping as described in Lemma 2.1 and let $x \in X$. If the pseudo-character of x in X is $\leq \mu$, the compact set $C = f^{-1}[x] \subseteq L$ is the intersection of at most μ-many open sets. Then Lemma 4.4 shows that C has an outer base consisting of at most μ-many open sets. Because f is a closed mapping, the point $x \in X$ has a neighborhood base with at most μ-many members, as required. $\square$

Corollary 4.6 *Suppose $X \in PIGO$ and suppose K is a compact subset of X. If K is the intersection of μ-many open sets in X, then K has an outer base with cardinality $\leq \mu$.*

Proof: The quotient space $Y = X/K$ in which K is identified to a point is also in PIGO. Now apply Proposition 4.5. $\square$

Example 4.7 *The McAuley's bow-tie space [30] is a stratifiable (and hence monotonically normal) first-countable space X containing a compact set K that is a G_δ-set in X and yet K does not have an outer base of cardinality ω (e.g., one may take K to be the unit interval on the x-axis). Therefore there are first-countable monotonically normal spaces which do not have the property of Corollary 4.6, and this property is strictly stronger than the character=pseudo-character property.*

Corollary 4.8 *If X is in PIGO and has countable cellularity, then X is first countable.*

Proof: Proposition 4.1 shows that X is hereditarily Lindelöf so that each point of X is a G_δ-set. Then Proposition 4.5 shows that X is first-countable. $\square$

A space X is *Lindelöf at infinity* if there is some compactification $c(X)$ with the property that $c(X) - X$ is a Lindelöf subspace of $c(X)$. Henriksen and Isbell [25] proved that a space is Lindelöf at infinity if and only if for each compact set $K \subseteq X$ there is a compact set L with $K \subseteq L \subseteq X$ where L has a countable outer base for its neighborhoods, and they deduced that any metric space is Lindelöf at infinity. We can extend their result as follows:

Corollary 4.9 *If X is perfect and a member of PIGO, then X is Lindelöf at infinity.*

Proof: If K is any compact subset of X, then K is a G_δ-set because X is perfect, so that K has a countable outer base for its neighborhoods by Corollary 3.6 (with $\mu = \omega$). It now follows from the characterization given by Henriksen and Isbell that X is Lindelöf at infinity. □

5 Paracompactness, metrization and dense metrizable subspaces in PIGO

Because every space in PIGO is monotonically normal, the Balogh-Rudin theorem [1] decides most questions about paracompactness in members of PIGO. That theorem asserts:

Theorem 5.1 *A monotonically normal space X is paracompact if and only if X does not contain a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal, and is hereditarily paracompact if it does not contain any subspace homeomorphic to a stationary set in a regular uncountable cardinal.*

Corollary 5.2 *The following properties of a monotonically normal space X are equivalent:*

a) X has a σ-disjoint base;

b) X has a σ-point-finite base;

c) X is quasi-developable.

Proof: We know that $a) \Rightarrow b) \Rightarrow c)$ for any space [2], [3]. To show that $c) \Rightarrow a)$, recall that no stationary set in a regular uncountable cardinal can be quasi-developable, so it follows from Theorem 5.1 that any quasi-developable monotonically normal space is hereditarily paracompact. But any hereditarily paracompact quasi-developable space has a σ-disjoint base. □

Another consequence of Theorem 5.1 is that many weak covering properties imply paracompactness in any monotonically normal space and hence in any member of PIGO, e.g., weak-θ-refinability, subparacompactness, being a perfect space (meaning that closed subsets are G_δ-sets), having a G_δ-diagonal, etc. See [29] for a lengthy list.

The D-space covering property was studied in [13]. A space X is a *D-space* if given any collection $\{N(x) : x \in X\}$ of open subsets of X with $x \in N(x)$ for every $x \in X$, there is a closed discrete subspace $D \subseteq X$ with the property that $\bigcup\{N(x) : x \in D\} = X$. The D-space property is known to be equivalent to paracompactness in GO-spaces [16] but whether the D-space property is equivalent to paracompactness in arbitrary monotonically normal spaces is not known. However, for spaces in PIGO, we have:

Proposition 5.3 *Suppose X is in PIGO. Then X is paracompact if and only if X is a D-space.*

Proof: First suppose X is a D-space. Because the D-space property is closed-hereditary and no stationary set in a regular uncountable cardinal can be a D-space, the Balogh-Rudin theorem (Theorem 4.1 above) shows that X must be paracompact.

Next, suppose X is paracompact. Because X is in PIGO, we have a GO-space L and a perfect irreducible mapping $f : L \to X$ as in Lemma 2.1. Being the perfect preimage of a paracompact space, L must be paracompact. Therefore L is a D-space by [16]. So X is the closed continuous image of a D-space and hence is a D-space [13]. $\square$

It is known that if L is a GO-space with a σ-closed-discrete dense set, then L is perfect (and hence paracompact). Whether every perfect GO-space has a σ-closed-discrete dense set is axiom-sensitive and is undecidable, at least for spaces of small cardinality [11]. In the larger class PIGO, we have

Proposition 5.4 *Suppose X is in PIGO and has a σ-closed-discrete dense set. Then X is perfect and paracompact.*

Proof: As in Lemma 2.1 we have a GO-space L and a perfect irreducible mapping $f : L \to X$. By Lemma 2.3, L has a σ-closed discrete dense subset, so that L is perfect and paracompact. Therefore so is the space X because X is a closed continuous image of L. $\square$

We next consider metrization of spaces in PIGO. In [9] the authors asked whether a semi-stratifiable space X must be metrizable if X has a monotonically normal compactification. Because every space with a monotonically normal compactification is in PIGO, our next result answers that question affirmatively.

Theorem 5.5 *Suppose X is in PIGO. If X is semi-stratifiable, then X is metrizable.*

Proof: Because $X \in PIGO$, we have a GO-space L and a perfect irreducible mapping $g : L \to X$ whose fibers have the special property mentioned in Lemma 2.1. Because X is semi-stratifiable, X has a G_δ-diagonal. By Lemma 2.2, L also has a G_δ-diagonal, so every fiber of the mapping g, being compact with a G_δ-diagonal, must be metrizable. It follows that each point of L is G_δ, so L is first-countable.

Let $R = \{a \in L : a \neq \min(L) \text{ and } [a, \to) \text{ is open in } L\}$. We will show that R is σ-closed discrete in L. Similarly, the set $\{a \in L : a \neq \max(L) \text{ and } (\leftarrow, a] \text{ is open in } L\}$ will be σ-closed discrete. Then it will follow from results of Faber (see Theorems 3.1 and 3.2 of [18]) that the space L is metrizable.

Because $X \in PIGO$, X is monotonically normal. Being monotonically normal and semi-stratifiable, X is stratifiable. Now a theorem of Heath [23] shows that X has a σ-discrete network $\mathcal{F} = \bigcup\{\mathcal{F}(n) : n < \omega\}$ whose members are closed sets. For each $F \in \mathcal{F}$ choose a point $y(F) \in g^{-1}[F]$. By Lemma 2.3, $M = \{y(F) : F \in \mathcal{F}\}$ is dense and σ-closed discrete in L, say $M = \bigcup_{n\in\omega} M(n)$, where each $M(n)$ is closed discrete.

For each $p \in R$, let $C(p) = g^{-1}[g(p)]$. Then $C(p)$ is compact. Because $p \in R$, the set $[p, \to)$ is open so that the set $C(p) - [p, \to) = (\leftarrow, p) \cap C(p)$ is also compact. Let $p^- = \sup(C(p) \cap (\leftarrow, p))$ if that set is nonempty (and then note that $p^- \in C(p)$), and otherwise we let p^- be $\leftarrow$. We claim that the interval (p^-, p) is not empty. If the set $C(p) - [p, \to) = (\leftarrow, p) \cap C(p)$ is nonempty, then $p^- \in C(p)$. If the interval $(p^-, p) = \emptyset$ then the special property of fibers of g is violated, so we conclude $(p^-, p) \neq \emptyset$ in this case. If the set $C(p) - [p, \to) = (\leftarrow, p) \cap C(p)$ is empty, then because p is not the left endpoint of L, we know that $(p^-, p) \neq \emptyset$.

Because M is dense in L, we may choose a point $y(p) \in (p^-, p) \cap M$. Then $[y(p), p)$ is closed because $p \in R$, so that $g([y(p), p))$ is closed in X, and $g(p) \notin g([y(p), p))$. Because $\mathcal{F}$ is a network for X, there is some $F_p \in \mathcal{F}$ with $g(p) \in F_p$ and $F_p \cap g([y(p), p)) = \emptyset$.

For $i, j < \omega$, let $R(i, j)$ be the set of all points $p \in R$ with $y(p) \in M(i)$ and $F_p \in \mathcal{F}(j)$. We will complete the proof by showing that each $R(i, j)$ is closed and discrete. For contradiction suppose some $R(i, j)$ is not closed and discrete. Then there are points $p_n \in R(i, j)$ and some point $y \in L$ with $p_n \to y$. Infinitely many points p_n lie on the same side of y. Let us assume that $p_n < y$ for all n and that the sequence p_n is strictly increasing; the case where $p_n > y$ is handled similarly.

Because $g(p_n) \in F_{p_n}$ we have $p_n \in g^{-1}[F_{p_n}]$ and because $p_n \in R(i, j)$ we know that the sets F_{p_n} all belong to the discrete collection $\mathcal{F}(j)$. It follows that for some $F_0 \in \mathcal{F}(j)$ we have $F_{p_n} = F_0$ for all sufficiently large n, say for $n \geq N$.

For $n \geq N$ we claim that $y(p_{n+1})$ cannot be less than or equal to p_n. For if $y(p_{n+1}) \leq p_n$ then we would have $p_n \in [y(p_{n+1}), p_{n+1})$ and then $g(p_n) \in g([y(p_{n+1}), p_{n+1})) \cap F_0 = g([y(p_{n+1}), p_{n+1})) \cap F_{p_{n+1}}$, so $g([y(p_{n+1}), p_{n+1})) \cap F_{p_{n+1}} \neq \emptyset$, contradicting the choice of $F_{p_{n+1}}$.

Therefore $p_n < y(p_{n+1}) < p_{n+1}$ showing that $y_{p_n} \to y$. But that is impossible because the points y_{p_n} all belong to the closed discrete set $M(i)$. Therefore, as claimed, the set $R(i, j)$ must be closed and discrete. $\Box$

Proposition 5.6 *Suppose $X \in PIGO$ has a σ-closed-discrete dense set and a point-countable base. Then X is metrizable.*

Proof: Let $f : L \to X$ be as in Lemma 2.1. By Lemma 2.3, L has a σ-closed-discrete dense set because X does.

Let $\mathcal{B}$ be a point-countable base for X. Let

$$\mathcal{C} = \{C \subset L : C \text{ is a convex component of } f^{-1}(B) \text{ for some } B \in \mathcal{B}\}.$$

We claim that $\mathcal{C}$ is a point-countable base for L. Clearly $\mathcal{C}$ is point-countable because $\mathcal{B}$ is. It remains to prove $\mathcal{C}$ is a base for L.

Let $x \in L$. If x is isolated, then by irreducibility, $\{x\} = f^{-1}(f(x))$ and so $f(x)$ is isolated in X. Thus $\{f(x)\} = B$ for some $B \in \mathcal{B}$ and $f^{-1}(B) = \{x\} \in \mathcal{C}$. So $\mathcal{C}$ contains a base at x.

Now suppose x has a local base of open intervals (a, b), where $a < x < b$. Fix such an interval (a, b). Since no fiber of f contains a nondegenerate convex set, there are points a' and b' in $L - f^{-1}(f(x))$ such that $a \leq a' < x < b' \leq b$. Choose $B \in \mathcal{B}$ such that $f(x) \in B$ and $B \cap \{f(a'), f(b')\} = \emptyset$. Let C be the convex component of $f^{-1}(B)$ containing x. Then C cannot contain a' or b'; it follows that $x \in C \subset (a, b)$.

Suppose $\{[x, y) : y > x\}$ is a base at x. If x is the least point of X, then $\mathcal{C}$ contains a base at x in an argument similar that of the previous paragraph. So assume x is not the least point of X, and let $K = f^{-1}(f(x))$. We claim that there is a point $l_x < x$ such that the interval $[l_x, x) \cap K = \emptyset$. If $K \cap (\leftarrow, x) = \emptyset$ this is obvious. If $K \cap (\leftarrow, x) \neq \emptyset$, then since $(\leftarrow, x)$ is closed, $K \cap (\leftarrow, x)$ is compact and $p = sup(K \cap (\leftarrow, x))$ is strictly less than x. Then any point l_x in the interval (p, x) is as desired. Choose $y' \in (x, y] - K$. Then $[l_x, x) \cup \{y'\}$ is closed and misses K, so $f([l_x, x) \cup \{y'\})$ is closed in X and misses $f(x)$, so we can choose $B \in \mathcal{B}$ with $f(x) \in B$ and $B \cap (f([l_x, x) \cup \{y'\})) = \emptyset$.

Let C be the convex component of $f^{-1}(B)$ containing x. Since $C \cap [l_x, x) = \emptyset$, it follows that x is the least point of C, and then since $y' \notin C$, we have $x \in C \subset [x, y)$. Thus $\mathcal{C}$ contains a base at x.

If $\{(y, x] : y < x\}$ is a base at x, then a similar argument shows that $\mathcal{C}$ contains a base at x. Thus $\mathcal{C}$ is a point-countable base for L. Now L is metrizable by Theorem 3.1 of [6], so X is too. □

Recall that a space X is a *β-space* if for each $x \in X$ and $n \in \omega$, one can assign an open set $g(n, x)$ containing x such that $x \in g(n, x)$ and if $p \in g(n, x_n)$ for each n, then $\{x_n : n \in \omega\}$ has a limit point.

Proposition 5.7 *Suppose X is in PIGO and is a β-space with a G_δ-diagonal. Then X is metrizable.*

Proof: Start with a GO-space L and a perfect irreducible mapping $g : L \to X$ as in Lemma 2.1. If the open sets $\{B(n, x) : x \in X, n < \omega\}$ witness the β-space property of X, then define $C(a, n) = g^{-1}[B(g(a), n)]$. It is easy to check that the collection $\{C(a, n) : a \in L, n < \omega\}$ witnesses the β-space property of L. By Lemma 2.2, we know that L has a G_δ-diagonal because X does. Consequently, the GO-space L is metrizable [5]. Because g is perfect, so is X. □

Proposition 5.8 *Suppose (X, τ) is in PIGO and has a G_δ-diagonal. Then there is a topology σ on X with $\sigma \subseteq \tau$ and such that (X, σ) is metrizable.*

Proof: Because (X, τ) has a G_δ-diagonal, the Balogh-Rudin [1] theorem (Theorem 5.1) shows that (X, τ) is paracompact. But any paracompact space with a G_δ-diagonal has a weaker metric topology. □

Proposition 5.9 *Suppose X is a member of PIGO. Then X has a dense metrizable subspace if and only if X has a dense subspace with a G_δ-diagonal.*

Proof: Because every subspace of a member of PIGO is also a member of PIGO, to prove the non-trivial half of the proposition it will be enough to show that every member of PIGO with a G_δ-diagonal has a dense metrizable subspace. Suppose X is such a space. Then there is a GO-space L and a perfect irreducible mapping $g : L \to X$ with the properties listed in Lemma 2.1. By Lemma 2.2, we know that L has a G_δ-diagonal because X does. But then, from Proposition 3.4 of [7], L has a dense metrizable subspace. A theorem of H.E. White [35] shows that a regular first-countable space has a dense metrizable subspace if and only if it has a σ-disjoint π-base, so we know that there is a σ-disjoint collection $\mathcal{P} = \bigcup\{\mathcal{P}(n) : n < \omega\}$ that is a π-base for L. For each $P \in \mathcal{P}$ let $P^* = \bigcup\{g^{-1}[x] : g^{-1}[x] \subseteq P\}$. Because g is irreducible each $P^* \neq \emptyset$ and because g is a closed mapping, each set $g[P^*]$ is a nonempty open subset of X. In addition, the collection $\{g[P^*] : P \in \mathcal{P}(n)\}$ is a pairwise disjoint collection.

Now suppose U is a non-empty open set in X. Then $g^{-1}[U]$, being a nonempty open set in L, must contain some set $P \in \mathcal{P}$ and then $P^* \subseteq P \subseteq g^{-1}[U]$ so that $g[P^*] \subseteq U$. Therefore the space X has a σ-disjoint π-base, so that if we apply White's theorem again, we see that X has a dense metrizable subspace. □

The authors of [4] noted that a GO-space is metrizable if and only if it is an F_{pp}-space, (i.e., each of its subspaces is a paracompact p-space in the sense of Arhangelskii), and asked whether the same is true for spaces with monotonically normal compactifications. We answer that question negatively by giving a compact monotonically normal space that is non-metrizable and is an F_{pp}-space.

Example 5.10 *There is a compact monotonically normal space that is non-metrizable and is an F_{pp}-space.*

Proof: Let $L = [0, \omega_1]$ and let C be the set of all limit ordinals in L. Then the quotient space $X = L/C$ obtained by collapsing all of C to a single point is in PIGO and therefore is monotonically normal. The space X is the one-point compactification of a discrete space of cardinality ω_1. Consider any subspace $Y \subseteq X$. If the unique limit point of X is not in Y, then Y is a discrete metric space, and if the unique limit point of X belongs to Y, then Y is compact. Hence X is an F_{pp}-space, and X is certainly not metrizable – it is not even first-countable. $\Box$

6 Products of PIGOs

While the class PIGO is well-behaved in some ways – it is a hereditary class and is preserved by perfect mappings, for example – it is badly behaved under products.

Proposition 6.1 *Suppose X and Y are in PIGO. If $X \times Y$ is in PIGO, then one of the following holds:*

a) at least one of X and Y is discrete;

b) neither X nor Y contains a countable set with a limit point;

c) both X and Y are metrizable.

Proof: Suppose $X \times Y$ is in PIGO. Suppose neither X nor Y is discrete and one space, say X, contains a countable set C with a limit point p. Let $D = C \cup \{p\}$. Then $D \times Y$, being a subspace of $X \times Y$, is also in PIGO and therefore is monotonically normal. Now apply Theorem 4.1 from [22] to show that the space Y must be stratifiable. Apply Theorem 5.5 to conclude that Y is metrizable. Because Y is not discrete, Y contains a countable set E that contains a limit point of itself. Then $X \times E$, being a subspace of $X \times Y$ is also monotonically normal, so applying Theorem 4.1 of [22] again shows that X must be stratifiable, and then applying Theorem 5.5 again shows that X must be metrizable. $\Box$

If X and Y are in PIGO and either a) or c) of the above theorem are satisfied, then $X \times Y$ is also in PIGO. But not so for b): if X is the space obtained from the ordinal space $[0, \omega_2]$ by isolating all points of countable cofinality, then X is a GO-space but X^2 is not hereditarily normal so not in PIGO. On the other hand, our next example shows that option b) is a real possibility.

Example 6.2 *There is a non-metrizable space X in PIGO with the property that $X \times X$ is also in PIGO.*

Proof: Let X be the set $[0, \omega_1]$ with all countable ordinals made discrete. Then X is a GO-space, so X is in PIGO. Consider the product space $Y = X^2$. We split X^2 along the diagonal as follows. Let $U = \{(\alpha, \beta) \in Y : \alpha \leq \beta \leq \omega_1\}$ and $V = \{(\gamma, \delta) \in Y : \delta < \gamma \leq \omega_1\}$. Then U and V are subsets of Y with only the point (ω_1, ω_1) in common. The set U is a GO-space under the lexicographic ordering with (ω_1, ω_1) as its top point and an analogous statement applies to V, which is a GO-space with (ω_1, ω_1) as its bottom point. Joining U and V together at the point (ω_1, ω_1) shows that X^2 is a GO-space and is therefore in PIGO. $\Box$

7 PILOTS vs. PIGO

In previous sections of our paper, we have studied spaces that are perfect images of generalized ordered spaces. Why did we focus on GO-spaces, rather than linearly ordered spaces? Suppose we are looking at a space X that has a monotonically normal compactification $c(X)$. According to Mary Ellen Rudin's solution of Nikiel's problem [32], there is a compact LOTS M and a continuous mapping $g : M \to c(X)$. If we restrict g to the subspace $g^{-1}[X]$, we obtain a perfect mapping. Then, as a step toward proving Lemma 2.1 we further restrict our mapping to give a perfect mapping that is irreducible. Notice that even though we start with a compact LOTS, this process of restricting to subspaces means that our domain will probably not be a LOTS, but it will be a GO-space.

But what if we could get a LOTS L and a perfect mapping from L onto our space X? Is that possible for every X in PIGO? For which spaces can we find a LOTS domain for our perfect mapping? Are there members of PIGO that are not perfect images of some LOTS?

If the space X is a perfect image of some LOTS, we will say that X belongs to the class *PILOTS* (= Perfect Image of Linearly Ordered Topological Space).

Proposition 7.1 *Any metric space belongs to the class PILOTS.*

Proof: Morita proved that any metric space X is the perfect image of some subspace of the metric space D^ω where D is a discrete space of suitable cardinality (see Problem 4.4.J of [17]). Herrlich proved that for any strongly zero dimensional metric space Y there is a linear order $<$ of Y whose open interval topology coincides with the given topology of Y (see Problem 6.3.2.f of [17]). Hence our result follows if we show that every subspace of D^ω is strongly 0-dimensional.

It is probably folklore that D^ω is hereditarily ultraparacompact (= every open cover has a pairwise-disjoint open refinement) and thus hereditarily strongly 0-dimensional. For the benefit of the reader, we outline a quick argument. For any finite sequence $\sigma = \langle d_0, d_1, ..., d_n\rangle$ of elements of D, let $[\sigma] = \{x \in D^\omega : x \text{ extends } \sigma\}$. Then $\mathcal{B} = \{[\sigma] : \sigma \in D^{<\omega}\}$ is the standard basis for D^ω. Given an open cover $\mathcal{U}$ of a subspace Y of D^ω, let $\mathcal{V}$ be the set of all nonempty sets of the form $[\sigma] \cap Y$, where $\sigma = \langle d_i\rangle_{i\leq n}$, such that

$$\exists U \in \mathcal{U}([\sigma] \cap Y \subseteq U) \text{ but } \nexists U \in \mathcal{U}([\sigma^-] \cap Y \subseteq U)$$

where $\sigma^- = \langle d_i\rangle_{i\leq n-1}$ (take σ^- to be the empty sequence when $n = 0$). It is straightforward to check that $\mathcal{V}$ is a pairwise-disjoint open refinement of $\mathcal{U}$. □

A direct consequence of Rudin's characterization of compact monotonically normal spaces as the continuous images of compact linearly ordered spaces gives us that every compact monotonically normal space (and hence every compact space in PIGO) is in PILOTS. This also gives us a natural class of spaces in PILOTS (and PIGO) that are neither metrizable nor linearly ordered: the one-point compactifications of uncountable discrete spaces. These spaces are monotonically normal: in the characterization of monotone normality labeled (c) in Section 1.1, let $V(p, U) = \{p\}$ if p is isolated, and $V(p, U) = U$ otherwise. It is easy to check that this satisfies condition (c).

We noted in Remark 3.5 that there are locally compact monotonically normal spaces that are not in PIGO. On the other hand, any locally compact space in PIGO must be in PILOTS. We will need the following lemma.

Lemma 7.2 *Let $(X, \sigma, \leq)$ be a 0-dimensional locally compact GO-space. Then there is a linear ordering $\preceq$ on X which induces σ, i.e., $(X, \sigma, \preceq)$ is a LOTS.*

Proof: *Claim 1. $\mathcal{B} = \{[a, b] : a, b \in X, [a, b]$ compact open$\}$ is a basis for X.* Let $p \in x$. If p is isolated, then $[p, p] \in \mathcal{B}$. Now suppose p has a local base of the form $\{[p, y) : y > p\}$. Fix $y > p$. By local compactness, we may assume $[p, y]$ is compact. A compact LOTS with a dense linear order is connected, so by 0-dimensionality, there is a point z with $p < z < y$ such that z has an immediate successor. Then $[p, z]$ is a compact open neighborhood of p contained in $[p, y)$. The case where p has a base of left half-open intervals can clearly be taken care of by an analogous argument, and if p has a base of open intervals simply apply the argument in both directions. This proves Claim 1.

For $p, q \in X$, define $p \sim q$ iff p and q are contained in some compact open interval $[a, b]$ (i.e., in a member of $\mathcal{B}$). Clearly $\sim$ is an equivalence relation, and each equivalence class is convex. Let $\mathcal{J}$ be the set of equivalence classes.

Claim 2. Each $J \in \mathcal{J}$ is a clopen subset of X which is a LOTS with respect to the restricted GO order. It follows from Claim 1 that each $J \in \mathcal{J}$ is open, hence also closed. Recall that any compact GO space is a LOTS with respect to the restricted GO order. Let $p \in J$. Then p is contained in some compact open interval $[a, b] = I \subseteq J$. If p is not an endpoint of I, or if it is an endpoint of J, then p has a base of open intervals with endpoints in I, hence in J (counting intervals with one "endpoint" equal to $\rightarrow$ or $\leftarrow$). Suppose p is an endpoint of I, e.g., suppose it's the right endpoint I, but not of J. Let $q \in J$ with $q > p$. Then $[p, q]$ is compact and p is relatively isolated in $[p, q]$ (since $[p, q] \cap I = \{p\}$), so p has an immediate successor p' in J. Then open intervals with p' as the right endpoint contain a base at p. This proves Claim 2.

Now we need to line up the members of $\mathcal{J}$ so that the induced order generates the topology σ on X. Call a LOTS J "open" if J has no least or greatest element, "closed" if it has both, and "right half-open" (respectively, "left half-open") if J has a least element but not a greatest element (respectively, a greatest element but not a least). In lining them up, we need to avoid, for example, placing an open J immediately above a closed or left half-open J'. To accomplish this, we may need to perform some "surgery" on the J's. Note that if $J \in \mathcal{J}$ is open, then by local compactness and 0-dimensionality, there is a compact open interval $[a, b]$ such that $J = (l_J, a'] \cup [a, b] \cup [b', r_J)$, where $a, b \in J$, and a' is the immediate predecessor of a and b' the immediate successor of b. Also, if $J \in \mathcal{J}$ is right half-open, then there is $b > l_J$ such that $[l_J, b]$ is compact open and $J = [l_J, b] \cup [b', r_J)$; an analogous statement holds if J is left half-open. Finally, we can "turn J around" via the order $x \preceq y$ iff $y \leq x$. Note that $(a, b)_{\leq} = (b, a)_{\preceq}$, $[a, b)_{\leq} = (b, a]_{\preceq}$, etc., where the subscript indicates the ordering being considered; hence the corresponding GO topologies are homeomorphic.

Claim 3. If $\mathcal{J}$ is finite, then there is a linear order on X which induces the topology σ. The proof is by induction on $|\mathcal{J}|$. The key case is $|\mathcal{J}| = 2$. Let $\mathcal{J} = \{J_1, J_2\}$. Suppose J_1 is open. If J_2 is also open, or is left half-open, put J_2 above J_1. If J_2 is right half-open, put J_2 below J_1. If J_2 is closed, find a compact open interval $[a, b]$ in J_1 such that $J_1 = (l_{J_1}, a'] \cup [a, b] \cup [b', r_{J_1})$, and stick J_2 between a' and a (or b and b').

The remaining cases are J_1 and J_2 both half-open, both closed, or one of each. If one of each, do a similar trick as for the case with J_1 open and J_2 closed. If both closed, put any one above the other. Finally, if both are half-open, we can by turning one around if necessary assume J_1 is left half-open and J_2 right half-open; then put J_2 above J_1. In all of these cases, it is easy to see that the indicated ordering is a linear order inducing the topology σ.

Now suppose Claim 3 holds with $|\mathcal{J}| = n \geq 2$, and let $\mathcal{J} = \{J_1, J_2, ..., J_{n+1}\}$. By the induction hypothesis, $J_1, J_2, ..., J_n$ can be combined to form a LOTS generating σ, and then by the case $n = 2$, $\bigcup_{i=1}^{n} J_i$ and J_{n+1} can be so combined as well. This proves Claim 3.

The next claim will finish the proof of the lemma.

Claim 4. If $\mathcal{J}$ is infinite, then there is a linear order on X which induces the topology σ. Let $|\mathcal{J}| = \kappa$. By performing surgeries as discussed prior to Claim 3, and by turning some around if necessary, we may assume every member of $\mathcal{J}$ is either closed or left half-open, and furthermore that κ-many members are closed. Let λ be the number of left half-open members.

For each $\alpha < \kappa$, we will define a LOTS L_α consisting of copies of countably many members of $\mathcal{J}$, and then put the lexicographic order on $L = \bigcup_{\alpha<\kappa}\{\alpha\} \times L_\alpha$. If $\alpha < \lambda$, let L_α be a left half-open J followed by an ω-sequence of closed J's. If $\lambda \leq \alpha < \kappa$, let L_α be a countably infinite collection closed J's arranged in the order type of the integers $\mathbb{Z}$. We can do all of this in such a way that each member of $\mathcal{J}$ is used exactly once. Then the lexicographic order on L induces the topology σ on the copy of each $J \in \mathcal{J}$, and so L is homeomorphic to X. This finishes the proofs of Claim 4 and of the lemma. □

Proposition 7.3 *Every locally compact space in PIGO is in PILOTS.*

Proof: Let X be a locally compact space in PIGO. Let $f : L \to X$ be perfect, where L a GO-space. Then f extends to a perfect map $F : L^* \to X^*$, where L^* is an ordered compactification of L and X^* is the one-point compactification of X. Let M be the "double-arrow version" of L^*, i.e., $M = L^* \times \{0, 1\}$ with the lexicographic order. Then M is a 0-dimensional LOTS, and the obvious map $g : M \mapsto L^*$ is perfect. Let $h = F \circ g : M \to X^*$. Then h is perfect map of M onto X^*. Let $Y = h^{-1}(X)$, and let k be h restricted to Y. Now X is open in X^* so Y is open in M. Thus Y is locally compact GO and 0-dimensional, and k is a perfect map of Y onto X. By the previous lemma, there is a linear order on Y making it a LOTS; thus X is in PILOTS. □

The following corollary is immediate from the above proposition and Proposition 3.4.

Corollary 7.4 *The following are equivalent for a locally compact space X:*

(i) X is in PIGO;

(ii) X is in PILOTS;

(iii) X has a monotonically normal compactification.

To see that there are spaces in PIGO that are not in PILOTS, we use the following metrization theorem.

Theorem 7.5 *Suppose $X \in PILOTS$. Then X is metrizable if and only if X has a G_δ-diagonal.*

Proof: To prove the non-trivial half of this proposition, suppose X is a space with a G_δ-diagonal and suppose we can show that there is a LOTS L and a perfect mapping $f : L \to X$ with the property that if $a < b$ in L have $f(a) = f(b)$, then there is some $c \in (a, b)$ with $f(c) \neq f(a)$. Then by Lemma 2.2, L has a G_δ-diagonal. Because L is a LOTS, we know that L is metrizable, and then so is its perfect image X.

Therefore, to complete the proof of Theorem 7.5 we must show that for each $X \in PILOTS$, we can find a LOTS and a perfect mapping with the properties described in the first paragraph of this proof. We do that in the next lemma. $\Box$

Lemma 7.6 *Suppose M is a LOTS and suppose there is a perfect mapping $g : M \to X$. Then there is a LOTS L and a perfect mapping $f : L \to X$ with the property that if $a < b$ in L and $f(a) = f(b)$, then for some $c \in (a, b)$ we have $f(c) \neq f(a)$.*

Proof: For each fiber $g^{-1}[x]$ of the mapping g, let $\mathcal{C}_x$ be the collection of all the maximal (with respect to $\subseteq$) elements of the family of all compact, convex subsets of M that are subsets of $g^{-1}[x]$. Some members of $\mathcal{C}_x$ might be singleton sets. Let $\mathcal{C} = \bigcup\{\mathcal{C}_x : x \in X\}$. Then $\mathcal{C}$ is a pairwise disjoint collection of compact, convex subsets of M and $\mathcal{C}$ covers M. Let M be the quotient space obtained by identifying each $J \in \mathcal{C}$ to a single point, and let $q : M \to L$ be the quotient mapping. If $<$ is the given linear order of M, define a linear order on $\mathcal{C}$ by the rule that for $J_1, J_2 \in \mathcal{C}$ we have $J_1 \prec J_2$ if and only if for every $x_i \in J_i$ we have $x_1 < x_2$. Then $\prec$ is a linear ordering of the set $\mathcal{C}$, and van Wouwe (Proposition 1.2.4 of [37]) has shown that the open interval topology of $\prec$ coincides with the quotient topology on L. Therefore L with its quotient topology and the order $\prec$ is a LOTS.

Define $f : L \to X$ by the rule that if $J \in \mathcal{C}$ then $f(J) = g(x)$ for any $x \in J$. Then $f \circ q = g$, making it easy to check that f is a continuous closed mapping with compact fibers, i.e., f is a perfect map. To complete the proof, suppose $J_a, J_b \in \mathcal{C}$ with $J_a \prec J_b$ and $f(J_a) = f(J_b)$. If for every $J \in \mathcal{C}$ with $J_a \prec J \prec J_c$ has $f(J) = f(J_a)$, then the entire segment of M from the least point of the compact set J_a to the greatest point of the compact set J_b is contained in a single fiber of g, showing that neither J_a nor J_b were maximal. Therefore, there must exist some $J_c \in (J_a, J_b)$ with $f(J_c) \neq f(J_a)$, as required. $\Box$

Corollary 7.7 *Suppose X is an non-metrizable space with a G_δ-diagonal. Then X is not in PILOTS. In particular, neither the Sorgenfrey line nor the Michael line are in PILOTS, even though, being GO-spaces, they are certainly in PIGO.* $\Box$

8 Questions

Most of the following questions ask whether a result known for GO-spaces can be proved for members of PIGO or for monotonically normal spaces in general. The references mentioned in several of the questions refer to the paper in which the corresponding result for GO-spaces can be found.

Q1) Proposition 5.4 above shows that any member of PIGO having a σ-closed-discrete dense set must be perfect. The converse is false if there is a Souslin space (= a non-separable GO-space having countable cellularity), the existence of which is independent of ZFC. Assuming there is no Souslin space, is it true that every perfect member of PIGO has a σ-closed-discrete dense subset?

Q2) Can 5.4 be generalized to say that every monotonically normal space with a σ-closed-discrete dense subset must be perfect?

Q3) Suppose X is in PIGO and has a σ-minimal base[1]. Is X paracompact? [10]

Q4) Suppose $X \in PIGO$ has an OIF (= open-in-finite) base. Is X metrizable? [12]

Q5) Suppose $X \in PIGO$ is a perfect space (= closed subsets of X are G_δ-sets). Is there a perfect GO-space M and a perfect mapping $g : M \to X$?

Q6) Suppose X is in PIGO and is separable. Is X monotonically Lindelöf? [8].

Q7) Which members of PIGO have monotonically normal compactifications?

Q8) Is every member of PIGO orthocompact? [29]

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