A note on η_1-spaces

by

Harold Bennett, Texas Tech University, Lubbock, TX 79409

Sheldon Davis, University of Texas at Tyler, Tyler, TX 75799

David Lutzer, University of Illinois, Urbana, IL 61801

Abstract: In this paper we study some topological properties of η_1-spaces, i.e., topological spaces that use the open-interval topology of the η_1-sets that were introduced by Hausdorff more than a century ago. We focus on paracompactness, normality of products, topological completeness of various kinds, and certain generalized metric properties such as the existence of a small diagonal. In many cases, we find an intimate relation to the Continuum Hypothesis(CH), e.g., that (CH) is equivalent to the statement that if X is an η_1-space of cardinality 2^ω, then X^n is hereditarily paracompact and monotonically normal for every finite n, and we show that CH is equivalent to the statement that every η_1-space of cardinality 2^ω is realcompact. In addition, we investigate the role of Hušek's small diagonal property, showing that an η_1-space X has a small diagonal if and only if each subset $S \subseteq X$ with $|S| \leq \omega_1$ is closed. Consequently, under CH, no η_1-space with cardinality 2^ω can have a small diagonal, and we show that that if CH fails, then is is undecidable whether each free ultrapower $\mathbb{R}^\omega/\mathcal{U}$ must have a small diagonal.

Key words and phrases: η_1-set, η_1-space, paracompact, monotonically normal, Dieudonné complete, realcompact, A-metric space, Baire space, $C_p(X)$, small diagonal, ultrapower, Continuum Hypothesis

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In memoriam: We dedicate this paper to the memory of our friend and colleague Phillip Zenor whose work has deeply influenced our research and careers.

1 Introduction

The goal of this paper is to study linearly ordered topological spaces $(X, <, \tau)$ that use the open interval topology $\tau = \tau(<)$ of an η_1-set $(X, <)$, and we call such spaces <u>η_1-spaces</u>. Recall that Hausdorff [17] defined an <u>η_1-set</u> to be a linearly ordered set $(X, <)$ with the property that if A and B are countable subsets of X having $a < b$ for every $a \in A$ and every $b \in B$, then some $x \in X$ has $a < x < b$ for each $a \in A$ and $b \in B$. As noted in [13], every η_1-set has cardinality $\geq 2^\omega$, and we will say that an η_1-set (or an η_1-space) is <u>small</u> if its cardinality is exactly 2^ω. In Section 2, we give examples of small η_1-sets, two of which are obviously linearly ordered topological fields. The Continuum Hypothesis (CH) has a crucial role to play in the study of small η_1-sets and small η_1-spaces. See [13] for a proof of the following:

Theorem 1.1 *If $2^\omega = \omega_1$ and if $(X, <)$ and $(Y, \prec)$ are small η_1-sets, then $(X, <)$ and $(Y, \prec)$ are order isomorphic and consequently the associated η_1-spaces $(X, \tau(<))$ and $(Y, \tau(\prec))$ are homeomorphic. In addition, if $2^\omega = \omega_1$ then any small η_1-space is homeomorphic to a topological field.*

Without CH, the situation is much more complicated.

Theorem 1.2 *For any free ultrafilter $\mathcal{U}$ on ω, the* <u>*ultrapower*</u> $\mathbb{R}^\omega/\mathcal{U}$ *(see 2.3 for definitions)is a small η_1-set and*

a) in any model of ZFC, either there is exactly one order-isomorphism class of ultrapowers $\mathbb{R}^\omega/\mathcal{U}$ or else there are 2^{2^ω} many non-isomorphic ultrapowers [11];

b) there is a model of set theory that satisfies $\omega_1 < 2^\omega$ in which there are different free ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ on ω such that the ultrapowers $\mathbb{R}^\omega/\mathcal{U}_1$ and $\mathbb{R}^\omega/\mathcal{U}_2$ are not order isomorphic and not homeomorphic [33], [7];

c) there is a model of set theory with the property that $\omega_1 < 2^\omega$ and there is a set Ψ containing 2^ω-many free ultrafilters on ω with the property that if $\mathcal{U}, \mathcal{V}$ are different members of Ψ, then the ultrapowers $\mathbb{R}^\omega/\mathcal{U}$ and $\mathbb{R}^\omega/\mathcal{V}$ have different cofinalities and are therefore not order isomorphic and not homeomorphic [1]. See also [5].

In later sections we will show that CH is equivalent to various topological statements about small η_1-spaces, e.g.,

Theorem 1.3 *The Continuum Hypothesis is equivalent to each of the following statements:*

a) for any small η_1-space X and any $n < \omega$, X^n is hereditarily paracompact (see 3.2);

b) for any small η_1-space X and any $n < \omega$, X^n is monotonically normal (see 4.8);

c) any small η_1-space is realcompact(see 5.5);

d) any small η_1-space is Dieudonné-complete(see 5.5);

e) for any small η_1-space X, the space X^2 is normal (see 4.9);

f) for any small η_1-spaces X and Y, the space $X \times Y$ is normal (see 4.9).

In Section 5, we examine properties related to completeness in η_1-spaces, e.g., the Baire category property, de Groot's subcompactness, Oxtoby's pseudocompleteness, and Choquet's weak α-favorably, and we show that any product of η_1-spaces is a Baire space. In Section 6 we examine Hušek's small diagonal property in η_1-spaces. We show that an η_1-space has a small diagonal if and only if each subset S with $|S| \le \omega_1$ is closed, and that in ZFC there are η_1-spaces that have a small diagonal (see 6.3) and other η_1-spaces that do not, but these examples are not small. In addition, the Continuum Hypothesis guarantees that no small η_1-space can have a small diagonal (see 6.4). In models where CH fails, the situation can be more complicated. We show in Example 6.7 that if CH fails, then there must be a small η_1-space (which is not an ultrapower) that does not have a small diagonal, and that if CH fails and we restrict attention to ultrapowers $\mathbb{R}^\omega/\mathcal{U}$ then it is undecidable whether each ultrapower $\mathbb{R}^\omega/\mathcal{U}$ must have a small diagonal.

2 Examples and basic properties of η_1-sets and spaces

To paraphrase Gillman and Jerison (p.177 of [13]), it is not at all obvious that η_1-sets exist. We begin with a few well-known examples that will be important in later sections.

Example 2.1 *Example via lexicographic orderings*

Consider the product set $X = \{0,1\}^{\omega_1}$ and its subset $Y = \{f \in X : \text{ for some } \alpha < \omega_1,\ f(\alpha) = 0 \text{ and for all } \beta \in (\alpha, \omega_1), f(\beta) = 1\}$. Use the lexicographic ordering $<$ on the set Y. Then $(Y, <)$ is an η_1-set with cardinality 2^ω. See pp. 185-189 of [13]. $\square$

Example 2.2 *Example via quotient rings*

Let $\mathbb{R}$ denote the usual space of real numbers and $C(\mathbb{R})$ be the ring of continuous functions from $\mathbb{R}$ to itself. Let $\mathcal{Z}$ be the collection of all zero-sets of functions in $C(\mathbb{R})$. There is an ultrafilter $\mathcal{F}$ contained in $\mathcal{Z}$ that contains the collection $\{[n, \rightarrow) : n < \omega\}$. Then the collection $M = \{f \in C(\mathbb{R}) : f^{-1}[0] \in \mathcal{F}\}$ is a maximal ideal in $C(\mathbb{R})$ so that the quotient ring $C(\mathbb{R})/M$ is a linearly ordered field. (Note that two cosets $f + M$ and $g + M$ are equal if and only if $f - g \in M$, i.e., $f(x) = g(x)$ for all x in some member of $\mathcal{F}$, and $f + M \prec g + M$ if and only if $f(x) < g(x)$ for all x in some member of $\mathcal{F}$.) This field contains a copy of the usual field of real numbers, namely $\{\hat{r} + M : r \in \mathbb{R}\}$ where $\hat{r} : \mathbb{R} \rightarrow \mathbb{R}$ is the constant function with value r. This field also contains the coset $i + M$ where $i(x) = x$ for all $x \in \mathbb{R}$ so that for each $r \in \mathbb{R}$ we have $\hat{r} + M < i + M$ and therefore $\{\hat{r} + M : r \in \mathbb{R}\}$ is a proper subset of $C(\mathbb{R})/M$ and $C(\mathbb{R})/M$ is non-Archimedian. This shows that, in the terminology of [13], M is a hyper-real ideal and $C(\mathbb{R})/M$ is a hyper-real field with cardinality 2^ω. See [13] for a proof that the field $C(\mathbb{R})/M$ is an η_1-set. $\square$

Example 2.3 *Example via ultrapowers*

Let $\mathcal{U}$ be any free ultrafilter of subsets of ω. Define that two functions $f, g \in \mathbb{R}^\omega$ have $f \equiv g$ provided $f(x) = g(x)$ for every x in some member of $\mathcal{U}$ and that two of the resulting equivalence classes have $cls(f) < cls(g)$ provided $f(x) < g(x)$ for all x in some member of $\mathcal{U}$. The resulting set of equivalence classes $\{cls(f) : f \in \mathbb{R}^\omega\}$ is the ultrapower $\mathbb{R}^\omega/\mathcal{U}$ and it follows from Los' theorem (or can easily be proved directly) that with point-wise operations, $\{cls(f) : f \in \mathbb{R}^\omega\}$ is a linearly ordered field with cardinality 2^ω and is an η_1-set. $\square$

While small η_1-sets (i.e., η_1-sets with cardinality 2^ω) are particularly interesting, there is no limit on the cardinality of η_1-sets.

Example 2.4 *There are η_1-sets with arbitrarily large cardinality.*

Proof: We give three examples. Fix a regular initial ordinal $\kappa \geq 2^\omega$ and an η_1-space X, and let Y_1 be the lexicographic product $\kappa \times X$. A more interesting example starts with $\kappa \geq 2^\omega$ and has $Y_2 = \{f \in \{0,1\}^\kappa : \text{some } \alpha < \kappa \text{ has } f(\alpha) = 0 \text{ and for each } \beta \in (\alpha, \kappa), f(\beta) = 1\}$. Then $|Y| \geq \kappa$. Use the lexicographic ordering $\prec$ of Y_2. If $A, B \subseteq Y_2$ have cardinality less than κ and have $a < b$ for each $a \in A$ and $b \in B$, then some $x \in X$ lies strictly between the sets A and B. Taking A and B to have cardinality ω we see that $(Y_2, \prec)$ is an η_1-set. See problem 13Q in [13]. As a third alternative, let $\mathcal{U}$ be a regular[1] ultrafilter on a non-measurable cardinal κ. Then the cardinality of the ultrapower $Y_3 = \mathbb{R}^\kappa/\mathcal{U}$ is $|\mathbb{R}^\kappa|$ (see [12]) and as in Example 2.3, this space is an η_1-set (or see Theorem 5.6 in [22]). $\square$

The next lemma lists basic properties of η_1-sets and η_1-spaces. Proofs of a), b), and c) in the following lemma can be found in [13] . Assertion d) follows from the definition of an η_1-set because if (a_n, b_n) is a decreasing sequence of open intervals in an η_1-space, then $\bigcap\{(a_n, b_n) : n < \omega\} \neq \emptyset$. Assertion e) follows from the fact that any η_1-space is a linearly ordered topological space (= LOTS); for details, see [25]. The proof of assertion f) is a modification of the proof of assertion c) found in [13]. Assertion g) holds because no strictly increasing countable sequence in an η_1-space X can have a limit in X.

Lemma 2.5 *Suppose $(X, <)$ is an η_1-set and τ is the open interval topology of $<$. Then:*

a) $(X, <)$ has no endpoints and no jumps (i.e., is dense ordered);

b) each countable linearly ordered set order-embeds in $(X, <)$;

c) the cardinality of X has $|X| \geq 2^\omega$;

d) every G_δ-subset of the space (X, τ) is open;

e) The space (X, τ) is hereditarily collectionwise normal, monotonically normal [18], and hereditarily countably paracompact;

f) there is a partition $\mathcal{J}$ of X where $\mathcal{J} \subseteq \tau$ and $|\mathcal{J}| = 2^\omega$ and therefore any topological space Y with $|Y| \leq 2^\omega$ is a continuous image of (X, τ);

g) No η_1-space can be Lindelöf, first-countable, or metrizable, and every (countably) compact subset of an η_1-space is finite. Because no connected subset of an η_1-space can have more than one point, no η_1-space can be connected or locally connected.

3 Paracompactness in η_1-spaces

As noted in Lemma 2.5, no η_1-space can be Lindelöf, but there are other important covering properties that an η_1-space might have. In this section, we consider the property of paracompactness. (Because any η_1-space X is a LOTS, paracompactness in X is equivalent to a family of other covering properties such as "every open cover has a point-countable open refinement".) We begin with an example showing that an η_1-space can fail to be paracompact.

Example 3.1 *In ZFC there is a non-paracompact η_1-space whose cardinality is $max(2^\omega, \omega_2)$.*

Proof: Let $(X, <, \tau)$ be the η_1-field of Example 2.3 and let 0 denote the zero element of X. We know that $|X| = 2^\omega$. Let S be any stationary set in ω_2 with the property that each $\alpha \in S$ has cofinality ω_1. (For example, the set $S = \{\alpha \in \omega_2 : cf(\alpha) = \omega_1\}$ is such a set.) Then $|S| = \omega_2$. Let

$$Y(S) = \{(\alpha, x) \in [0, \omega_2) \times X : \mathrm{cf}(\alpha) < \omega_1\} \cup (S \times \{0\}).$$

[1] An ultrafilter $\mathcal{U}$ on κ is *regular* if there is a subcollection $\mathcal{E} \subseteq \mathcal{U}$ that has $|\mathcal{E}| = \kappa$ and is point-countable. For any κ there is a regular ultrafilter on κ and provided $\mathcal{U}$ is a regular ultrafilter, the ultraproduct $\mathbb{R}^\kappa/\mathcal{U}$ has cardinality c^κ [12].

Let $\prec$ be the lexicographic ordering of $Y(S)$. Then $(Y(S), \prec)$ is an η_1-set and the set $Z = \{(\alpha, 0) \in Y(S) : \alpha \in S\}$ is a closed subspace of $(Y(S), \prec, \tau)$ that is homeomorphic to the stationary subspace S of ω_2. Therefore $(Y(S), \prec, \mathcal{T})$ cannot be paracompact [10]. Note that $|Y(S)| = \max(\omega_2, 2^\omega)$ and that this example makes no assumption about the relative sizes of ω_2 and 2^ω. $\square$

Recall that a small η_1-space is an η_1-space with the smallest possible cardinality, i.e., with cardinality 2^ω.

Proposition 3.2 *The following are equivalent:*

a) the Continuum Hypothesis(CH);

b) every small η_1-space is hereditarily paracompact;

c) every small η_1-space is paracompact;

d) for every small η_1-space X, X^n is hereditarily paracompact for each $n < \omega$.

Proof: To prove a) $\Rightarrow$ b), note that if the LOTS $(X, <, \tau)$ is not hereditarily paracompact, then there is a stationary subspace of some regular uncountable cardinal κ that embeds as a subspace of X [10]. Because X is small, (CH) gives $|X| = 2^\omega = \omega_1$ so that $\kappa = \omega_1$. Therefore the stationary subspace S must contain non-trivial convergent sequences. But X is an η_1-space so that X does not contain any nontrivial convergent sequences. This contradiction shows that b) holds.

That b) implies c) is trivial. To prove that c) implies a), consider the contrapositive and suppose a) fails, i.e., suppose $\omega_1 < 2^\omega$. Then $\omega_2 \leq 2^\omega$ so that $\max(\omega_2, 2^\omega) = 2^\omega$. Therefore the space of Example 3.1 is a non-paracompact η_1-space of cardinality 2^ω, contradicting c). Hence c) implies a).

Obviously assertion d) implies assertion b), and therefore implies assertion a). To see that a) implies d), we use Corollary 4.8 below which shows that every ultrapower of $\mathbb{R}$ has $(\mathbb{R}^\omega/\mathcal{U})^n$ hereditarily paracompact for every $n < \omega$ and then use the fact that, under CH, every small η_1-space X is homeomorphic to the ultrapower $\mathbb{R}^\omega/\mathcal{U}$ (see Theorem 1.1), so that X^n is hereditarily paracompact. $\square$

Proposition 3.2 shows that the small η_1-spaces of Examples 2.1, 2.2, and 2.3 are paracompact provided we assume CH. But even more is true: the spaces of Examples 2.2 and 2.3 are paracompact in ZFC, and so are their finite powers, as will be shown in the next section (see Corollary 4.8).

Remark 3.3 There is an alternate proof of a) implies c) in our Proposition 3.2 based on Bankston's Theorem 6.1 in [2]. The ultraproduct of a collection of topological spaces $\{(X_i, \mathcal{T}_i) : i \in \omega\}$ with respect to an ultrafilter $\mathcal{U}$ on ω is formed as follows. Let Y be the usual product $Y = \Pi\{X_i : i < \omega\}$ of the sets X_i and let q be the quotient function from Y to the ultraproduct set $\Pi\{X_i : i \in \omega\}/\mathcal{U}$. Next, endow Y with the box-product topology and use the quotient topology defined by q for the set $\Pi\{X_i : i \in \omega\}/\mathcal{U}$. This gives the ultraproduct of the spaces $(X_i, \mathcal{T}_i)$. One can check that if each X_i is the usual linearly ordered space $\mathbb{R}$ of real numbers, then Bankston's ultraproduct topology coincides with the LOTS topology on $\mathbb{R}^\omega/\mathcal{U}$ described in Example 2.3. Bankston proved that the Continuum Hypothesis is equivalent to the statement "Provided each space $(X_i, \mathcal{T}_i)$ is regular and has weight $\leq 2^\omega$, then for any ultrafilter $\mathcal{U}$ on ω, the topological ultraproduct $\Pi\{(X_i, \mathcal{T}_i) : i \in \omega\}/\mathcal{U}$ is paracompact." Thus the Continuum Hypothesis (which is statement (a) of 3.2) implies that the ultrapower $Y = \mathbb{R}^\omega/\mathcal{U}$ of Example 2.3 is paracompact, and then Theorem 1.1 shows that, under CH, every η_1-space of cardinality $2^\omega = \omega_1$ is homeomorphic to Y and consequently is paracompact.

4 Products of ultrapowers, η_1-spaces, and A-metric spaces

In this section we study finite powers of η_1-spaces, forcing on the properties of normality, paracompactness, and monotone normality. We use these results in Proposition 3.2 in the previous section, and to prove parts b), e), and f) of Theorem 1.3.

Because any η_1-space X is a LOTS, any η_1-space X is normal, collectionwise normal, and monotonically normal (see Definition 4.6 below). We begin with an example showing that squares of η_1-spaces can fail to be normal.

Example 4.1 *In ZFC, there is an η_1-space with cardinality $\max(\omega_2, 2^\omega)$ whose square is not normal.*

Proof: Any stationary subset of ω_2 can be split into two disjoint stationary subsets[2]. Let S and T be two disjoint stationary subsets of ω_2 with the property that each $\alpha \in S \cup T$ has cofinality ω_1. As in Example 3.1, make η_1-sets $X(S)$ and $X(T)$ that contain, respectively, copies of S and T as closed subsets. Let $Y = \Big(\{0\} \times X(S)\Big) \cup \Big(\{1\} \times X(T)\Big)$ be the lexicographically ordered η_1-set where each element of $X(S)$ precedes each element of $X(T)$. Then $|Y| = \max(\omega_2, 2^\omega)$ and $S \times T$ is a closed subset of Y^2. Because S and T are disjoint stationary subsets of ω_2 we know that $S \times T$ is not normal [24]. Therefore, neither is Y^2. $\square$

However, the situation is completely different if the η_1-space X is a linearly ordered topological field such as $C(\mathbb{R})/M$ in Example 2.2 or an ultrapower $\mathbb{R}^\omega/\mathcal{U}$ as in Example 2.3. Our arguments involve a metric-like structure that we call an A-metric where $(A, +, <)$ is a linearly ordered Abelian group. This idea has been studied under many names: see Remark 4.10, below.

Definition 4.2 *Let $(A, +, <)$ be a linearly ordered Abelian group. By an <u>A-metric</u> on a set X, we mean a function $d : X \times X \to A$ that satisfies the usual metric-like properties, i.e., for each $x, y, z \in X$,*

a) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;

b) $d(x, y) = d(y, x)$;

c) $d(x, z) \leq d(x, y) + d(y, z)$.

An <u>A-metric space</u> is a topological space X with an A-metric $d : X \times X \to A$ having the property that if $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ (where $\epsilon \in A$ and $\epsilon > 0$), then $\{B(x, \epsilon) : \epsilon \in A, \epsilon > 0\}$ is an open neighborhood base at x.

Lemma 4.3 *Suppose X is a A-metric space. Then so is every subspace of X and every product space X^n for finite $n \geq 1$.*

[2] In fact, there is a collection of ω_2-many pairwise disjoint stationary subsets of a given stationary set such as $\{\alpha < \omega_2 : cf(\alpha) = \omega_1\}$. See Theorem 12.5 of [15] for Solovay's theorem [35], or use a slight modification of the Ulam matrix technique described in [31] [36].

Proof: For a subspace $Y \subseteq X$, simply restrict the given A-metric function to the subset Y^2 of X^2. For the product space X^n for a finite n, define a function $D : X^n \times X^n \to A$ by the rule

$$D((x_1, \cdots, x_n), (y_1, \cdots, y_n)) = d(x_1, y_1) + \cdots + d(x_n, y_n)$$

where $d : X \times X \to A$ is the given A-metric for X. $\Box$

Lemma 4.4 *For each $n < \omega$, each of the following is an A-metic space:*

a) the space $(\mathbb{R}^\omega/\mathcal{U})^n$ where $\mathcal{U}$ is any free ultrafilter on ω;

b) the space $\Big(C(\mathbb{R})/M\Big)^n$ with $C(\mathbb{R})$ and M as in Example 2.2;

c) the space X^n where X is any linearly ordered topological Abelian group X.

Proof: To prove (a), write $X = \mathbb{R}^\omega/\mathcal{U}$. Note that $(X, +, <)$ is a linearly ordered Abelian group and define $d : X \times X \to X$ by the rule that $d(a, b) = b - a$ if $a \le b$ and $d(a, b) = a - b$ otherwise. It is easy to verify that d is an A-metric compatible with the open-interval topology of X so that by Lemma 4.3 the space X^n is also an A-metric space for each finite n. The proofs of (b) and (c) are analogous. $\Box$

Many of the proofs in the study of A-metric spaces closely resemble the proofs of analogous statements about metric spaces, provided one takes a little extra care, e.g., when trying to compute the distance from a point of X to a subset of X. We give an example in the next Lemma and Proposition. Suppose X is an A-metric space with $d : X \times X \to A$, where $(A, +, <)$ is a linearly ordered Abelian group. Let A^+ be the Dedekind completion of the linearly ordered set $(A, <)$. Because the ordering of A^+ extends the ordering $<$ of A, we use the same symbol $<$ to denote the orderings of both A and A^+. For any nonempty $C \subset X$ and any $x \in X$, let

$$dist(x, C) = \inf\{d(x, a) : a \in C\}$$

where the infimum is taken in the set A^+. Then $dist(x, C) \in A^+$ and can be compared to various elements of A or of A^+, as in statements like $\epsilon < dist(x, C)$ and $dist(x, C) < dist(x, D)$. However $dist(x, C)$ is probably not an element of the group A so that operations such as $a + dist(b, C)$ or $2 * dist(b, C)$ are probably not defined. Nevertheless, with a little care we can avoid such problems.

Lemma 4.5 *Let X be an A-metric space over the linearly ordered Abelian group $(A, +, <)$ and let $d : X \times X \to A$ be an A-metric as defined above. Let $C \subseteq X$ and $\epsilon > 0$ in A. Then:*

a) the set $V = \{x \in X : dist(x, C) > \epsilon\}$ is open so that $\{x \in X : dist(x, C) \le \epsilon\}$ is closed;

b) the set $W = \{x \in X : dist(x, C) < \epsilon\}$ is open so that $\{x \in X : dist(x, C) \ge \epsilon\}$ is closed.

Proof: If the set $P = \{a \in A : a > 0\}$ has a first element ϵ_0, then for each $x \in X$, the set $B(x, \epsilon_0) = \{x\}$ so that X is discrete and therefor assertions a) and b) are trivial. So assume that P has no first element. Let $x_0 \in V$, so that $dist(x_0, C) > \epsilon$. Because $(A, <)$ has no jumps, there are points $\alpha, \beta \in A$ having $dist(x_0, C) > \alpha > \beta > \epsilon$. Then for all $z \in C$, $d(x_0, z) \ge dist(x_0, C) > \alpha > \beta > \epsilon$. Let $\delta = \alpha - \beta$. Then $\delta > 0$. Consider any $y \in B(x_0, \delta)$. Then $d(x_0, y) < \delta$. Suppose there is some

$z_0 \in C$ having $d(y, z_0) \leq \beta$. Because $z_0 \in C$, $d(x_0, z_0) \geq dist(x_0, C) > \alpha > \beta > \epsilon$. The triangle inequality gives

$$d(x_0, z_0) \leq d(x_0, y) + d(y, z_0) \leq d(x_0, y) + \beta < \delta + \beta = (\alpha - \beta) + \beta = \alpha < dist(x_0, C) \leq d(x_0, z_0)$$

so that $d(x_0, z_0) < d(x_0, z_0)$ and that is impossible. Therefore, for each $z \in C$ we have $d(y, z) > \beta > \epsilon$ showing that $B(x_0, \delta) \subseteq V$. Therefore V is open and its complement is closed.

Next consider any $x \in W$. Then $dist(x, C) < \epsilon$ so there are points $g, h \in A$ with $dist(x, C) < h < g < \epsilon$. Let $\delta = g - h$ and suppose $y \in B(x, \delta)$. There is some $c \in C$ with $dist(x, C) \leq d(x, c) < h$. Then

$$d(y, a) \leq d(y, x) + d(x, c) < \delta + h = (g - h) + h = g < \epsilon$$

showing that $B(x, \delta) \subseteq W$. Therefore W is open, and its complement is closed. □

There is a strengthening of normality called monotone normality that was introduced by Borges and studied in [18]. Monotone normality is a property shared by many classes of generalized metric spaces, and also by ordered spaces.

Definition 4.6 *A space X is monotonically normal if for each pair (C, D) of disjoint closed sets in X, there is an open set $G(C, D)$ with the property that $C \subseteq G(C, D) \subseteq cl\Big(G(C, D)\Big) \subseteq X - D$ and with the monotonicity property that if (C_1, D_1) and (C_2, D_2) are pairs of disjoint closed sets with $C_1 \subseteq C_2$ and $D_1 \supseteq D_2$, then $G(C_1, D_1) \subseteq G(C_2, D_2)$.*

The next result is already known (see p. 283 of [28]). Our proof is particularly transparent.

Proposition 4.7 *Every A-metric space is monotonically normal .*

Proof: Suppose (C, D) is a pair of disjoint closed sets in the A-metric space X. Define

$$G(C, D) = \{x \in X : dist(x, C) < dist(x, D)\}.$$

Then $G(C, D)$ is open because if $x_0 \in G(C, D)$ we may choose $\epsilon \in A$ with $dist(x_0, C) < \epsilon < dist(x_0, D)$. Apply both parts of Lemma 4.5 to find a δ with the property that

$$B(x_0, \delta) \subseteq \{x \in X : dist(x, C) < \epsilon\} \cap \{x \in X : \epsilon < dist(x, D)\}.$$

Then $B(x_0, \epsilon) \subseteq G(C, D)$, as required. Next $C \subseteq G(C, D)$ because if $x \in C$ then x is not in the closed set D so we have $dist(x, C) = 0 < dist(x, D)$. Further, $G(C, D) \cap G(D, C) = \emptyset$ so that $cl_X(G(C, D)) \subseteq X - D$. Finally, suppose (C_1, D_1) and (C_2, D_2) are pairs of disjoint closed sets with $C_1 \subseteq C_2$ and $D_2 \subseteq D_1$. We must show that $G(C_1, D_1) \subseteq G(C_2, D_2)$. For contradiction, suppose that is not true, and choose $x \in G(C_1, D_1) - G(C_2, D_2)$. Then we have

(1) $dist(x, C_1) < dist(x, D_1)$ and

(2) $dist(x, D_2) \leq dist(x, C_2)$

Because $C_1 \subseteq C_2$ and $D_2 \subseteq D_1$ we know that

(3) $dist(x, C_2) \leq dist(x, C_1)$ and $dist(x, D_1) \leq dist(x, D_2)$.

To verify the first statement of (3) note that $\{d(x,a) : a \in C_1\} \subseteq \{d(x,a) : a \in C_2\}$ so that $\inf(\{d(x,a) : a \in C_2\}) \leq inf(\{d(x,a) : a \in C_1\})$, i.e., $dist(x, C_2) \leq dist(x, C_1)$. The second statement in (3) is proved analogously. But then we have

$$dist(x, D_2) \leq dist(x, C_2) \leq dist(x, C_1) < dist(x, D_1) \leq dist(x, D_2)$$

which gives $dist(x, D_2) < dist(x, D_2)$ and that is impossible. □

We can now summarize the main results in this section as follows.

Corollary 4.8 *If X is the ultrapower $\mathbb{R}^\omega/\mathcal{U}$ where $\mathcal{U}$ is a free ultrafilter on ω or if X is a linearly ordered topological field such as $C(\mathbb{R})/M$ of Example 2.2, then:*

a) in ZFC, for each $n < \omega$, X^n is hereditarily paracompact and monotonically normal;

b) under the Continuum Hypothesis (CH), if X is any small η_1-space (i.e., $|X| \leq 2^\omega$), then every finite power of X is hereditarily paracompact and monotonically normal.

Proof: To prove assertion (a), fix $n \geq 1$. From Proposition 4.3 we know that each X^n is an A-metric space where A is the linearly ordered Abelian group $A = \mathbb{R}^\omega/\mathcal{U}$. Therefore X^n is monotonically normal by Proposition 4.7. Next note that X^n is a topological vector space over the field $F = \mathbb{R}^\omega/\mathcal{U}$ (and therefore an additive group), and recall the theorem of Buziakova and Vural [6] which states that any monotonically normal topological group is hereditarily paracompact. Therefore X^n is hereditarily paracompact. A second proof of assertion (a) is based on the theorem of Balogh and Rudin [3] which states that a monotonically normal space is hereditarily paracompact if and only if it contains no subspace that is homeomorphic to a stationary set in an uncountable regular initial ordinal. Given the Balogh-Rudin theorem, combined with the easy proof that no stationary subset of an uncountable regular initial ordinal can be an A-metric space, Proposition 4.7 completes the proof of the first part of assertion a). The proof of the second part, where $X = C(\mathbb{R})/M$, is analogous.

To prove assertion b), recall that under CH, Theorem 1.1 shows that every small η_1-space is homeomorphic to the ultrapower $\mathbb{R}^\omega/\mathcal{U}$ where $\mathcal{U}$ is any free ultrafilter on ω, so that b) follows from a). □

Corollary 4.9 *The following statements are equivalent:*

a) the Continuum Hypothesis.

b) If X and Y are small η_1-spaces, then $X \times Y$ is normal.

c) If X is a small η_1-space, then X^2 is normal.

Proof: Suppose (a) holds and that X and Y are small η_1-spaces. Then there is a single free ultrafilter $\mathcal{U}$ on ω with the property that both X and Y are homeomorphic to the ultrapower $\mathbb{R}^\omega/\mathcal{U}$. But then $X \times Y$ is homeomorphic to $(\mathbb{R}^\omega/\mathcal{U})^2$ and that space is normal by Corollary 4.8. Hence (a) implies (b). Trivially (b) implies (c). To show that (c) implies (a), we verify the contrapositive. Suppose that (c) holds and (a) does not. Then $\omega_2 \leq 2^\omega$ so that Example 4.1 gives an η_1-space X whose square is not normal and whose cardinality is $\max(\omega_2, 2^\omega) = 2^\omega$ and that contradicts(c). □

Remark 4.10 *Historical links.* What we called an A-metric space has been studied under many names. They are what Sikorski [34] called "ω_μ-metrizable spaces," and what Juhasz [21] called "κ-metric spaces," both meaning that the space has a well-ordered uniformity $\{U_\alpha : \alpha < \omega_\mu\}$ satisfying $U_\beta \subseteq U_\alpha$ whenever $\alpha < \beta < \kappa = \omega_\mu$. It is known ([34], [28], [29]) that this uniformity property is equivalent to having the space's topology defined by a metric-like function with values in an ordered Abelian group, as in the above definition of A-metric spaces. In case X has a countable uniformity, then the linearly ordered group A is the usual group of real numbers. In case $\kappa > \omega$, then one can use the group $A = \mathbb{Z}^\kappa$ with the lexicographic order. Note that if X is an A-metric space and κ is the co-initiality of the interval $(0, \rightarrow)$ in A, then X is a κ-metric space in the sense of [21]. Juhasz proved in [21] that any A-metric space (which he called a κ-metric space) is paracompact. Because any subspace Y of an A-metric space X is also an A-metric space, as is any finite power X^n of X, Juhasz actually proved that if X is a A-metric space, then each finite power X^n of X is hereditarily paracompact. Later, Vaughan introduced the class of κ-stratifiable spaces in [37] and [38]. Any A-metric space is κ-stratifiable where κ is the co-initiality of the set of positive members of $(A, <)$. Vaughan [37], [38] and Harris [16] proved that any κ-stratifiable space is hereditarily paracompact, and that for each finite $n \geq 1$, if X is κ-stratifiable, then so is X^n. This gives another proof of that statement that any finite power of any A-metric space is hereditarily paracompact, a result that is more general than our Proposition 4.8.

We have proved that if X is an η_1-field, then X^n is paracompact for each finite $n \geq 1$. By contrast, the space given in Example 4.1 has a non-normal square and fails to be paracompact. This leads to the following questions.

Question 4.11 *Suppose X is a paracompact η_1-space that is not necessarily an η_1-field. Without CH, is X^2 normal, paracompact, or monotonically normal? What about X^n for each finite n?*

Question 4.12 *Suppose X is an η_1-space and X^2 is normal. Is X paracompact? (Note that by Proposition 4.13, if X^2 is hereditarily normal then X is paracompact.)*

The following general result is related to Question 4.12.

Proposition 4.13 *Suppose X is a LOTS, a GO-space or, more generally, any monotonically normal space. If X^2 is hereditarily normal, then X is hereditarily paracompact.*

Proof: We prove the contrapositive. Suppose X is not hereditarily paracompact. By the Balogh-Rudin theorem in [3] there is an uncountable regular initial ordinal λ and a stationary subset $S \subseteq \lambda$ that is homeomorphic to a subspace of X (for GO-spaces, see the easier reference [10]). By Solovay's theorem [35] (see Theorem 12.5 of [15]) there are disjoint stationary subsets S_1 and S_2 of λ with $S_1 \cup S_2 \subseteq S$. Then S_1 and S_2 are homeomorphic to subspaces of X so that $S_1 \times S_2$ is a subspace of X^2. Because $S_1 \cap S_2 = \emptyset$, the space $S_1 \times S_2$ is not normal, so X^2 is not hereditarily normal. □

Question 4.14 *Suppose X is an A-metric space. Is X^ω an A-metric space? paracompact? normal? In particular, if $\mathcal{U}$ is a free ultrafilter on ω and $X = \mathbb{R}^\omega/\mathcal{U}$ as in Example 2.3, is X^ω paracompact?*

Question 4.15 *Suppose $(X, \tau, <)$ is a small η_1-space. Modify the topology τ by making using all sets of the form $[\alpha, \beta)$ as a base for a new topology σ (i.e., σ is the Sorgenfrey modification of τ). Is the space $(X, \sigma)^2$ normal?*

5 Completeness properties in η_1-spaces

A space X is (countably) subcompact provided there is a base $\mathcal{B}$ of nonempty open subsets of X such that if $\mathcal{F} \subseteq \mathcal{B}$ is a (countable) regular filter base[3], then $\bigcap \mathcal{F} \neq \emptyset$. This property is a considerable strengthening of the Baire-space property (which asserts that if G_n is a sequence of dense open sets in X, then $\bigcap\{G_n : n \geq 1\}$ is dense in X).

Proposition 5.1 *Any η_1-space is countably subcompact and is pseudo-complete in the sense of [32].*

Proof: Let $\mathcal{B}$ be the base whose members are all nonempty bounded open intervals of X and suppose $\mathcal{F} \subseteq \mathcal{B}$ is a countable regular filterbase. Index $\mathcal{F}$ as $\mathcal{F} = \{I(n) : n \geq 1\}$ where $I(n) = (a_n, b_n)$. Let $A = \{a_n : n \geq 1\}$ and $B = \{b_n : n \geq 1\}$. We claim that for each $m, n,\ a_m < b_n$. If not, then for some m, n we have $b_n \leq a_m$. Choose a member $I(k) \in \mathcal{F}$ with $I(k) \subseteq I(m) \cap I(n)$ and a point $x \in I(k)$. Then $a_m < x < b_m$ and $a_n < x < b_n$ which, combined with $b_n \leq a_m$ gives $x < b_n \leq a_m < x$ and that is impossible. If we apply the definition of an η_1-set to the countable subsets A and B, we find some $y \in X$ with the property that $a_m < y < b_n$ for every $m, n \geq 1$. Then $y \in \bigcap \mathcal{F}$ as required.

To show that X is pseudo-complete in the sense of [32] we need a sequence of π-bases $\mathcal{P}(n)$ with the property that if $P(n) \in \mathcal{P}(n)$ has $cl(P(n+1)) \subseteq P(n)$ then $\bigcap\{P(n) : n < \omega\} \neq \emptyset$. With $\mathcal{B}$ as above, let $\mathcal{P}(n) = \mathcal{B}$ for each n. $\square$

Corollary 5.2 *The product of any number of η_1-spaces is a Baire space.*

Proof: This statement is true for countably subcompact spaces [14]. $\square$

Proposition 5.3 *Under the Continuum Hypothesis(CH), no small η_1-space (i.e., with cardinality 2^ω) can be subcompact.*

Proof: Assume CH. Let X be an η_1-space of cardinality $2^\omega = \omega_1$, and suppose X is subcompact with respect to a base $\mathcal{B}$. Index $X = \{x_\alpha : \alpha < \omega_1\}$. Choose $B(0) \in \mathcal{B}$ with $x_0 \notin B(0)$. Now suppose $\alpha < \omega_1$ and we have chosen $B(\beta) \in \mathcal{B}$ for each $\beta < \alpha$ in such a way that if $\beta < \gamma < \alpha$ then $cl(B(\gamma)) \subseteq B(\beta)$ and $x_\beta \notin B(\beta)$. Then the collection $\{B(\beta) : \beta < \alpha\}$ is a regular filter base of members of $\mathcal{B}$, so that $\bigcap\{B(\beta) : \beta < \alpha\} \neq \emptyset$. Because $\alpha < \omega_1$, the set $\bigcap\{B(\beta) : \beta < \alpha\}$ is a G_δ-set and therefore is open by Lemma 2.5. Therefore some $B(\alpha) \in \mathcal{B}$ has $cl(B(\alpha)) \subseteq \bigcap\{B(\beta) : \beta < \alpha\}$ and $x_\alpha \notin B(\alpha)$. This induction gives the regular filterbase $\mathcal{F} = \{B(\alpha) : \alpha < \omega_1\} \subseteq \mathcal{B}$, and $\bigcap \mathcal{F} \subseteq B(\alpha) \subseteq X - \{x_\alpha\}$ for each $\alpha < \omega_1$, so $\bigcap \mathcal{F} = \emptyset$. This contradiction shows that X cannot be subcompact. $\square$

In fact, there are η_1-spaces of arbitrarily large cardinality that are not subcompact. The key result concerning κ-saturated models is Theorem 6.6 in [23].

Proposition 5.4 *For each regular cardinal κ, there is a κ-saturated model of the of the theory of dense linear orders without endpoints that has cardinality 2^κ. This model gives and η_1-space $(X, < \tau)$ that is λ-subcompact for each $\lambda < \kappa$ but not subcompact.*

[3] $\mathcal{F}$ is a regular filter base if for each $B_1, B_2 \in \mathcal{F}$ there is some $B_3 \in \mathcal{F}$ with $\mathrm{cl}(B_3) \subseteq B_1 \cap B_2$.

Proof: The first statement follows from Theorem 6.6 of [23]. Given the existence of the κ-saturated linear order $(X,<)$ that has cardinality 2^κ, then for each cardinal $\lambda<\kappa$ every G_λ-set in $(X,<,\tau)$ is open so that a slight modification the proof used in Proposition 5.3 shows that X cannot be subcompact. Furthermore, if $\mathcal{B}$ is the collection of all nonempty open intervals of X, then $\bigcap\mathcal{F}\neq\emptyset$ for each regular filterbase $\mathcal{F}$ whose cardinality is $\lambda<\kappa$, so X is λ-subcompact. □

A different kind of completeness property is realcompactness. A space X is realcompact if it can be embedded as a closed subspace of some product of real lines. A weaker completeness property is called Dieudonné completeness which is characterized as follows: a space X is Dieudonné-complete if it can be embedded as a closed subset of a product of completely metrizable spaces [9]. The next result establishes parts c) and d) of Theorem 1.3.

Proposition 5.5 *The following are equivalent:*

a) The Continuum Hypothesis (CH)

b) every small η_1-space (i.e., having cardinality 2^ω) is realcompact

c) every small η_1-space is Dieudonné complete.

Proof: First, suppose that (CH) holds. Let X be an η_1-space of cardinality $2^\omega=\omega_1$. Then X is paracompact by Proposition 3.2, and Katětov showed that a paracompact space X is realcompact if and only if each closed discrete subspace $D\subseteq X$ is realcompact ([9], Problem 5.5.10). Mackey and Hewitt showed that a discrete space is realcompact if and only if its cardinality is non-measurable ([9] Problem 3.11D), and Ulam [36] (see also [31]) proved that ω_1 (which is $|X|$) cannot be a measurable cardinal. Therefore, assuming $2^\omega=\omega_1$ we see that any closed discrete subspace D of X is realcompact, as required. Therefore a) implies b).

Any realcompact space is Dieudonné complete, so b) implies c)

To show that c) implies a), suppose every η_1-space of cardinality 2^ω is Dieudonné-complete. Because any Dieudonné-complete LOTS is paracompact ([20], [10], Problem 8.5.13-(j) in [9]), Proposition 3.2 shows that the Continuum Hypothesis holds. □

Whether or not a given η_1-space X has any of the completeness properties listed above, the space $C_p(X)$ of all continuous real-valued functions on X with the pointwise topology always has many very strong completeness properties. Because every compact subset of the η_1-space X is finite, our next results also apply to $C_k(X)$, the space of real-valued functions on X with the compact-open topology. Our next result shows that for any η_1-space X, $C_p(X)$ has properties much stronger than being a Baire space (= any intersection of countably many dense open sets is dense) and can be used to distinguish between certain strengthenings of the Baire-space property. See [26] for definitions of the terms used in our next result.

Proposition 5.6 *For any η_1-space X, $C_p(X)$ is a Baire space. In fact, $C_p(X)$ has each of the following stronger properties:*

a) $C_p(X)$ is pseudo-complete in the sense of Oxtoby;

b) $C_p(X)$ is weakly α-favorable in the sense of Choquet;

c) $C_p(X)$ has non-void intersection with every non-void G_δ-subset of the product space $\mathbb{R}^X$.

However, $C_p(X)$ is neither Čech-complete nor subcompact in the sense of de Groot.

Proof: Because every countable subset of X is closed, assertions (a), (b), and (c) follow directly from Theorem 8.4 in [26]. Because X is not a discrete space, it follows from Theorem 8.6 of [26] that $C_p(X)$ is not Čech-complete and from [27] that X is not subcompact. □

6 Diagonal conditions in η_1-spaces

The hypothesis that the diagonal set $\Delta_X = \{(x, x) : x \in X\}$ is a G_δ-subset of X^2 is a component of most metrization theorems[4]. But because any G_δ-subset of any η_1-space is open (Proposition 2.5), it is clear that no η_1-space can have a G_δ-diagonal. A weaker diagonal condition was introduced by Hušek in [19].

Definition 6.1 *A space X has a small diagonal if for each uncountable subset $T \subseteq X^2 - \Delta_X$ there is an open subset U of X^2 with the property that $\Delta_X \subseteq U$ and $|T - U| > \omega$.*

For a study of the role of small diagonals in ordered spaces, see [4].

Proposition 6.2 *An η_1-space X has a small diagonal if and only if every subset $S \subseteq X$ having $|S| \leq \omega_1$ is closed in X.*

Proof: First suppose that there is a non-closed subset $S \subseteq X$ with $|S| \leq \omega_1$. Because X cannot contain a non-trivial convergent countable sequence, every countable subset of X is closed. Therefore $|S| = \omega_1$ and there is a limit point p of S and a transfinite strictly monotone sequence $\{x(\alpha) : \alpha < \omega_1\}$ of points of S that converges to p. Let $T = \{(x(\alpha), x(\alpha + 1)) \in X^2 : \alpha < \omega_1\}$. Then $T \subseteq X^2 - \Delta_X$ and $|T| = \omega_1$. If U is any open set in X^2 with $\Delta_X \subseteq U$, then there is an open set V in X with $(p, p) \in V^2 \subseteq U$ and therefore there is some $\beta < \omega_1$ with $x(\alpha) \in V$ for all $\alpha > \beta$. But then $T - U \subseteq \{(x(\alpha), x(\alpha + 1)) : \alpha < \beta\}$ is a countable set, so that X does not have a small diagonal. Contraposition shows that if X has a small diagonal, then every subset $S \subseteq X$ with $|S| \leq \omega_1$ must be closed in X.

Conversely, suppose every subset $S \subseteq X$ having $|S| \leq \omega_1$ is closed in X, and let $T \subseteq X^2 - \Delta_X$ be any uncountable set. Replacing T with a suitable subset, we may assume that $|T| = \omega_1$. For $i = 1, 2$, let π_i be projection from X^2 onto the i^{th} coordinate. Then the subsets $S_i = \pi_i[T]$ of X each have cardinality $\leq \omega_1$ and therefore the subspace $S = S_1 \cup S_2$ of X is closed in X, as is each of its subsets. Therefore, S is closed and discrete. Therefore the product space $S \times S$ is a closed discrete subspace of X^2. Let $W = X^2 - T$. Then W is an open subspace of X^2 that contains the diagonal Δ_X and because $T \cap W = \emptyset$, we see that $T - W$ is uncountable, as required to show that X has a small diagonal. □

Corollary 6.3 *In ZFC, there are η_1-spaces that have a small diagonal.*

Proof: As in the second part of Example 2.4, if we take a regular cardinal $\kappa > 2^\omega$ we can construct a linearly ordered set $(Y_2, <)$ with the property that given two sets $A, B \subseteq Y_2$ each with cardinality $< \kappa$ and having $a < b$ for each $a \in A$ and $b \in B$, some point of Y_2 lies between A and B. Then

[4]For example Okuyama [30] proved that a space is metrizable if and only if it has a G_δ-diagonal, is paracompact, and is a p-space in the sense of Arhangelskii.

$(Y_2, <)$ is an η_1-space, and no set of cardinality ω_1 has a limit point in Y_2. Now Proposition 6.2 shows that Y_2 has a small diagonal. $\Box$

The situation for small η_1-spaces, i.e., η_1-spaces of cardinality 2^ω, is more interesting and involves the Continuum Hypothesis. We begin with:

Corollary 6.4 *If the Continuum Hypothesis holds, then no η_1-space of cardinality 2^ω can have a small diagonal. (In particular, the spaces of Examples 2.1, 2.2, and 2.3 cannot have a small diagonal.) Consequently, if some small η_1-space has a small diagonal, then $\omega_2 \leq 2^\omega$.*

Proof: Under CH, any small η_1-space has cardinality ω_1 so that it has many non-closed subsets of cardinality ω_1. Now apply 6.2. The final sentence in the corollary is the contrapositive of the first. $\Box$

Without CH, the situation is more complicated, as our next results show. Recall the definition of the special cardinal d. We begin with a partial ordering $<^*$ on the set ω^ω defined by the rule that for some $n \in \omega$, $f(k) < g(k)$ for all $k \geq n$. The cardinal d (called the dominating number) is the smallest cardinal of a cofinal subset of the poset $(\omega^\omega, <^*)$. See [8].

Corollary 6.5 *If some ultrapower $\mathbb{R}^\omega/\mathcal{U}$ has a small diagonal, then the cofinality of $\mathbb{R}^\omega/\mathcal{U}$ is greater than ω_1 and the dominating number $d > \omega_1$.*

Proof: For contradiction, suppose $d = \omega_1$, where d is the least cardinality of a cofinal subset of the poset $(\omega^\omega, <^*)$. There is an order-preserving mapping from $(\omega^\omega, <^*)$ onto a cofinal subset of $\mathbb{R}^\omega/\mathcal{U}$ so that $\mathbb{R}^\omega/\mathcal{U}$ has cofinality ω_1. But then the function $x \to x^{-1}$ shows that the set $(0, \infty)$ in $\mathbb{R}^\omega/\mathcal{U}$ has co-initiality ω_1 so that $(0, \infty)$ contains a non-closed subset of $\mathbb{R}^\omega/\mathcal{U}$ having cardinality ω_1. Now apply Proposition 6.2. $\Box$

Corollary 6.6 *Assume CH fails. Whether some ultrapower $\mathbb{R}^\omega/\mathcal{U}$ has a small diagonal is undecidable because:*

> *a) there is a model $\mathcal{M}_1$ of (ZFC + notCH) in which no ultrapower $\mathbb{R}^\omega/\mathcal{U}$ can have a small diagonal; and*
>
> *b) there is a model $\mathcal{M}_2$ of (ZFC + notCH) in which some ultrapower $\mathbb{R}^\omega/\mathcal{U}$ has a small diagonal.*

Proof: To prove a), recall that there is a model $\mathcal{M}_1$ of ZFC with $\omega_1 = d < 2^\omega$ (see Theorem 5.1 in [8]). In that model, for any free ultrafilter $\mathcal{U}$ on ω, there is an order-preserving mapping from the poset $(\omega^\omega, <^*)$ onto a cofinal subset of the ultrapower $\mathbb{R}^\omega/\mathcal{U}$, so that the ultrapower also has cofinality ω_1 as does its subset $(\leftarrow, 0)$ where 0 is the zero element of $\mathbb{R}^\omega/\mathcal{U}$. By Proposition 6.2, $\mathbb{R}^\omega/\mathcal{U}$ cannot have a small diagonal. (This argument was suggested in a private communication by Ilias Farah.)

To prove b) recall Roitman's proof [33] that there is a model $\mathcal{M}_2$ of ZFC in which CH fails and there are many ultrafilters on ω that give an ultrapower of $\mathbb{R}$ having cofinality $> \omega_1$. (See also [7].) For any such ultrapower $\mathbb{R}^\omega/\mathcal{U}$ no subset with cardinality ω_1 can have a limit point so that every subset of cardinality ω_1 is closed. Now apply Proposition 6.2. $\Box$

However, if we look at small η_1-spaces in general (rather than restricting ourselves to ultrapowers) examples are easier to find and we have:

Example 6.7 *If CH fails, then there is an η_1-space of cardinality 2^ω that does not have a small diagonal.*

Proof: Suppose CH fails. Then $\omega_1 < \omega_2 \leq 2^\omega$. Let $X = \mathbb{R}^\omega/\mathcal{U}$ for a free ultrafilter $\mathcal{U}$, as in Example 2.3 and let

$$Y = ([0,\omega_1) \times X) \cup \{(\omega_1, 0)\} \cup ((\alpha, x) : \omega_1 < \alpha < \omega_2 \textit{ and } x \in X).$$

With the lexicographic order, Y is an η_1-space of cardinality $max(\omega_2, 2^\omega) = 2^\omega$ and the set $S = \{(\alpha, x) : \alpha < \omega_1\}$ is a set of cardinality ω_1 that is not closed. By Proposition 6.2, Y cannot have a small diagonal. $\square$

Remark 6.8 Diagonal conditions are familiar components of metrization theorems, e.g., Okuyama's theorem [30] that a space is metrizable if and only if it has a G_δ-diagonal, is paracompact, and is a p-space in the sense of Arhangelskii. We have studied paracompactness in η_1-spaces in Section 3, and this section discusses diagonal conditions in η_1-spaces. In passing we note that no η_1-space can be a p-space.

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