Measurements and G_δ-Subsets of Domains

by

Harold Bennett, Mathematics Department, Texas Tech University, Lubbock, TX 79409

and

David Lutzer, Mathematics Department, College of William and Mary, Williamsburg, VA 23187

Abstract: In this paper we study domains, Scott domains, and the existence of measurements. We use a space created by D.K. Burke to show that there is a Scott domain P for which $\max(P)$ is a G_δ-subset of P and yet no measurement μ on P has $\ker(\mu) = \max(P)$. We also correct a mistake in the literature asserting that $[0, \omega_1)$ is a space of this type. We show that if P is a Scott domain and $X \subseteq \max(P)$ is a G_δ-subset of P, then X has a G_δ-diagonal and is weakly developable. We show that if $X \subseteq \max(P)$ is a G_δ-subset of P, where P is a domain but perhaps not a Scott domain, then X is domain-representable, first-countable, and is the union of dense, completely metrizable subspaces. We also show that there is a domain P such that $\max(P)$ is the usual space of countable ordinals and is a G_δ-subset of P in the Scott topology. Finally we show that the kernel of a measurement on a Scott domain can consistently be a normal, separable, non-metrizable Moore space.

Key Words and Phrases: domain-representable, Scott-domain-representable, measurement, Burke's space, developable spaces, weakly developable spaces, G_δ-diagonal, Čech-complete space, Moore space, ω_1, weakly developable space, sharp base, AF-complete

MR Subject Classifications: Primary = 54D35; Secondary = 54E30, 54E52, 54E99, 06B35, 06F99

1 Introduction

Domains and Scott domains are special kinds of partially ordered sets (posets)[1]. Any domain $(P, \sqsubseteq)$ has a set of maximal elements, denoted $\max(P)$, and these maximal elements are often thought of as being ideal elements that are approximated by lower elements of the poset P. Non-maximal elements of P can be thought of as giving partial information about the maximal elements above them, with $a \sqsubseteq b$ meaning that b provides information that is at least as precise as the information provided by a. A familiar example is the "interval domain for the set of real numbers," i.e., the poset $\mathcal{I}$ whose members are all closed, bounded intervals of the set $\mathbb{R}$ of real numbers, including degenerate intervals of the form $[a, a] = \{a\}$ where $a \in \mathbb{R}$, and whose partial order $\sqsubseteq$ is reverse inclusion. Clearly $\max(\mathcal{I}) = \{[a, a] : a \in \mathbb{R}\}$. Thus, from the point of view of set theory, it makes sense to identify $\max(\mathcal{I})$ with the set of real numbers.

But there is more. Any domain $(P, \sqsubseteq)$ has a natural topology called the Scott topology, and $\max(P)$ is a dense subspace of P in that topology. A topological space X is said to be *(Scott) domain representable* if there is a (Scott) domain P such that X is homeomorphic to $\max(P)$ with the relative Scott topology. Determining which spaces are (Scott) domain-representable is known as "the representation question." See [18] and [4] for surveys.

Sometimes a domain $(P, \sqsubseteq)$ will carry an additional structure called a *measurement*, a concept introduced by Keye Martin in [16] and [17]. A measurement is a special kind of function μ from P to the non-negative real numbers where $\mu(p)$ is often thought of as providing a numerical measure of the amount

[1] See Section 2 for technical definitions.

of uncertainty in the information provided by the element $p \in P$, with $\mu(p) = 0$ meaning that p provides perfectly precise information. For example, in the Interval Domain $\mathcal{I}$ for the real numbers, the diameter function $\mu([a, b]) = b - a$ is a measurement and a given $[a, b]$ locates each of the maximal elements $\{c\}$ with $c \in (a, b)$ to within an error of at most $\epsilon = b - a$. See Section 2 for the rather technical definition of a measurement.

The technical properties of a measurement μ on a domain P make it easy to prove that $\ker(\mu)$, the set of all $a \in P$ with $\mu(a) = 0$, is a subset of $\max(P)$, and is a G_δ-subset of P (see Lemma 2.4.1 of [16]). A special version of the representation question asks:

> (Measurement Question) [18]: For which topological spaces X can we find a domain (or Scott domain) $(P, \sqsubseteq)$ and a measurement μ on P so that X is homeomorphic to $\ker(\mu) \subseteq \max(P)$? A more restrictive version of this question asks when we can find a domain (or Scott domain) P and a measurement μ on P such that $X = \max(P) = \ker(\mu)$.

Martin proved (Theorem 4.7 of [15]) that for domains, the two parts of the measurement problem are the same: if $X = \ker(\mu) \subseteq P$ for some measurement μ on a domain P, then for some domain Q and some measurement ν on Q, we have $X = \ker(\nu) = \max(Q)$. However, it is not clear whether Q will be a Scott domain provided P is a Scott domain. Partial solutions of the Measurement Question are announced in [18] and [19] and these partial solutions involve topological properties that are well-known in set-theoretic topology – Moore spaces, G_δ-diagonals, etc. The authors of [18] asked:

> (MMR Question): Is there a Scott domain $(P, \sqsubseteq)$ so that $\max(P)$ is a G_δ-subset of P and yet there is no measurement μ on P with $\ker(\mu) = \max(P)$?

A negative answer to the MMR Question was announced in [19], where it was claimed that there is a Scott domain P having the usual space $[0, \omega_1)$ of countable ordinals as $\max(P)$ and having the additional property that $\max(P)$ is a G_δ-subset of P. This would have solved the MMR Question in the negative, because results announced in [18] and in [19] show that $[0, \omega_1)$ cannot be $\ker(\mu)$ for any measurement on a domain. Unfortunately, we now know that $[0, \omega_1)$ cannot be $\max(P)$ for any Scott domain having $\max(P)$ a G_δ-subset of P, as the following theorem and corollary show.

Theorem 1.1 *Suppose that $(P, \sqsubseteq)$ is a Scott domain and that $X \subseteq \max(P)$ is a G_δ-subset of P with the Scott topology. Then :*

a) *there is a sequence $\langle \mathcal{G}(n) \rangle$ of open covers of X such that if $x \in G_n \in \mathcal{G}(n)$ for each n, then $\bigcap\{G_n : n \geq 1\} = \{x\}$, so that X has a G_δ-diagonal;*

b) *there is a sequence $\langle \mathcal{G}(n) \rangle$ of open covers of X such that if $x \in G_n \in \mathcal{G}(n)$, then the collection $\{\bigcap\{G_i : 1 \leq i \leq n\} : n \geq 1\}$ is a local base at x, i.e., the space X is weakly developable in the sense of [2] and therefore has a base of countable order in the sense of [20];*

c) *there is a sequence $\langle \mathcal{G}(n) \rangle$ of open covers of the space X such that if $\mathcal{F}$ is a centered collection[2] of non-empty closed subsets of X, and if for each $n \geq 1$ some $G_n \in \mathcal{G}(n)$ and some $F_n \in \mathcal{F}$ have $F_n \subseteq G_n$, then $\bigcap \mathcal{F} \neq \emptyset$, i.e., the space X is AF-complete in the sense of [2];*

d) *if X is completely regular, then X is Čech-complete;*

[2]i.e., a collection with the finite intersection property

e) *if X is T_3 and θ-refinable* [3]*, then X is a complete Moore space, and if X is paracompact then X is completely metrizable.*

Corollary 1.2 *Suppose that P is a Scott domain and that $X \subseteq \max(P)$ is a G_δ-subset of P. If the space X is regular and countably compact, then X is a compact metrizable space.*

Corollary 1.2 shows that $[0, \omega_1)$ cannot have the properties claimed in [19]. Both Theorem 1.1 and Corollary 1.2 are proved in Section 3 of our paper.

The main result in Section 4 of our paper shows that a space constructed by Burke [6] gives a negative answer to the MMR Question. Burke's space is a locally compact Hausdorff space with a G_δ-diagonal that is not a Moore space and is not θ-refinable. We describe a Scott domain P with Burke's space as $\max(P)$ and we show that $\max(P)$ is a G_δ-subset of P. Then we invoke a theorem of Martin [16] about spaces that are kernels of measurements to show that Burke's space cannot be the kernel of a measurement on P. Similar arguments show that the space Ψ of [10] is also $\max(P)$ for some Scott domain P having $\max(P)$ a G_δ-subset of P, and therefore the conclusion in part (b) of Theorem 1.1 cannot be strengthened to assert that the space X must have a sharp base in the sense of [2].

The proof of Theorem 1.1 uses the Scott domain hypothesis at several crucial points, and it is natural to ask what can be said about the space $X \subseteq \max(P)$ in case X is a G_δ-subset of P where P is a domain but perhaps not a Scott-domain. Such spaces exist: in Section 5 we show that there is a domain P such that $[0, \omega_1) = \max(P)$ is a G_δ-subset of P and yet (from Corollary 1.2) there is no Scott domain Q such that $[0, \omega_1) \subseteq \max(Q)$ is a G_δ-subset of Q. In Section 5 we investigate this situation and prove:

Theorem 1.3 *Suppose that $(P, \sqsubseteq)$ is a domain and that in the Scott topology on P, the set $X \subseteq \max(P)$ is a G_δ-subset of P. Then with the relative Scott topology, X is first-countable and domain representable, and X is a union of dense G_δ-subspaces, each of which is completely metrizable.*

Our final section lists a sequence of open questions that may interest topologists and domain theorists.

<u>Relation to the literature:</u> K. Martin (Theorem 4.1 of [15]) has shown that the following assertions about a space X are equivalent:

a) there is a domain D with $X \subseteq \max(D)$ where X is a G_δ-subset of D

b) there is a domain E such that $X = \max(E)$ and $\max(E)$ is a G_δ-subset of E.

Nevertheless, to be consistent with other parts of the literature, we state many of our results with what appears to be the more general hypothesis (a).

Chapter 5 of [16] contains many results that are related to our Theorems 1.1 and 1.3, but different in subtle ways. For example, Martin's Theorem 5.7.1 [16] shows that if P is a Scott domain and if $X \subseteq \max(P)$ is paracompact and is a G_δ-subset of P, then X is metrizable. We obtain that result as a corollary of using the theory of weak developments [2] to X. As another example, Martin's Theorem 5.7.2 [16] asserts that if D is a Scott domain and if $X \subseteq \max(D)$ is a G_δ-subset of D, then some subspace $Y \subseteq X$ is dense in X, is a G_δ-subset of X, and is completely metrizable. Our Theorem 1.3 is proved by very different methods under weaker hypotheses (we do not assume that P is a Scott domain) and gives stronger conclusions.

Throughout this paper the symbols $\mathbb{R}, \mathbb{P}$, and $\mathbb{Q}$ denote the usual spaces of real, irrational, and rational numbers.

[3] θ-refinable = submetacompact; see Section 2 for the definition.

2 Basic definitions

Let $(P, \sqsubseteq)$ be a partially ordered set (poset). For any $p \in P$ let $\uparrow(y) := \{z \in P : y \sqsubseteq z\}$ and $\downarrow(y) := \{x \in P : x \sqsubseteq y\}$. The *supremum* of a nonempty bounded set S in P is the least of all the upper bounds of S, if such exists. A subset $E \subseteq P$ is *directed* if it is non-empty and has the property that if $e_1, e_2 \in E$ then some $e_3 \in E$ has $e_1, e_2 \sqsubseteq e_3$. The poset P is a *directed complete partial order* (dcpo) if $\sup(E) \in P$ whenever E is a directed subset of P.

In any poset $(P, \sqsubseteq)$ one can define an auxiliary relation $\ll$ as follows: $a \ll b$ means that for any directed set E with $b \sqsubseteq \sup(E)$, there is some $e \in E$ with $a \sqsubseteq e$. We use the notation $\Uparrow(a) := \{b \in P : a \ll b\}$ and $\Downarrow(b) := \{a \in P : a \ll b\}$. The poset P is *continuous* if for each $b \in P$, the set $\Downarrow(b)$ is directed and has $\sup(\Downarrow(b)) = b$. A *domain* is a continuous dcpo. A *Scott domain* is a domain S with the added property that any finite bounded subset $F \subseteq S$ has $\sup(F) \in S$. Because $F = \emptyset$ is a finite bounded subset of S, it follows that a Scott domain must have a minimum element (which is $\sup(\emptyset)$). The following lemma gives an equivalent way to look at Scott domains.

Lemma 2.1 *A domain $(D, \sqsubseteq)$ is a Scott domain if and only if D has a minimum element 0_D and whenever $p, q \in P$ have $p, q \sqsubseteq r$ for some $r \in P$, then $\sup(p, q)$ exists in D.*

The requirement in Lemma 2.1 that $(D, \sqsubseteq)$ have a minimum element is a minor restriction, provided our goal is to study $\max(D)$. If D does not already have a minimum element, we let $D^+ := D \cup \{0_D\}$ where $0_D \notin D$ and we extend $\sqsubseteq$ by making the new element 0_D lie below each element of D. Then $\max(D^+) = \max(D)$, both as sets and as topological spaces.

However, as the referee pointed out to us, the ability to add a minimum element to a domain D without changing $\max(D)$ has some important consequences, because the product of any family of (Scott) domains will be a (Scott) domain provided each of the domains has a minimum element (see Exercise I-2.18 in [9]). Therefore (Scott) domain representability is arbitrarily productive, and the property of being representable as a G_δ-subset in a (Scott) domain is countably productive.

One of the most important technical results about domains is the Interpolation Lemma:

Lemma 2.2 *[18] Suppose that $a \ll c$ are points of a domain P. Then some $b \in P$ has $a \ll b \ll c$.*

The Interpolation Lemma shows that the collection $\{\Uparrow(p) : p \in P\}$ is a base for a topology on P that is called the *Scott topology.* A topological space X is *(Scott) domain representable* if there is a (Scott) domain $(P, \sqsubseteq)$ such that X is homeomorphic to $\max(P)$ when $\max(P)$ carries the relative Scott topology.

Let $[0, \infty)^*$ be the set $[0, \infty)$ with the reverse order. Then $[0, \infty)^*$ is a domain and has a Scott topology (which is not the same as the usual topology). A *measurement* [18] on a domain $(P, \sqsubseteq)$ is a function $\mu : P \to [0, \infty)^*$ that satisfies:

i) μ is continuous when both P and $[0, \infty)^*$ carry their Scott topologies.

ii) if $\mu(x) = 0$ and if $\langle p_n \rangle$ is a sequence of elements of $\Downarrow(x)$ having $\lim_{n\to\infty} \mu(p_n) = 0$, then $\{p_n : n \geq 1\}$ is a directed set whose supremum is x.

Let $\ker(\mu) := \{x \in P : \mu(x) = 0\}$. It is easy to see that if μ is a measurement on a domain P, then $\ker(\mu) \subseteq \max(P)$ and that $\ker(\mu)$ will be a G_δ-subset of P [16].

There is a sequence of properties from classical set-theoretic topology that will be important in this paper. See [2] for more details. The definitions of *G_δ-diagonal, weakly developable, and AF-complete* are given as part of Theorem 1.1. A space X is *developable* if there is a sequence $\langle \mathcal{G}(n) \rangle$ of open covers of X such that if $x \in G_n \in \mathcal{G}(n)$ then $\{G_n : n \geq 1\}$ is a neighborhood base at x. A base $\mathcal{B}$ for the space X is said to be a *base of countable order* (BCO) if the collection $\{B_n : n \geq 1\} \subseteq \mathcal{B}$ is a local base at $x \in X$ whenever $x \in B_n \in \mathcal{B}$ and B_{n+1} is a proper subset of B_n for each n. Clearly any developable space is weakly developable, and it is proved in [2] that any weakly developable space has a BCO and has a G_δ-diagonal. A space X is θ-refinable (also known as submetacompact) if for each open cover $\mathcal{U}$ of X there is a sequence of $\langle \mathcal{V}(n) \rangle$ of open covers of X such that each $\mathcal{V}(n)$ refines $\mathcal{U}$ and such that for each $x \in X$, some $n = n(x)$ has the property that $\{V \in \mathcal{V}(n) : x \in V\}$ is finite [20].

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We begin with a Scott domain $(P, \sqsubseteq)$ and a space $X \subseteq \max(P)$ that is a G_δ-subset of P in the Scott topology. Write $X = \bigcap\{D_n : n \geq 1\}$ where $D_{n+1} \subseteq D_n$ are Scott open sets.

As shown in [2], assertion (a) of the theorem actually follows from assertion (b). In addition, Martin's Proposition 5.3.5 [16] proves (a) under weaker hypotheses.

To prove assertion (b), let $\mathcal{G}(n) := \{\Uparrow(p) \cap X : p \in D_n\}$. Each $\mathcal{G}(n)$ is an open cover of $X \subseteq \max(P)$. Suppose for each $i \geq 1$ that $x \in G_i = \Uparrow(p_i) \cap X \in \mathcal{G}(i)$ where $p_i \in D_i$. Then each set $\{p_1, \cdots, p_n\}$ is bounded by x so that, P being a Scott domain, some $q_n \in P$ has $q_n = \sup\{p_1, \cdots, p_n\}$. Then $\{q_n : n \geq 1\}$ is a directed subset of P so that $\sup\{q_n : n \geq 1\} \in P$. Furthermore, $p_k \sqsubseteq q_k \sqsubseteq \sup\{q_n : n \geq 1\}$ for each k so that $p_k \in D_k$ gives $\sup\{q_n : n \geq 1\} \in D_k$. Therefore $\sup\{q_n : n \geq 1\} \in \bigcap\{D_k : k \geq 1\} = X$. But we also know that $\sup\{q_n : n \geq 1\} \sqsubseteq x \in X$ so that $\sup\{q_n : n \geq 1\} = x$. Now consider any relative Scott neighborhood $\Uparrow(r) \cap X$ of x. Because $r \ll x = \sup\{q_n : n \geq 1\}$ we know that $r \sqsubseteq q_m$ for some m. But then $\Uparrow(q_m) \subseteq \Uparrow(r)$ and we have $\bigcap\{\Uparrow(p_i) : 1 \leq i \leq m\} \subseteq \Uparrow(q_m) \subseteq \Uparrow(r)$ showing that $\bigcap\{G_i : 1 \leq i \leq m\} \subseteq \Uparrow(r) \cap X$ as required to prove assertion (b).

To prove assertion (c), notice that we can replace $\mathcal{F}$ with the collection $\hat{\mathcal{F}}$ of all finite intersections of members of $\mathcal{F}$. Therefore, with no loss of generality, we may assume that $\mathcal{F}$ is closed under finite intersections.

Let $E := \{p \in P : \text{ for some } F \in \mathcal{F}, F \subseteq \Uparrow(p) \cap X\}$. We claim that E is a directed set. For let $p_1, p_2 \in E$. Choose $F_i \in \mathcal{F}$ with $F_i \subseteq \Uparrow(p_i) \cap X$ for $i = 1, 2$. Because $\mathcal{F}$ is closed under finite intersections, the set $F_3 = F_1 \cap F_2 \in \mathcal{F}$. Let $p \in F_3$. Then $p \in \Uparrow(p_1) \cap \Uparrow(p_2)$ so $p_i \sqsubseteq p$ for $i = 1, 2$. Because P is a Scott domain, some $p_3 \in P$ has $p_3 = \sup\{p_1, p_2\}$. Then $F_3 \subseteq \Uparrow(p_1) \cap \Uparrow(p_2) \subseteq \Uparrow(p_3)$ so that $p_3 \in E$. Hence E is directed. Therefore some $r \in P$ has $r = \sup(E)$.

We claim that $r \in X$. By hypothesis on $\mathcal{F}$, for each $n \geq 1$ we may choose $F_n \in \mathcal{F}$ and $r_n \in D_n$ with $F_n \subseteq \Uparrow(r_n) \cap X$. Then $r_n \in E$ so that $r_n \sqsubseteq \sup(E) = r$. Because $r_n \sqsubseteq r$ and $r_n \in D_n$ we know that $r \in D_n$ so that $r \in \bigcap\{D_n : n \geq 1\} = X$, as claimed.

We claim that $r \in \bigcap \mathcal{F}$. If not, then some $\hat{F} \in \mathcal{F}$ has $r \notin \hat{F}$. Because $\hat{F}$ is closed in X, there is some $s \in P$ with $r \in \Uparrow(s) \cap X \subseteq X - \hat{F}$. Then we have $s \ll r = \sup\{r_n : n \geq 1\}$ so that for some n we have $s \sqsubseteq r_n$. Then $F_n \subseteq \Uparrow(r_n) \cap X \subseteq \Uparrow(s) \cap X \subseteq X - \hat{F}$. But then $F_n, \hat{F}$ are disjoint members of the centered collection $\mathcal{F}$ and that is impossible. Therefore $r \in \bigcap \mathcal{F}$ as required by (c).

Assertion (d) now follows directly, because the AF-completeness property characterizes Čech-completeness for completely regular spaces.

The first part of assertion (e) follows from assertion (b) plus the fact that a θ-refinable space with a BCO must be a Moore space [20], and that a Čech-complete Moore space is Moore-complete [1]. The second part of assertion (e) follows from the first part plus the fact that any paracompact Moore space must be metrizable. Then apply assertion (c) to conclude that X is a Čech-complete metrizable space, and hence is completely metrizable. □

Corollary 3.1 *Suppose $(P, \sqsubseteq)$ is a Scott domain and that $X \subseteq \max(P)$ is a G_δ-subset of P with the Scott topology. If X is a countably compact regular space then X is metrizable. In particular, the usual space $[0, \omega_1)$ is not homeomorphic to $\max(P)$ for any Scott domain P in which $\max(P)$ is a G_δ-set.*

Proof: Chaber [7] has proved that a countably compact T_3 space with a G_δ-diagonal must be metrizable. Now apply part (a) of Theorem 1.1. □

4 Examples

In this section we show that a space of D.K. Burke [6] can be used to give a negative answer to the MMR Question from the Introduction. One part of this proof uses the easy part of a result announced in [18] and [19]. To the best of our knowledge, no proof has ever appeared, so we provide it here.

Lemma 4.1 *Suppose that $\mu : P \to [0, \infty)^*$ is a measurement on a domain P. Then $\ker(\mu)$ is a developable T_1-space.*

Proof: Write $X = \ker(\mu)$. As noted in the Introduction, a result of Martin of [15] shows $\ker(\mu) \subseteq \max(P)$. For any domain P, the subspace $\max(P)$ is a T_1-space, so that X is also T_1.

Recall that $[0, \infty)^*$ is the set $[0, \infty)$ with the order reversed, and carries the Scott topology in which basic neighborhoods of 0 are sets of the form $[0, \frac{1}{n})$. For each $n \geq 1$ and each $x \in X$ we know that $\mu(x) = 0 \in [0, \frac{1}{n})$ so that there is some $p(x, n) \in P$ with $p(x, n) \ll x$ and $\mu[\Uparrow(p(x, n))] \subseteq [0, \frac{1}{n})$. Defining the points $p(x, n)$ recursively, we may assume that $p(x, n) \ll p(x, n+1) \ll x$ for each n. Let $G(x, n) := \Uparrow(p(x, n)) \cap X$. Then $\mathcal{G}(n) := \{G(x, n) : x \in X\}$ is an open cover of X.

To show that the sequence $\langle \mathcal{G}(n) \rangle$ is a development for X, fix a point x in a basic open set $\Uparrow(q) \cap X$, and consider any choice of $G(y_n, n) \in \mathcal{G}(n)$ with $x \in G(y_n, n)$. We will show that for some $n \geq 1$, we have $G(y_n, n) \subseteq \Uparrow(q) \cap X$. We know that $q \ll x$. Because $x \in G(y_n, n) = \Uparrow(p(y_n, n)) \cap X$ we know that $p(y_n, n) \ll x$. By choice of $p(y_n, n)$ we know that $\lim\{\mu(p(y_n, n)) : n \geq 1\} = 0$ and from $x \in \ker(\mu)$ we know that $\mu(x) = 0$. Now the definition of a measurement (see Section 2) tells us that $\{p(y_n, n) : n \geq 1\}$ is a directed set and its supremum is x. Because $q \ll x = \sup\{p(y_n, n) : n \geq 1\}$ we have some $n \geq 1$ with $q \sqsubseteq p(y_n, n)$. But then $\Uparrow(p(y_n, n)) \subseteq \Uparrow(q)$ showing that $G(y_n, n) \subseteq \Uparrow(q) \cap X$, as required. □

Example 4.2 *There is a locally compact Hausdorff space X that has a G_δ-diagonal and is not developable, and a Scott domain $\mathcal{P}$ such that X is homeomorphic to $\max(\mathcal{P})$ where $\max(\mathcal{P})$ is a G_δ-subset of $\mathcal{P}$, but there is no measurement $\mu : \mathcal{P} \to [0, \infty)^*$ with $\ker(\mu) = \max(\mathcal{P})$.*

Proof: We want to thank the referee for suggestions that substantially improved the approach to this example that we used in an earlier version of this paper. We use a space described by Burke in [6]. We only need part of Burke's construction and we change his notation somewhat. Let Z be the usual Cantor set in the unit interval. Let $\mathcal{A}$ be a family of countably infinite subsets of Z that is maximal with respect to the following two properties:

B1) if A_1, A_2 are distinct members of $\mathcal{A}$, then $A_1 \cap A_2$ is finite;

B2) each $A \in \mathcal{A}$ has a unique cluster point in $\mathbb{R}$.

Burke actually had a third property is his list but noted that the third property was needed only for a related example.

Let $\mathcal{A} = \{A_i : i \in I\}$ where I is an index set that is disjoint from Z. For each $i \in I$, let z_i be the unique cluster point of A_i in $\mathbb{R}$.

Burke's space is $X = I \cup Z$, with each point of Z being isolated and where a point $i \in I$ has basic neighborhoods of the form $N(i, F) = \{i\} \cup (A_i - F)$ where F is a finite subset of A_i. As Burke proved, this space is locally compact, Hausdorff, and has a G_δ-diagonal, but is not developable.

First we describe a new neighborhood base for the non-isolated points of X. For any open interval (a, b) in $\mathbb{R}$, let $(a, b)^* := (a, b) - \{\frac{a+b}{2}\}$. For each $i \in I$ and $\epsilon > 0$, let

$$M(i, \epsilon) = \{i\} \cup ((z_i - \epsilon, z_i + \epsilon)^* \cap A_i) .$$

Because z_i is the unique cluster point of A_i in $\mathbb{R}$, and because A_i is a subset of the compact set Z, the set $A_i - (z_i - \epsilon, z_i + \epsilon)$ must be finite. Therefore, each set $M(i, \epsilon)$ is open in Burke's space. Because only a finite number of points of A_i are removed when making Burke's set $N(i, F)$, it is easy to see that every set $N(i, F)$ contains some $M(i, \frac{1}{n})$. Therefore, in Burke's space, the collection $\{M(i, \frac{1}{k}) : k \geq 1\}$ is an open neighborhood base at i for each $i \in I$.

Let $\mathcal{P}$ be the collection of all nonempty compact subsets of the locally compact space X and let $\sqsubseteq$ be reverse inclusion. Then $\max(\mathcal{P}) = \mathcal{I} \cup \mathcal{Z}$ where $\mathcal{I} = \{\{i\} : i \in I\}$ and $\mathcal{Z} = \cup\{\{z\} : z \in Z\}$, and for $K_1, K_2 \in \mathcal{P}$ we have $K_1 \ll K_2$ if and only if $K_2 \subseteq \mathrm{Int}(K_1)$. Because X is locally compact, general theory shows that $(\mathcal{P}, \sqsubseteq)$ is a Scott domain that represents X under the mapping that sends each $i \in I$ to $\{i\}$ and each $z \in Z$ to $\{z\}$. Thus we may identify X with $\max(\mathcal{P})$.

Next we show that $\max(\mathcal{P})$ is a G_δ-subset of $\mathcal{P}$. Let $\mathcal{D}(n) := \mathcal{Z} \cup \bigcup\{\Uparrow(M(i, \frac{1}{n})) : i \in I\}$. Because $\{z\} \ll \{z\}$ for each $z \in Z$, each $\{z\}$ is isolated in $\mathcal{P}$ so that the set $\mathcal{D}(n)$ is a Scott-open subset of $\mathcal{P}$. Clearly $\max(\mathcal{P}) \subseteq \bigcap\{\mathcal{D}(n) : n \geq 1\}$. To verify the reverse inclusion, suppose $P_0 \in \bigcap\{\mathcal{D}(n) : n \geq 1\}$ and $P_0 \notin \max(\mathcal{P})$. Then $|P_0| \geq 2$ and for each $n \geq 1$ there is an $i_n \in I$ with $P_0 \in \Uparrow(M(i_n, \frac{1}{n}))$. In case $P_0 \cap I \neq \emptyset$, fix $i_0 \in P_0 \cap I$. Then $i_0 \in P_0 \subseteq M(i_n, \frac{1}{n})$ gives $i_n = i_0$ for each n so that $P_0 \subseteq \bigcap\{M(i_0, \frac{1}{n}) : n \geq 1\} = \{i_0\}$ and that is impossible because $|P_0| \geq 2$. Therefore $P_0 \cap I = \emptyset$, which gives $P_0 \subseteq Z$. But then $P_0 \subseteq M(i_n, \frac{1}{n}) - \{i_n\}$ so that P_0 has diameter $< \frac{2}{n}$ for each n and that is impossible because $|P_0| \geq 2$. Therefore, $\bigcap\{\mathcal{D}(n) : n \geq 1\} = \max(\mathcal{P})$ as required.

It remains only to show that there is no measurement $\mu : \mathcal{P} \to [0, \infty)^*$ with $\ker(\mu) = \max(\mathcal{P})$. If such a measurement existed, then Lemma 4.1 would tell us that Burke's space $X = \max(\mathcal{P})$ would be developable. But that is exactly the property that Burke's space does not have, so that the proof of Example 4.2 is complete. □

Remark 4.3 In many examples in the literature, if there is a way to define the diameter of a member of a domain P in such a way that the elements of $\max(P)$ are exactly the elements of P having diameter zero, then $\max(P)$ turns out to be the kernel of a measurement. (See, for example, the diameter measurement on the interval domain $\mathcal{I}$ in the Introduction.) The referee pointed out that the usual distance function in $\mathbb{R}$ gives a diameter measure for elements of the domain $\mathcal{P}$ in Example 4.2 above, and yet (as we show) there is no measurement μ with $\max(\mathcal{P}) = \ker(\mu)$. To explain this rather subtle point, fix any $z \in Z$ and then choose $z_k \in (z - \frac{1}{k}, z + \frac{1}{k})$ with $z_k \notin \{z, z_1, \cdots, z_{k-1}\}$. This is possible because the Cantor set is dense-in-itself. Let $F_k := \{z, z_k\}$. Then $F_k \in \mathcal{P}$ and $diam(F_k) < \frac{1}{k}$ so that $diam(F_k) \to 0$. Because $z \ll z$ we know that $F_k \in \Downarrow(z)$. If the diameter function gave rise to a measurement on $\mathcal{P}$, then the set $\{F_k : k \geq 1\}$ would need to be directed, and that is clearly not the case.

Remark 4.4 In Example 2.17 of [11], Gruenhage described a locally compact, Hausdorff, sub-metrizable space that is not a Moore space. Like Burke's space, this space is homeomorphic to $\max(P)$ for some Scott domain P, where $\max(P)$ is a G_δ-subset of P, but it is not the kernel of any measurement on a domain.

Remark 4.5 A property called a "sharp base" is studied in [2] and is stronger than "weakly developable". As proved in [2], the space Ψ of [10] does not have a sharp base. However, a proof similar to the one given for Example 4.2 shows that Ψ is a G_δ-subset of a Scott domain. Consequently, one cannot strengthen assertion (b) of Theorem 1.1 to assert that if P is a Scott domain and $\max(P)$ is a G_δ-subset of P, then $\max(P)$ must have a sharp base in the sense of [2].

Generalized ordered spaces (GO-spaces) have been an important source of examples in set theoretic topology (e.g., the Michael line or the Sorgenfrey line) and it is known from [8] that any GO-space constructed on $\mathbb{R}$ will be Scott-domain representable. This suggests that GO-spaces on $\mathbb{R}$ might be a valuable source of pathological examples related to the measurement problem. However, our next corollary shows that pathological GO-spaces will not have a major role to play in that problem.

Corollary 4.6 *Suppose that X is a GO-space and that $X \subseteq \max(P)$ for some Scott domain P where X is a G_δ-set in P. Then X is completely metrizable.*

Proof: Part (a) of Theorem 1.1 shows that X has a G_δ-diagonal and is therefore paracompact [13], so that part (e) of the theorem forces X to be completely metrizable. □

Question (x) of [16] asked whether there is a measurement on a (Scott) domain whose kernel is normal and not metrizable. Our next result shows that the answer is "Consistently, yes." Probably this is known, but we have not been able to find it in the literature.

Example 4.7 *If there is a Q-set (an uncountable subspace of $\mathbb{R}$ in which every subset is a relative G_δ) then there is a separable, normal, non-metrizable Moore space X, a Scott domain $\mathcal{D}$, and a measurement μ on $\mathcal{D}$ such that $X = \ker(\mu) = \max(\mathcal{D})$.*

Sketch of Proof: Let Y be a Q-set in $\mathbb{R}$ and let

$$X := (Y \times \{0\}) \cup \left(\mathbb{R} \times \{\frac{1}{n} : n \geq 1\}\right),$$

topologized so that each point $(x, \frac{1}{n})$ has its usual Euclidean neighborhoods and so that neighborhoods of each point $(y, 0)$ are sets of the form $T(y, n) \cap X$ where $T(y, n) \subseteq \mathbb{R} \times [0, \infty)$ is a vertical isosceles triangle with:

i) $(y, 0)$ is a vertex if $T(y, n)$

ii) the vertex angle at $(y, 0)$ goes from $\frac{\pi}{2} - \frac{1}{2n}$ to $\frac{\pi}{2} + \frac{1}{2n}$ and the height of the triangle is $\frac{1}{n}$;

iii) the intersection of $T(y, n)$ with each horizontal line $H_n := \mathbb{R} \times \{\frac{1}{n}\}$ is an open interval on H_n.

The space X is a separable, normal, non-metrizable Moore space. To define $\mathcal{D}$, let $C(y, n)$ be the closure of $T(y, n)$ in X (which is the same as the closure of $T(y, n)$ in the Euclidean topology). Let $\mathcal{D}$ be the collection of all $C(y, n)$ together with all sets that are finite unions of sets of the form $[a, b] \times \{\frac{1}{n}\}$ where $a \leq b$, and all sets of the form $\{(y, 0)\}$ for $y \in Y$. Let $\sqsubseteq$ be reverse inclusion. Then in $(\mathcal{D}, \sqsubseteq)$ we have $D_1 \ll D_2$ if and only if $D_2 \subseteq \mathrm{Int}_X(D_1)$, and $(\mathcal{D}, \sqsubseteq)$ is a Scott domain with $\max(\mathcal{D})$ homeomorphic to X under the mapping $\{x\} \to x$. Let $diam(D)$ denote the usual Euclidean diameter of a set $D \in \mathcal{D}$ and define $\mu(D) = diam(D)$. Then μ is a measurement on D and $\ker(\mu) = \max(D)$, as required. □

5 What to do without Scott

Theorem 1.1 studied properties of a space $X \subseteq \max(P)$ in case P is a Scott domain and X is a G_δ-subset of P. The proof of Theorem 1.1 used the Scott-domain property in several simple, but apparently unavoidable, ways. This section proves Theorem 1.3, which explores what can be said about a space $X \subseteq \max(P)$ that is a G_δ-subset of a domain P where P is not necessarily a Scott domain. Then we present an example showing that there is a domain (but not a Scott domain) P in which the space $[0, \omega_1) = \max(P)$ is a G_δ-subset of P. Finally, we characterize spaces that are the kernels of a measurement on a domain.

To prove Theorem 1.3, we begin with a domain $(P, \sqsubseteq)$ where $X \subseteq \max(P)$ is a G_δ-subset of P, and we write $X = \bigcap\{D_n : n \geq 1\}$ where $D_{n+1} \subseteq D_n$ and each D_n is open in the Scott topology of P.

K. Martin's Proposition 5.7.1 [16] shows that the space X will be first countable. Martin also shows that if $X = \max(P)$, then X is a Baire space. We can prove more: the subspace $\max P$ of P is certainly domain-representable and then Theorem 3.2 from [3] shows that X is also domain representable (but using some other domain).

To complete the proof of Theorem 1.3, fix any $x_0 \in X$. We will show that the point x_0 belongs to a dense G_δ-subset T of X that is completely metrizable. Let $E_1 \subseteq D_1$ have the property that $\{\Uparrow(p) \cap X : p \in E_1\}$ is a maximal pairwise disjoint collection of non-empty subsets of X and $x_0 \in E_1$. Suppose $n \geq 1$ and E_n is given with $x_0 \in E_n$. For each $p \in E_n$ let $E_{n+1}(p)$ be a subset of $D_{n+1} \cap \Uparrow(p)$ such that $\{\Uparrow(q) \cap X : q \in E_{n+1}(p)\}$ is a maximal pairwise disjoint collection of nonempty subsets of $\Uparrow(p) \cap X$. Make sure that the point $x_0 \in E_{n+1}(x_0)$ and let $E_{n+1} = \bigcup\{E_{n+1}(p) : p \in E_n\}$. Then for each $n \geq 1$ the set $O_n := \bigcup\{\Uparrow(p) \cap X : p \in E_n\}$ is a dense open set in X so that, X being a Baire space [14], the set $T = \bigcap\{O_n : n \geq 1\}$ is dense in X. Also, note that $x_0 \in T$.

For each $n \geq 1$ let $\mathcal{G}(n) := \{\Uparrow(p) \cap T : p \in E_n\}$. Each $\mathcal{G}_n$ is a pairwise-disjoint relatively open cover of T. Hence each $\mathcal{G}(n)$ is a discrete collection in the subspace T. We show that $\mathcal{G} := \bigcup\{\mathcal{G}(n) : n \geq 1\}$ is a base for T. Let $x \in T$ and suppose U is a relatively open subset of $\max(P)$ with $x \in U$. Choose $\hat{q} \in P$ with $x \in \Uparrow(\hat{q}) \cap X \subseteq U$. Then $\hat{q} \ll x$ so that the Interpolation Lemma yields some $q \in P$ with $\hat{q} \ll q \ll x$. Then $\Uparrow(q) \cap T \subseteq \uparrow(q) \cap T \subseteq \Uparrow(\hat{q}) \cap T \subseteq U$.

Because $x \in T$, there is a unique sequence $p_n \in E(n)$ with $x \in \Uparrow(p_n)$. Because of the way that $E(n+1)$ was constructed from $E(n)$, and because $x \in \Uparrow(p_{n+1}) \cap \Uparrow(p_n)$, we know that $p_n \ll p_{n+1}$. Hence the set $S := \{p_n : n \geq 1\}$ is directed so that some point $r \in P$ has $r = \sup(S)$. Because x is an upper bound for F, we know that $r \sqsubseteq x$. Because $p_n \in D_n$ and $p_n \sqsubseteq \sup(S) = r$, we know that $r \in D_n$ for each n and therefore $r \in \bigcap\{D_n : n \geq 1\} = X \subseteq \max(P)$. Then $r \sqsubseteq x$ gives $r = x$.

At this point we have $q \ll x = \sup(S)$ so that some $p_n \in S$ has $q \sqsubseteq p_n$. Then $x \in \Uparrow(p_n) \cap T \subseteq \uparrow(q) \cap T \subseteq U$. Because $\Uparrow(p_n) \cap T \in \mathcal{G}$ we see that $\mathcal{G}$ is a base for T.

Because each collection $\mathcal{G}(n)$ is a pairwise-disjoint relatively open cover of T, each member of each $\mathcal{G}(n)$ is both closed and open in T. Therefore, the subspace T of X is regular so that the Bing-Nagata-Smirnov metrization theorem now guarantees that the subspace T is metrizable.

Because, as noted in the second paragraph of this proof, X is domain representable and T is a G_δ-subspace of X, it follows from Theorem 3.2 of [3] that T is also domain representable. But any domain-representable metrizable space is completely metrizable, and this completes the proof of Theorem 1.3. $\square$

Our next example shows that there are spaces of the type studied in Theorem 1.3 that are not of the type studied in Theorem 1.1.

Example 5.1 *There is a domain P such that* $\max(P)$ *is the usual space* $[0, \omega_1)$ *of countable ordinals and* $\max(P)$ *is a G_δ-subset of P, and there is another domain Q such that* $\max(Q)$ *is* $[0, \omega_1)$ *and* $\max(Q)$ *is* not *a G_δ-subset of Q.*

Proof: Let Lim be the set of limit ordinals in $[0, \omega_1)$. For each $\lambda \in Lim$ choose a strictly increasing sequence $\alpha(n, \lambda)$ of non-limit ordinals whose supremum is λ. For each $n < \omega$, level n of the poset P is the set

$$P(n) := \{([\alpha(n, \lambda), \lambda], n) : \lambda \in Lim\} \cup \{(\{\beta\}, n) : \beta \in [0, \omega_1) - Lim\},$$

and level ω of P is the set

$$P(\omega) := \{(\{\gamma\}, \omega) : \gamma < \omega_1\}.$$

Let $P := \bigcup\{P(n) : n \leq \omega\}$. For any ordered pair (u, v), let $\pi_1(u, v) = u$ and $\pi_2(u, v) = v$. Partially order P by the rule that $p \sqsubseteq q$ if and only if one of the following happens:

a) $p = q$, or

b) $p \neq q$ and $\pi_1(q) \subseteq \pi_1(p)$ and $\pi_2(p) < \pi_2(q)$.

In other words, for distinct $p, q \in P$, $\sqsubseteq$ is reverse inclusion in the first coordinate and strict increase in the second coordinate. It is straightforward to prove that $(P, \sqsubseteq)$ is a domain with $\max(P) = P(\omega)$ and that in the relative Scott topology, $\max(P)$ is a copy of $[0, \omega_1)$. One must check that $p \ll p$ for each $p \in P - P(\omega)$ while $q \ll q$ is false for every $q \in P(\omega)$. To see that $\max(P)$ is a G_δ-subset of P, let

$$D(n) := \bigcup\{\Uparrow(p) : p \in P(n)\}.$$

Then $D(n)$ is a Scott open set and $\max(P) \subseteq \bigcap\{D(n) : n \geq 1\}$. To show that $\max(P) = \bigcap\{D(n) : n \geq 1\}$, let $q \in \bigcap\{D(n) : n \geq 1\}$. Then $\pi_2(q) \geq n$ for each n so that $\pi_2(q) = \omega$, as required. We now have a domain P with $\max(P) = [0, \omega_1)$ where $\max(P)$ is a G_δ-subset of P.

Next we note that the space $X = [0, \omega_1)$ is locally compact and Hausdorff, and therefore is homeomorphic to $\max(Q)$ where Q is the collection of all non-empty compact subsets of $[0, \omega_1)$ ordered by reverse inclusion. Because Q is a Scott domain, Corollary 1.2 shows that $\max(Q)$ cannot be a G_δ-subset of Q. □

Finally, we extend a theorem from [5] to characterize regular spaces that are the kernel of a measurement on a domain. The equivalence of (a) and (e) in the theorem was probably what the authors of [18] had in mind when they announced that there is "a completeness condition $\mathcal{C}$ such that a T_1 space is developable with completeness condition $\mathcal{C}$ iff it is the kernel of a measurement on a continuous dcpo" but they did not identify what condition $\mathcal{C}$ is.

Theorem 5.2 *For a T_3-space X, the following are equivalent:*

a) X has a development $\langle \mathcal{G}(n) \rangle$ such that $\mathcal{G}(n+1) \subseteq \mathcal{G}(n)$ and such that if $G_n \in \mathcal{G}(n)$ has $\mathrm{cl}_X(G_{n+1}) \subseteq G_n$ for each n, then $\bigcap\{G_n : n \geq 1\} \neq \emptyset$, i.e. X is a Rudin-complete developable space;

b) X is developable and subcompact, i.e. X has a base $\mathcal{B}$ such that if $\mathcal{C} \subseteq \mathcal{B}$ has the property that for each $C_1, C_2 \in \mathcal{C}$ some $C_3 \in \mathcal{C}$ has $\mathrm{cl}(C_3) \subseteq C_1 \cap C_2$, then $\bigcap \mathcal{C} \neq \emptyset$;

c) X is developable and the non-empty player has a winning strategy in the strong Choquet game on X (see [5] for definitions);

d) X is developable and domain-representable;

e) there is a domain P and a measurement μ on P such that X is homeomorphic to $ker(\mu) = \max(P)$.

Sketch of proof: Statements (a), (b), (c), and (d) are equivalent in the light of [5]. Lemma 4.1 shows that any space satisfying (e) must be developable so that assertion (e) clearly implies (d). To complete the proof, suppose X satisfies (b). Then there is a development $\langle \mathcal{G}(n) \rangle$ for X with $\mathcal{G}(n) \subseteq \mathcal{B}$ and $\mathcal{G}(n+1) \subseteq \mathcal{G}(n)$. Define $P(\omega) := \{(\{x\}, \omega) : x \in X\}$ and for $n < \omega$ let $P(n) := \{(G, n) : G \in \mathcal{G}(n)\}$. Let $P = \bigcup\{P(n) : 1 \leq n \leq \omega\}$. Partially order P by the rule that $(G_1, n_1) \sqsubseteq (G_2, n_2)$ if an only if either $G_1 = G_2$ and $n_1 = n_2$ or else $\mathrm{cl}(G_2) \subseteq G_1$ and $n_1 < n_2$. Because X is developable, for any directed subset $E \subseteq P$, either E contains a maximal element or else $\sup(E) = \bigcap\{\pi_1(e) : e \in E\}$ is a singleton set, where $\pi_1(e)$ denotes the first coordinate of the ordered pair $e \in E$. Whenever $(G, n) \in P$ with $n < \omega$ we have $(G, n) \ll (G, n)$, and it never happens that $e \ll e$ where $e \in P(\omega)$. Consequently, $(P, \sqsubseteq)$ is a domain with $\max(P) = P(\omega)$, and the function $x \to (\{x\}, \omega)$ is a homeomorphism from X onto $\max(P)$. We define a function μ on P by the rule that $\mu(G, n) = \frac{1}{n}$ whenever $n < \omega$, and $\mu((\{x\}, \omega) = 0$. Once again using the fact that X is developable, we show that μ is a measurement on P, as required. □

6 Open Questions

The following questions remain open and are likely to be of interest to both topologists and domain theorists.

Question 6.1 *(For metric spaces) Suppose X is completely metrizable. In a comment just after Example 4.3 of [15], Martin noted that for any complete metric space X, there is a domain D_X and a measurement μ_X on D_X with $X = \ker(\mu_X) = \max(D_X)$. We ask whether there is a Scott domain P such that X is homeomorphic to $\max(P)$ and $\max(P)$ is a G_δ-subset of P (with the Scott topology). We emphasize that, in this question, P must be a Scott domain. In [16], Martin asked whether every complete metric space can be embedded as a dense G_δ-subset of a Scott domain, an apparently easier question that also remains open. Martin's Proposition 5.7.2 in [16] provides an affirmative answer to his apparently easier question in case X is a complete* separable *metric space.*

Question 6.2 *(For Moore spaces) Is it true that for each Scott-domain representable Moore space Y there must be a Scott domain P such that $\max(P)$ is homeomorphic to Y and is a G_δ-subset of P? Is it true that a Scott-domain representable Moore space must be the kernel of some measurement on some Scott domain? Theorem 5.2 above shows that a Scott-domain representable Moore space must be the kernel of a measurement on some domain P, but the domain that we construct is not likely to be a Scott domain. A related result is Theorem 4.12 of [18] announcing that every Čech-complete Moore space Y is $\ker(\mu)$ for some measurement μ_Y on some domain D_Y, and it is known that any Scott-domain-representable, completely regular Moore space is Čech complete [14]. But we do not know whether the domain D_Y in which $Y = \ker(\mu_Y)$ is a Scott domain.*

Question 6.3 *Which spaces are $\max(P)$ for some Scott domain P, where $\max(P)$ is a G_δ-subset of P? Our Theorems 1.1 and 1.3 give necessary conditions, and results in [12] give necessary conditions for the Scott-domain question.*

Finally, recall a classical result: if a completely regular space X is a G_δ-subset of some compact Hausdorff space, then X is a G_δ subset of every compact Hausdorff space in which X densely embeds. In an earlier version of this paper, we asked whether there is a domain-theoretic analog of that classical assertion. We asked which spaces Y must be G_δ-subsets of every domain or Scott domain P with $Y = \max(P)$. This question might sound natural, but it does not have a reasonable answer. Consider a one-point space. The linearly ordered sets $P = [0, \omega]$ and $Q = [0, \omega_1]$ are both Scott domains (in their usual order) and $\max(P)$ and $\max(Q)$ are each the one-point space. However, $\max(P)$ is a G_δ-subset of P while $\max(Q)$ is not a G_δ-subset of Q. Also, note that in the Scott domain $Q = [0, \omega_1]$, $\max(Q)$ is completely metrizable and not a G_δ-subset of Q. This answers question (viii) of Chapter 5 of [16].

References

[1] Aarts, J. and Lutzer. D., Completeness properties designed for recognizing Baire spaces, *Dissertationes Mathematicae* 114(1974).

[2] Alleche, B., Arhangelskii, A., and Calbrix, J., Weak developments and metrization, *Topology Appl.* 100(2000), 23-38.

[3] Bennett, H. and Lutzer, D., Domain representable spaces, *Fundamenta Math.*, 189(2006), 255-268.

[4] Bennett, H., and Lutzer, D., Domain representable spaces and completeness, submitted.

[5] Bennett, H., Lutzer, D., and Reed, G.M., Domain representability and the Choquet game in Moore and BCO spaces, *Topology and Appl.* 155(2008), 445-458.

[6] Burke, D., A non-developable locally compact Hausdorff space with a G_δ-diagonal, *General Topology and its Applications* 2(1972), 287-291.

[7] Chaber, J., Conditions which imply compactness in countably compact spaces, *Bull. Acad. Pol. Sci. Ser. Math.* 24(1976), 993-998.

[8] Duke, K. and Lutzer, D., Scott domain representability of a class of generalized ordered spaces, *Topology Proc.*, to appear.

[9] Gierz, G., Hoffman, K.H., Keimal, K., Lawson, J.D., Mislove, M., and Scott, D., *Continuous Lattices and Domains*, vol. 93 in *Encyclopedia of Mathematics and its Applications*, Cambridge, 2003.

[10] Gillman, L. and Jerrison, M., *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

[11] Gruenhage, G., Generalized metric spaces, in *Handbook of Set-Theoretic Topology* ed. by. K. Kunen and J. Vaughan, North-Holland, Amsterdam, 1984.

[12] Kopperman, R., Kunzi, H., and Waszkiewicz, P., Bounded complete models of topological spaces, Topology Appl. 139(2004), 285- 297.

[13] Lutzer, D., On generalized ordered spaces, *Dissertationes Mathematicae* 89 (1971).

[14] Martin, K., Topological games in domain theory, *Topology Appl.* 129(2003), 177-186.

[15] Martin, K., Ideal models of spaces, *Theoretical Computer Science* 305(2003), 277-297.

[16] Martin, K., A foundation for computation, Ph.D. Thesis, Tulane University, New Orleans, 2000.

[17] Martin, K., The measurement process in domain theory, Proc. 27th Internat. Colloq. on Automata, Languages, and Programming in *Lecture Notes in Computer Science* 1853(2000), 116-126.

[18] Martin, K., Mislove, M., and Reed, G.M., Topology and domain theory, pp. 371-394 in *Recent Progress in General Topology, II* ed. by M. Hušek and J. van Mill, Elsevier, Amsterdam, 2002.

[19] Reed, G.M., Measurements on domains and topology, *Electronic Notes in Theoretical Computer Science* 40(2001), 303.

[20] Wicke, H., and Worrell, J., Characterizations of developable topological spaces, *Canad. J. Math.* 17 (1965). 820-830.