SELECTED ORDERED SPACE PROBLEMS

HAROLD BENNETT AND DAVID LUTZER

1. Introduction

A *generalized ordered space* (a GO-space) is a triple $(X, \tau, <)$ where $(X, <)$ is a linearly ordered set and τ is a Hausdorff topology on X that has a base of order-convex sets. If τ is the usual open interval topology of the order $<$, then we say that $(X, \tau, <)$ is a *linearly ordered topological space* (LOTS). Besides the usual real line, probably the most familiar examples of GO-spaces are the Sorgenfrey line, the Michael line, the Alexandroff double arrow, and various spaces of ordinal numbers. In this paper, we collect together some of our favorite open problems in the theory of ordered spaces. For many of the questions, space limitations restricted us to giving only definitions and references for the question. For more detail, see [5]. Notably absent from our list are problems about orderability, about products of special ordered spaces, about continuous selections of various kinds, and about C_p-theory, and for that we apologize. Throughout the paper, we reserve the symbols $\mathbb{R}, \mathbb{P}$, and $\mathbb{Q}$ for the usual sets of real, irrational, and rational numbers respectively.

2. A few of our favorite things

The most important open question in GO-space theory is Maurice's problem, which Qiao and Tall showed [19] is closely related to several other old questions of Heath and Nyikos [5]. Maurice asked whether there is a ZFC example of a perfect GO-space that does not have a σ-closed-discrete dense subset. A recent paper [9] has shown that a ZFC example, if there is one, cannot have local density $\leq \omega_1$, and what remains is:

Question 1. *Let $\kappa > \omega_1$ be a cardinal number. Is it consistent with* ZFC *that any perfect GO-space with local density $\leq \kappa$ must have a σ-closed-discrete dense set? Equivalently, is it consistent with* ZFC *that every perfect non-Archimedean space with local density $\leq \kappa$ is metrizable? Is it consistent with* ZFC *that every perfect GO-space with local density $\leq \kappa$ and with a point-countable base is metrizable?* **1001 ?**

Question 2 (The GO-embedding problem)**.** *Let $\kappa > \omega_1$ be a cardinal number. Is it consistent with* ZFC *that every perfect GO-space X with local density $\leq \kappa$ embeds in some perfect LOTS? (Note: the embedding map h is not required to be monotonic and $h[X]$ is not required to be open, or closed, or dense, in the perfect LOTS.)* **1002 ?**

Let $\mathcal{M}$ be the class of all metric spaces. A space X is *cleavable over* $\mathcal{M}$ [1] if for each subset $A \subseteq X$, there is a continuous f_A from X into some member of $\mathcal{M}$ such that $f_A(x) \neq f_A(y)$ for each $x \in A$ and $y \in X \setminus A$. The property *cleavable over* $\mathcal{S}$ is analogously defined, where $\mathcal{S}$ is the class of all separable metrizable spaces. It is known [2] that the following properties of a GO space X are equivalent: (a) X is cleavable over $\mathcal{M}$; (b) X has a weaker metrizable topology; (c) X has a G_δ-diagonal; (d) there is a σ-discrete collection $\mathcal{C}$ of cozero subsets of X such that if $x \neq y$ are points of X, then some $C \in \mathcal{C}$ has $|C \cap \{x, y\}| = 1$.

For a GO-space X with cellularity $\leq \mathfrak{c}$, each of the following is equivalent to cleavability of X over $\mathcal{S}$: (a) X has a weaker separable metric topology; (b) X has a countable, point-separating cover by cozero sets; (c) X is *divisible by cozero sets*, i.e., for each $A \subseteq X$, there is a countable collection $\mathcal{C}_A$ of cozero subsets of X with the property that given $x \in A$ and $y \in X \setminus A$, some $C \in \mathcal{C}_A$ has $x \in C \subseteq Y \setminus \{y\}$. However, for each $\kappa > \mathfrak{c}$ there a LOTS X that is cleavable over $\mathcal{S}$ and has $c(X) = \kappa$ (Example 4.8 of [2]). Therefore we have:

? 1003 **Question 3.** *Characterize GO-spaces that are cleavable over $\mathcal{S}$ without imposing restrictions on the cardinal functions of X.*

? 1004 **Question 4.** *Characterize GO-spaces that are divisible by open sets (in which the collection $\mathcal{C}$ mentioned in (c) above consists of open sets, but not necessarily cozero-sets).*

? 1005 **Question 5.** *Characterize GO spaces that are* cleavable over $\mathbb{R}$*, i.e., in which the cleaving functions f_A can always be taken to be mappings into $\mathbb{R}$.*

Cleavability over $\mathbb{P}$ and $\mathbb{Q}$ has already been characterized in [2]. For compact, connected LOTS, see [12].

? 1006–1007 **Question 6.** *Arhangelskii has asked whether a compact Hausdorff space X that is cleavable over some LOTS (or GO-space) L must itself be a LOTS. What if X is zero-dimensional?*

Arhangelskii proved that if a compact Hausdorff space X is cleavable over $\mathbb{R}$, then X is embeddable in $\mathbb{R}$. Buzyakova [12] showed the same is true if $\mathbb{R}$ is replaced by the lexicographic product space $\mathbb{R} \times \{0, 1\}$.

? 1008 **Question 7** (Buzyakova)**.** *Is a compact Hausdorff space X that is cleavable over an infinite homogeneous LOTS L must be embeddable in L?*

A topological space X is *monotonically compact* (resp., *monotonically Lindelöf*) if for each open cover $\mathcal{U}$ it is possible to choose a finite (resp., countable) open refinement $r(\mathcal{U})$ such that if $\mathcal{U}$ and $\mathcal{V}$ are any open covers of X with $\mathcal{U}$ refining $\mathcal{V}$, then $r(\mathcal{U})$ refines $r(\mathcal{V})$. It is known that any compact metric space is monotonically compact and that any second countable space is monotonically Lindelöf and that any separable GO-space is hereditarily monotonically Lindelöf [11].

? 1009–1012 **Question 8.** *Is every every perfect monotonically Lindelöf GO-space separable? Is every hereditarily monotonically Lindelöf GO-space separable? If there is a*

Souslin line, is there a compact Souslin line that is hereditarily monotonically Lindelöf and is there is a Souslin line that is not monotonically Lindelöf?

Question 9. *If Y is a subspace of a perfect monotonically Lindelöf GO-space X, must Y be monotonically Lindelöf?* **1013 ?**

Question 10. *(a) Is the lexicographic product space $X = [0,1] \times \{0,1\}$ monotonically compact? (b) If X is a first-countable compact LOTS, is X monotonically Lindelöf? (c) If X is a first-countable monotonically compact LOTS, is X metrizable?* **1014–1016 ?**

Studying properties of a space X off of the diagonal means studying properties of the space $X^2 \setminus \Delta$. For example one can show that a GO-space X is separable if and only $c(X^2 \setminus \Delta) = \omega$.

Question 11 ([8])**.** *Is it true that a GO-space is separable if $X^2 \setminus \Delta$ has a dense Lindelöf subspace? If X is a Souslin space, can $X^2 \setminus \Delta$ have a dense Lindelöf subspace?* **1017–1018 ?**

Question 12. *In* ZFC*, is there a non-metrizable, Lindelöf LOTS X that has a countable rectangular open cover of $X^2 \setminus \Delta$ (i.e., a collection $\{U_n \times V_n : n < \omega\}$ of basic open sets in X^2 with $\bigcup\{U_n \times V_n : n < \omega\} = X^2 \setminus \Delta$)?* **1019 ?**

Under CH or $\mathfrak{b} = \omega_1$ the answer to Question 12 is affirmative [7].

Question 13. *Suppose X is a LOTS that is first-countable and hereditarily paracompact off of the diagonal (i.e., $X^2 \setminus \Delta$ is hereditarily paracompact). Must X have a point-countable base? Is it possible that a Souslin space can be hereditarily paracompact off of the diagonal?* **1020–1021 ?**

We note that if one considers GO-spaces rather than LOTS in Question 13, then there is a consistently negative answer. Under CH, Michael [17] constructed an uncountable dense-in-itself subset X of the Sorgenfrey line $\mathbb{S}$ such that X^2 is Lindelöf. Because $\mathbb{S}^2$ is perfect, X^2 is perfect and Lindelöf, i.e., hereditarily Lindelöf. But X cannot have a point-countable base.

Let $\mathbb{P}_{\mathbb{S}}$ be the set of irrational numbers topologized as a subspace of the Sorgenfrey line. It is known [6] that X is domain-representable, being a G_δ-subset of the Sorgenfrey line. In fact, the Sorgenfrey line is representable using a Scott domain [13] .

Question 14. *Is the G_δ subspace $\mathbb{P}_{\mathbb{S}}$ of $\mathbb{S}$ also Scott-domain-representable?* **1022 ?**

Question 15 (Suggested by R. Buzyakova)**.** *Suppose X is a GO-space that is countably compact but not compact and that compact subset of X is a metrizable G_δ-subspace of X. Must X have a base of countable order* [23]*?* **1023 ?**

Mary Ellen Rudin [20] proved that every compact monotonically normal space is a continuous image of a compact LOTS. Combining her result with results of Nikiel proves a generalized Hahn-Mazurkiewicz theorem, namely that a topological space X is a continuous image of a compact, connected LOTS if and only if X is compact, connected, locally connected, and monotonically normal.

Question 16. *Is there an elementary submodels proof of the generalized Hahn-Mazurkiewicz theorem above?* **1024 ?**

A space X is *weakly perfect* if for each closed subset $C \subseteq X$ there is a G_δ-subset D of X with $D \subseteq C = \mathrm{cl}_X(D)$.

? 1025 **Question 17** ([3])**.** *Suppose Y is a subspace of a weakly perfect GO-space X. Must Y be weakly perfect?*

? 1026 **Question 18** ([4])**.** *Suppose that X is a Lindelöf GO-space that has a small diagonal and that can be p-embedded in some LOTS. Must X be metrizable?*

? 1027–1028 **Question 19** ([14])**.** *Let $<$ be the usual ordering of $\mathbb{R}$. For which subsets $X \subseteq \mathbb{R}$ is there a tree T and linear orderings of the nodes of T so that (a) no node of T contains an order isomorphic copy of $(X, <)$, and (b) $(X, <)$ is order isomorphic to the branch space of T? (Both $\mathbb{R}$ and $\mathbb{P}$ are representable in this way, but $\mathbb{Q}$ is not.) Which $F_{\sigma\delta}$-subsets of $\mathbb{R}$ are order isomorphic to the branch space of some countable tree?*

An *Aronszajn line* is a linearly ordered set that has cardinality ω_1, contains no order-isomorphic copy of ω_1, no copy of ω_1 with the reverse order, and no order isomorphic copy of an uncountable set of real numbers. Such things exist in ZFC.

? 1029–1030 **Question 20** ([15])**.** *Can an Aronszajn line be Lindelöf in its open interval topology without containing a Souslin line? If an Aronszajn line has countable cellularity in its open interval topology, must the Aronszajn tree from which the line comes contain a Souslin tree?*

? 1031 **Question 21** (Gruenhage and Zenor)**.** *Suppose X is a LOTS with a σ-closed-discrete dense set and a continuous function $\Psi\colon (X^2 \setminus \Delta) \times X \to \mathbb{R}$ such that if $x \neq y$ are points of X, then $\Psi(x, y)(x) \neq \Psi(x, y)(y)$. (Note that this is weaker than the existence of a continuous separating family.) Must X be metrizable?*

References

[1] Arhangelskii, A., The general concept of cleavability of a topological space, *Topology and its Appl.*, 44(1992), 27-36.(MR 1173240)

[2] Bennett, H., Byerly, R., and Lutzer, D., Cleavability in ordered spaces, *ORDER*, 18(2001), 1-17. (MR 1844513)

[3] Bennett, H., Heath, R., and Lutzer, D. GO-spaces with σ-closed-discrete dense sets, *Proceedings Amer. Math. Soc.*, 129(2000), 931-999.(MR 1707135)

[4] Bennett, H. and Lutzer,D., Diagonal conditions in ordered spaces, *Fundamenta Mathematicae* 153(1997), 99-123. (MR 1461451)

[5] Bennett, H. and Lutzer, D., Recent developments in the topology of ordered spaces, in *Recent Progress in General Topology II* ed. by M Hušek and J. van Mill, Elsevier, Amsterdam, 2002.(MR 1969994)

[6] Bennett, H. and Lutzer, D., Domain-representable spaces, *Fundamenta Mathematicae*, 189(2006), 255-268.(MR 2213622)

[7] Bennett, H. and Lutzer, D., Off-diagonal metrization theorems, *Topology Proceedings*, 22(1997), 37-58.(MR 1657898)

[8] Bennett, H. and Lutzer, D., Continuous separating families in ordered spaces and strong base conditions, *Topology and its Applications*, 119(2002), 305-314.(MR 1888675)

[9] Bennett, H. and Lutzer, D., On a question of Maarten Maurice, *Topology and its Applications*, 153(2006), 1631-1638.

[10] Bennett, H. and Lutzer, D., Metrizably fibered generalized ordered spaces, *Topology and its Applications*, 130(2003), 283-290. (MR 1978892)

[11] Bennett, H., Lutzer, D., and Matveev, M., The monotone Lindelöf property and separability in ordered spaces, *Topology and its Applications*, 151(2005), 180-186. (MR 2139751)

[12] Buzyakova, R., On cleavability of continua over LOTS, *Proceedings Amer. Math. Soc.*, 132(2004), 2171-2181.(MR 2053992)

[13] Duke, K., and Lutzer, D., Scott-domain-representability of certain ordered spaces, to appear.

[14] Funk, W., and Lutzer, D., Branch space representations of lines, *Topology and its Applications*, 151(2005), 187-214. (MR 2139752)

[15] Funk, W., and Lutzer, D., Lexicographically ordered trees, *Topology and it Applications*, 152(2005), 275-300.(MR 2164390)

[16] Gruenhage, G. and Zenor, P., Continuously Urysohn spaces, to appear.

[17] Michael, E., Paracompactness and the Lindelöf property in finite and countable Cartesian products, *Compositio Math*, 23(1971), 199-214.(MR 0287502)

[18] Qiao, Y-Q, Martin's Axiom does not imply that perfectly normal Non-Archimedean spaces are metrizable, *Proceedings Amer. Math. Soc.*, 129(2000), 1179-1188.(MR 1610777)

[19] Qiao, Y-Q and Tall, F., Perfectly normal non-metrizable non-Archimedean spaces are generalized Souslin lines, *Proceedings Amer. Math. Soc.*, 131(2003), 3929-3936. (MR 1999943)

[20] Rudin, M. E., Nikiel's conjecture, *Topology and it Applications* 116 (2001) 305–331. (MR 1857669)

[21] Shi, W-X, Extensions of perfect GO-spaces with σ-discrete dense sets, *Proc. Amer. Math. Soc.*, 127(1999), 615-618. (MR 1468203)

[22] Tkachuk, V., A glance at compact spaces which map "nicely" onto the metrizable spaces, *Topology Proc.*, 19(1994), 321-334. (MR 1369767)

[23] Wicke, H. and Worrell, J., Characterizations of developable topological spaces, *Canadian J. Math.*, 17(1965), 820-830. (MR 0182945)