Semi-stratifiable Spaces with Monotonically Normal Compactifications

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Abstract: In this paper we use Mary Ellen Rudin's solution of Nikiel's problem to investigate metrizability of certain subsets of compact monotonically normal spaces. We prove that if H is a semi-stratifiable space that can be covered by a σ-locally-finite collection of closed metrizable subspaces and if H embeds in a monotonically normal compact space, then H is metrizable. It follows that if H is a semi-stratifiable space with a monotonically normal compactification, then H is metrizable if it satisfies any one of the following: H has a σ-locally finite cover by compact subsets; H is a σ-discrete space; H is a scattered; H is σ-compact. In addition, a countable space X has a monotonically normal compactification if and only if X is metrizable. We also prove that any semi-stratifiable space with a monotonically normal compactification is first-countable and is the union of a family of dense metrizable subspaces. Having a monotonically normal compactification is a crucial hypothesis in these results because R.W. Heath has given an example of a countable non-metrizable stratifiable (and hence monotonically normal) group. We ask whether a first-countable semi-stratifiable space must be metrizable if it has a monotonically normal compactification. This is equivalent to "If X is a first-countable stratifiable space with a monotonically normal compactification, must H be metrizable?"

Key words: monotonically normal space, monotonically normal compactification, Rudin's solution of Nikiel's problem, metrizable space, countable subspace, σ-discrete space, semi-stratifiable, stratifiable, scattered, dense metrizable subset.

MR Subject Classifications: Primary = 54D15; Secondary = 54D30, 54D35; 54F05

********draft of August 5*********

1 Introduction

Research since the 1970s shows that there are close parallels between generalized ordered (GO-) spaces and monotonically normal spaces, particularly in the theories of cardinal functions and of paracompactness (see [1]). In this paper we investigate the extent to which metrization theory for subsets of compact monotonically normal spaces resembles metrization theory for GO-spaces, i.e., for subspaces of compact linearly ordered spaces.

One of the most celebrated results in recent set-theoretic topology is Mary Ellen Rudin's solution of Nikiel's problem [17] and it is our primary tool in this study. Rudin proved:

Theorem 1.1 *Any compact monotonically normal space is the continuous image of a compact linearly ordered topological space.*

In Section 4 of this paper we combine ordered space techniques with Theorem 1.1 to prove:

Proposition 1.2 *Suppose H is a subspace of a compact monotonically normal space (equivalently, suppose H has a monotonically normal compactification). Then:*

a) there is a GO-space $(Z, \tau, \preceq)$ and a perfect irreducible mapping g from Z onto H with the property that no fiber of g contains a jump of $(Z, \preceq)$ (see(4.2));

b) if each point of H is a G_δ-set, then each fiber of the mapping g is metrizable (see(4.3));

c) if H has a G_δ-diagonal, then so does Z (see 4.4)).

We use Proposition 1.2 to study which metrization theorems for GO-spaces can be extended to the larger category of spaces with monotonically normal compactifications.

The basic metrization theorem for GO-spaces that might generalize to monotonically normal spaces (because it does not mention the order of the GO-space) appears in [15]:

Theorem 1.3 *A GO-space X is metrizable if and only if X is semi-stratifiable.*

Easy examples show that Theorem 1.3 does not generalize to arbitrary monotonically normal spaces because there are many non-metrizable spaces that are stratifiable [2], [3], [10], and stratifiable spaces are exactly the semi-stratifiable monotonically normal spaces. However, because every GO-space has a GO-compactification (which, of course, is monotonically normal), it is natural to ask the following more interesting question:

Question 1: Suppose a space H is semi-stratifiable and has a monotonically normal compactification. Must H be metrizable?

As a preliminary step toward that question, in this paper we prove the following result.

Theorem 1.4 *Suppose H is a semi-stratifiable space with a monotonically normal compactification. Then:*

a) the space H is the union of a family of dense metrizable subspaces (see (3.4));

b) the space H is first-countable (see (3.4));

In addition, H is metrizable if any one of the following holds:

c) if there is a σ-locally finite cover of H by closed metrizable subsets (see (3.1));

d) if there is a σ-locally-finite cover of H by compact subsets (see 3.2(a));

e) if $H = \bigcup\{H_n : n \geq 1\}$ where each H_n is a closed discrete subset[1] *of H (see 3.2(b));*

f) if H is scattered (see 3.2(c));

g) if H is σ-compact (see 3.2(d));

h) if H is countable (see 3.2(e)).

[1]In the literature, spaces that are countable unions of closed discrete subspaces are called "σ-discrete spaces." If a σ-discrete space H has a monotonically normal compactification, then it must be stratifiable, so that a theorem of Gruenhage [8] shows that H must be at least M_1. Our result shows that if a σ-discrete space H has a monotonically normal compactification, then H is even more than M_1.

Theorem 1.4 can be used to show that certain spaces do not have monotonically normal compactifications. For example, because a countable, first-countable space is metrizable, (1.4) shows that (as pointed out by Collins [4]) the non-metrizable countable stratifiable group constructed by R.W. Heath in [10] cannot have a monotonically normal compactification. In addition, our results can also be seen as part of the study of spaces that embed in compact monotonically normal spaces, a research program suggested by Collins in [4].

Because any semi-stratifiable space having a monotonically normal compactification must be monotonically normal and hence stratifiable (see [13]), Theorem 1.4 (b) shows that Question 1 is equivalent to:

Question 2: Suppose H is a first-countable stratifiable space with a monotonically normal compactification. Must H be metrizable?

Remark 1.5 Our paper considers spaces that have a monotonically normal compactification and shows that, under certain additional hypotheses, the spaces are metrizable. An interesting contrast is provided by the paper [14] in which the authors show that some, but not all, metrizable spaces have monotonically normal compactifications.

Our paper is organized as follows. Section 2 gives relevant definitions, and Section 3 gives the proof of Theorem 1.4 by using a sequence of ordered space lemmas that appear in Section 4 where Proposition 1.2 is proved. The key idea in Section 4 is a process that could be called a "compressing jumps construction."

Undefined terms are as in [6]. Intervals in a linearly ordered set $(X, \leq)$ are denoted by $[a, b]$, (a, b), etc., and all spaces are at least Hausdorff.

The authors would like to thank Peter Collins, Paul Gartside, Gary Gruenhage, and Peter Nyikos for e-mails that helped us develop our ideas.

2 Definitions

Recall that a space is *semi-stratifiable* [5] if for each open set U there is a sequence $S(n, U)$ of closed sets with the property that $U = \bigcup\{S(n, U) : n \geq 1\}$ and with the property that if $U \subseteq V$ are open, then $S(n, U) \subseteq S(n, V)$ for each $n \geq 1$.

For a space $(X, \mathcal{T})$, let $\mathcal{P} = \{(A, U) : A$ is closed and U is open and $A \subseteq U\}$. Recall that a Hausdorff space $(X, \mathcal{T})$ is *monotonically normal* if there is a function $G : \mathcal{P} \to \mathcal{T}$ with the properties that

a) $A \subseteq G(A, U) \subseteq \mathrm{cl}(G(A, U)) \subseteq U$ and

b) if $(A_i, U_i) \in \mathcal{P}$ with $A_1 \subseteq A_2$ and $U_1 \subseteq U_2$, then $G(A_1, U_1) \subseteq G(A_2, U_2)$.

A space $(X, \mathcal{T})$ is *stratifiable* [2] if there is a function $G : \omega \times \mathcal{T} \to \mathcal{T}$ such that for each open $U, U = \bigcup\{G(n, U) : n \in \omega\} = \bigcup\{cl(G(n, U)) : n \in \omega\}$ and with the property that if U, V are open with $U \subseteq V$, then $G(n, U) \subseteq G(n, V)$ for each n.

It is easy to see that a space is stratifiable if and only if it is monotonically normal and semi-stratifiable. The following relations are well-known:

$$X \text{ is metrizable} \Rightarrow X \text{ is stratifiable } \Rightarrow X \text{ has a } \sigma-\text{discrete network} \Rightarrow X \text{ is semi}-\text{stratifiable}$$

and none of those implications can be reversed.

A *generalized ordered space* (GO-space) is a triple $(X, \sigma, \leq)$ where $\leq$ is a linear ordering of X and σ is a Hausdorff topology on X that has a base of order-convex sets. If the topology σ coincides with the open-interval topology of the ordering $\leq$, then X is a *linearly ordered topological space* (LOTS).

Monotone normality is a very strong property: it is hereditary and implies collectionwise normality (see [13]). The most familiar types of monotonically normal spaces are the stratifiable spaces of Borges [2] and GO-spaces. However, there are monotonically normal spaces that are neither stratifiable nor GO.

3 Proof of Theorem 1.4

In this section we prove a sequence of propositions that will establish Theorem 1.4. The proofs make extensive use of some ordered space constructions that are given in Section 4. Part (c) of 1.4 (about σ-locally-finite covers by closed metrizable subsets) is the pivotal part of Theorem 1.4 and we prove it first.

Proposition 3.1 *Suppose H is a semi-stratifiable space with a monotonically normal compactification K. If there is a σ-locally-finite covering of H by closed metrizable subsets, then H is metrizable.*

Proof: According to Theorem 1.1, there is a compact LOTS L and a continuous mapping F from L onto K. The restriction of F to the subspace $X = F^{-1}[H]$ is a perfect mapping and there is a relatively closed subset Y of $X = F^{-1}[H]$ such that the restriction of F to Y is a perfect irreducible mapping from Y onto H. Denote the restriction of F to the subspace Y by f.

Let σ be the topology that Y inherits from L and let $\leq$ be the linear ordering of Y obtained by restricting the given ordering of the LOTS L. Then $(Y, \sigma, \leq)$ is a GO-space.

The fibers of f might contain jumps of $(Y, \leq)$ (i.e., intervals $[a, b]$ of $(Y, \leq)$ with $(a, b) = \emptyset$) and that complicates their topological structure. Lemma 4.1 allows us to collapse each jump of $(Y, \leq)$ that is contained in some fiber of f, thereby obtaining a new GO-space $(Z, \tau, \preceq)$ and a natural projection $\pi : Y \to Z$ that is continuous, perfect, and irreducible. Lemma 4.2 gives a well-defined function $g(z) = f[\pi^{-1}[z]]$ from Z onto H that is continuous, perfect, and irreducible and whose fibers do not contain any jumps of $(Z, \preceq)$. According to Lemma 4.4, because H has a G_δ-diagonal, so does Z.

Now suppose that $\mathcal{H} = \bigcup\{\mathcal{H}_n : n \geq 1\}$ is a cover of H by closed metrizable subspaces, where each $\mathcal{H}_n$ is locally finite in H. If $T \in \mathcal{H}_n$, then $g^{-1}[T]$ is closed in Z and is a perfect preimage of a metric space, so that $g^{-1}[T]$ is a paracompact p-space in the sense of Arhangelskii (see [9]). But then $g^{-1}[T]$, being a paracompact p-space with a G_δ-diagonal, is metrizable ([2], [9]), and therefore has a network $\mathcal{N}_T$ that is σ-locally-finite in $g^{-1}[T]$.

Because $\mathcal{H}_n$ is locally-finite in H, we see that $\{g^{-1}[T] : T \in \mathcal{H}_n\}$ is a locally finite collection of closed subsets in Z. Therefore, $\mathcal{M} = \bigcup\{\bigcup\{\mathcal{N}_T : T \in \mathcal{H}_n\} : n \geq 1\}$ is a σ-locally-finite collection of subsets of Z, and $\mathcal{M}$ is a network for Z. But then Z is semi-stratifiable, so that Theorem 1.3 shows that Z is metrizable. Because $g : Z \to H$ is a perfect mapping, it follows that H is metrizable. $\square$

Corollary 3.2 *Suppose H is a semi-stratifiable space with a monotonically normal compactification. Then H is metrizable if any one of the following holds:*

a) $H = \bigcup\{\mathcal{H}_n : n \geq 1\}$ where $\mathcal{H}_n$ is a locally finite collection of compact subsets of H;

b) $H = \bigcup\{H_n : n \geq 1\}$ where each H_n is a closed, discrete subspace of H;

c) H is a scattered;

d) H is σ-compact;

e) H is countable.

Proof: Recall that any compact subset of a semi-stratifiable space is metrizable [5]. Therefore a) follows immediately from Proposition 3.1. For b), note that the collection of all singletons in H_n is a locally finite collection. For d), note that if $H = \bigcup\{H_n : n \geq 1\}$ where each H_n is compact, then $\mathcal{H}_n = \{H_n\}$ is the kind of σ-locally-finite cover described in a). For e), the set of all singleton subsets of H is a σ-locally finite cover of H so that a) applies. Finally, to prove c), recall that in [16], Nyikos proved that any scattered semi-stratifiable space is σ-closed-discrete and therefore assertion a) gives assertion c). $\square$

Remark 3.3 *:* (1) In an e-mail, Paul Gartside pointed out another proof of assertion e): If H is countable and has a monotonically normal compactification K, then K is separable and therefore hereditarily Lindelöf by [7], so that K must be first-countable. Therefore H is first-countable and countable, showing that H is second-countable.

(2) Because any countable metric space embeds in the usual space of rational numbers and therefore has a linearly ordered compactification, part e) of Corollary 3.2 can be restated as "A countable regular space H has a monotonically normal compactification if and only if H is metrizable."

Proposition 3.4 *Suppose H is a semi-stratifiable space that has a monotonically normal compactification. Then H is a union of a family of dense metrizable subspaces, and H is first-countable.*

Proof: The semi-stratifiable space H inherits monotone normality from its monotonically normal compactification and therefore H is stratifiable. A theorem of R. W. Heath [11] [12] shows that H has a network $\mathcal{N}$ that is σ-closed-discrete in H. Fix any point $p \in H$. Choose one point out of each member of $\mathcal{N}$, making sure that if $p \in N \in \mathcal{N}$, then the point chosen from N is the point p. The set of chosen points is a dense subspace $H_p \subseteq H$ that contains p, and that is σ-closed-discrete in H. Because H_p is a σ-closed-discrete space that has a monotonically normal compactification, part a) of Corollary 3.2 shows that H_p is metrizable. Therefore $H = \bigcup\{H_p : p \in H\}$ is a union of dense metrizable subspaces.

To complete the proof, consider any fixed $p \in H$ and consider the dense metrizable subset H_p of H constructed above. Because H is regular and H_p is a first-countable dense subspace that contains p, we see that H is also first-countable at p. $\square$

4 Ordered space constructions and the proof of 1.2

In this section, we present propositions and some corollaries that prove Proposition 1.2 in Section 1 and that are used in Section 3 to give the proof of Theorem 1.4 from Section 1.

Recall that a jump in a linearly ordered set $(Y, \leq)$ is a closed interval $[a, b]$ with $(a, b) = \emptyset$. We describe a construction that collapses certain jumps of a GO-space $(Y, \sigma, \leq)$.

Collapsing Jumps Construction: Let $(Y, \sigma, \leq)$ be a GO-space. Suppose $\mathcal{J}$ is a pairwise disjoint collection of jumps of $(Y, \leq)$.

1) Define $Z = \mathcal{J} \cup \{\{y\} : y \in Y - \bigcup \mathcal{J}\}$. We will use the Greek letters α, β, γ and δ to denote points of Z. These points of Z are also subsets of Y. For any $\alpha \in Z, \min(\alpha)$ is the smallest point of the set α, and $\max(\alpha)$ is the largest point of α. Because α is a finite subset of Y, both $\min(\alpha)$ and $\max(\alpha)$ exist.

2) For $\alpha, \beta \in Z$, define $\alpha \preceq \beta$ if either $\alpha = \beta$ or if $\alpha \neq \beta$ and there exist $a \in \alpha$ and $b \in \beta$ with $a < b$ in Y. Then $\preceq$ is a linear ordering of Z.

3) Define the natural projection $\pi : Y \to Z$ by $\pi(y) = J$ is $y \in J \in \mathcal{J}$ and $\pi(y) = \{y\}$ otherwise. Then π is weakly increasing, i.e., if $y_1 < y_2$ in Y, then $\pi(y_1) \preceq \pi(y_2)$ in Z.

4) Let τ be the topology on Z having all sets of the following three forms as a subbase:

(i) (α, β) where $\alpha \prec \beta$;

(ii) $[\alpha, \rightarrow)$ provided $[\min(\alpha), \rightarrow) \in \sigma$;

(iii) $(\leftarrow, \beta]$ provided $(\leftarrow, \max(\beta)] \in \sigma$.

Lemma 4.1 *With notation as in the Collapsing Jumps Construction, $(Z, \tau, \preceq)$ is a GO-space and the function $\pi : (Y, \sigma) \to (Z, \tau)$ is a continuous, closed mapping whose fibers have at most two points.* $\square$

The proof of Lemma 4.1 is straightforward. Continuity of π is most easily proved by considering the π-inverse image of subbasic open sets of τ.

Recall the hypothesis of Proposition 1.2. We have a space H that is contained in a compact monotonically normal space K. Rudin's Theorem 1.1 gives a compact LOTS L and a continuous $F : L \to K$. We let $X = F^{-1}[H]$. Then the restriction $F|_X$ is a perfect mapping from X onto H so there is a closed subspace $Y \subseteq X$ with the property that the restriction of F to Y gives a perfect irreducible mapping of the GO-space Y onto H. We write $f = F|_Y$ and now we have the situation described in Lemma 4.2.

Lemma 4.2 *Suppose H is a topological space, $(Y, \sigma, \leq)$ is a GO-space, and $f : Y \to H$ is an irreducible perfect mapping of Y onto H. Let $\mathcal{J}$ be the collection of all jumps $[a, b]$ of Y having $f(a) = f(b)$. Then*

a) $\mathcal{J}$ is a pairwise disjoint collection;

b) if $(Z, \tau, \preceq)$ is the GO-space constructed by collapsing all members of $\mathcal{J}$, then there is an irreducible perfect mapping g from (Z, τ) onto H that satisfies $f(y) = g(\pi(y))$ for all $y \in Y$;

c) if $\alpha \prec \beta$ in Z with $(\alpha, \beta) = \emptyset$, then $g(\alpha) \neq g(\beta)$ so that no fiber of g contains a jump of $(Z, \prec)$.

Proof: To prove (a), note that if there were adjacent jumps $[a, b]$ and $[b, c]$ in $\mathcal{J}$, then $f(a) = f(b) = f(c)$ so that the points a, b, c would belong to a single fiber $f^{-1}[h]$ where $h = f(a) \in H$. But then $\{b\} = (a, \rightarrow) \cap (\leftarrow, c)$, showing that the non-degenerate fiber $f^{-1}[h]$ contains an isolated point of Y, contrary to hypothesis that f is irreducible. Therefore $\mathcal{J}$ is a pairwise disjoint collection, so that the Collapsing Jumps construction can be applied to $\mathcal{J}$.

To prove (b), for each $\alpha \in Z$, choose any $a \in \alpha$ and define $g(\alpha) = f(a)$. Because f is constant on each set $\alpha \in Z$, we see that g is well-defined. To show that g is continuous, suppose D is a closed subset of H. Then $f^{-1}[D]$ is closed in Y and because π is a closed mapping, $\pi[f^{-1}[D]]$ is closed in (Z, τ). But $\pi[f^{-1}[D]] = g^{-1}[D]$, so that g is continuous. To show that g is a closed mapping, suppose C is a closed set in Z. Then $\pi^{-1}[C]$ is closed in Y so that $f[\pi^{-1}[C]]$ is closed in H. But $f[\pi^{-1}[C]] = g[C]$, as required. Next, to show that g has compact fibers, note that for each $h \in H, g^{-1}[h] = \pi[f^{-1}[h]]$ is compact because $f^{-1}[h]$ is compact and π is continuous.

To show that g is irreducible, suppose $U \subseteq Z$ is a nonempty open set. Then $\pi^{-1}[U]$ is a nonempty open set in Y so there is some $h_1 \in H$ with $f^{-1}[h_1] \subseteq \pi^{-1}[U]$ because f is irreducible. But then $g^{-1}[h_1] = \pi[f^{-1}[h_1]] \subseteq U$, showing that g is irreducible.

To prove (c), suppose $\alpha \prec \beta$ in Z and that $(\alpha, \beta) = \emptyset$ with $g(\alpha) = g(\beta)$. There are several cases to consider. First, suppose $|\alpha| = |\beta| = 1$, say $\alpha = \{a\}, \beta = \{b\}$. Then in Y, $(a, b) = \emptyset$ and $f(a) = g(\alpha) = g(\beta) = g(b)$ showing that $[a, b] \in \mathcal{J}$. But then $\alpha = \pi(a) = \pi(b) = \beta$ contrary to $\alpha \prec \beta$. Therefore, it cannot happen that both α and β are singletons. Without loss of generality, suppose $\alpha = \{a_1, a_2\}$. Let $b = \min(\beta)$. Then $a_1 < a_2 < b$ and $(a_1, a_2) = \emptyset = (a_2, b)$, showing that a_2 is an isolated point of Y. Note that $f(b) = g(\beta) = g(\alpha) = f(a_1) = f(a_2)$ so that a_2 is an isolated point of Y that is contained in a single non-degenerate fiber of f, contradicting irreducibility of f. Therefore we cannot have $(\alpha, \beta) = \emptyset$ and $g(\alpha) = g(\beta)$, as claimed. $\square$

Lemma 4.3 *Suppose each point of the space H is a G_δ-subset of H and suppose $g : (Z, \tau, \preceq) \rightarrow H$ is the perfect irreducible onto mapping in Lemma 4.2. Then each fiber of g is metrizable.*

Proof: Fix $h \in H$ and open sets $U_n \subseteq H$ having $\{h\} = \bigcap\{U_n : n \geq 1\}$. Write $M = g^{-1}[h]$. Each $g^{-1}[U_n]$ is open in Z and $g^{-1}[h] = \bigcap\{g^{-1}[U_n] : n \geq 1\}$. Let $\mathcal{V}_n$ be the collection of all convex components of the set $g^{-1}[U_n]$. Suppose $\alpha \neq \beta$ are points of M. We may suppose $\alpha \prec \beta$. Because $\alpha, \beta \in M$, from part (c) of Lemma 4.2 we know that $(\alpha, \beta) \neq \emptyset$. Because g is irreducible, we know that the open set $(\alpha, \beta) \not\subseteq M$. Choose $\gamma \in (\alpha, \beta) - M$. Then there is some n with $\gamma \notin g^{-1}[U_n]$ so that $\gamma \notin St(\alpha, \mathcal{V}_n)$. Because members of $\mathcal{V}_n$ are convex, it follows that $\beta \notin St(\alpha, \mathcal{V}_n)$. Now we know that the only point of M that belongs to $\bigcap\{St(\alpha, \mathcal{V}_n) : n \geq 1\}$ is the point α. Therefore the collections $\mathcal{W}_n = \{V \cap M : V \in \mathcal{V}_n\}$ are relatively open covers of M and show that M has a G_δ-diagonal. Because M is compact, Sneider's theorem (see [6]) shows that M is metrizable. $\square$

Lemma 4.4 *Suppose the space H has a G_δ-diagonal and suppose $g : (Z, \tau, \preceq) \rightarrow H$ is a perfect irreducible onto mapping as in Lemma 4.2. Then each fiber of g is metrizable and the space Z has a G_δ-diagonal.*

Proof: Let $\{\mathcal{U}_n : n \geq 1\}$ be a G_δ-diagonal sequence of open covers of H. Let $\mathcal{W}_n$ be the collection of all convex components of all sets of the form $g^{-1}[U]$ for $U \in \mathcal{U}_n$. Each $\mathcal{W}_n$ is an open cover of Z. Suppose $\alpha \neq \beta$ are points Z; we may suppose $\alpha \prec \beta$. If $g(\alpha) \neq g(\beta)$ then for some $n, g(\beta) \notin St(g(\alpha), \mathcal{U}_n)$ so that $\beta \notin St(\alpha, \mathcal{W}_n)$. The remaining case is where $g(\alpha) = g(\beta)$. Then there is a single fiber M of g with $\alpha, \beta \in M$ and a minor modification of the proof of Lemma 4.3 shows that for some $n \geq 1, \beta \notin St(\alpha, \mathcal{W}_n)$. Therefore, Z has a G_δ-diagonal. □

Remark: In Lemma 4.4, if we knew that $(Z, \tau, \preceq)$ were a LOTS, or locally a LOTS, then we could conclude that Z is metrizable, and that would give a positive answer to Questions 1 and 2. Unfortunately, all we know is that Z is a GO-space, making Lemma 4.4 somewhat less useful for our purposes because the existence of a G_δ-diagonal in a GO-space does not give metrizability as the Sorgenfrey and Michael lines show. However, we use Lemma 4.4 in the proof of Proposition 3.1.

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