On a Question of Maarten Maurice

Harold Bennett, Texas Tech University, Lubbock, TX 79409
David Lutzer, College of William and Mary, Williamsburg, VA 23187-8795

May 19, 2005

Abstract

In this note we give ZFC results that reduce the question of Maarten Maurice about the existence of σ-closed-discrete dense subsets of perfect generalized ordered spaces to the study of very special Baire spaces, and we discuss the current status of the question for spaces with small density. Work of Shelah, Todorčevic, Qiao, and Tall shows that Maurice's problem is undecidable for generalized ordered spaces of local density ω_1.

MR Classifications: Primary 54F05; Secondary 54E52, 54D65, 54A35.

1 Introduction

Recall that a space X is *perfect* if each closed subset of X is a G_δ-set in X. It is well-known that, among generalized ordered spaces, any space with a σ-closed-discrete dense subset must be perfect [8]. Probably the most challenging and important open problem in the theory of generalized ordered (GO)-spaces asks about the converse of that assertion, and was posed by Maarten Maurice (see [17]) in the 1970s:

In ZFC, is there a perfect GO-space that does not have a σ-closed-discrete dense subset?

Maurice's question includes the phrase "in ZFC" for good reason: if a Souslin space exists, it would provide a counterexample, and Souslin spaces do exist in some models of ZFC (e.g., under $V = L$). Therefore, another way to phrase Maurice's question would be "Is there a model of ZFC in which each perfect GO-space has a σ-closed-discrete dense subset?"

Although Maurice's question was originally posed as an ordered space question, it turned out to be a particularly sharp version of a broader open question (see "Open Problem 9" in [9]) that asks whether there is any ZFC example of a T_3-space that is perfect and does not have a σ-closed-discrete dense subset.

Recent decades have seen significant progress on Maurice's question. An important paper by Qiao and Tall [13] showed that Maurice's question is equivalent to another question posed by Peter Nyikos

a) (Nyikos) *In ZFC, is there a non-metrizable, perfect, non-Archimedean space?*

and examined related questions such as

b) (Heath) *In ZFC, is there a non-metrizable perfect GO-space with a point-countable base?*

In Section 2 of this note we present ZFC decomposition theorems that link the questions of Maurice, Heath, and Nyikos to the study of first-category subsets of certain special Baire spaces. In Section 3, we will summarize consistency results of Shelah, Todorčevic, Qiao, and Tall that provide models which cannot contain any space of small size or small density that would be a counterexample to Maurice's question. We summarize the results as follows:

Theorem: *It is undecidable in ZFC whether every perfect GO-space of local density ω_1 must have a σ-closed-discrete dense subset, and whether every perfect GO-space with local density ω_1 and a point-countable base must be metrizable.*

The questions of Maurice, Nyikos, and Heath remain open for spaces with local density $> \omega_1$.

Work of W-X Shi links Maurice's question to a more technical open question that asks whether each perfect generalized ordered space can be topologically embedded in some perfect linearly ordered space. We discuss this issue in Section 3, below.

Recall that a *GO-space* is a triple $(X, <, \tau)$ where $(X, <)$ is a linearly ordered set and τ is a Hausdorff topology on X that has a base of order-convex sets. If τ is the usual open-interval topology of $<$, then $(X, <, \tau)$ is a *linearly ordered topological space* (LOTS). Čech [6] proved that GO-spaces are exactly those spaces that embed topologically in some LOTS. Also recall that in any perfect space, every relatively-discrete set (i.e., a set that contains no limit point of itself) is σ-closed-discrete.

The authors would like to thank the referee for insightful comments on an earlier version of this paper.

2 The ZFC structure of perfect GO-spaces

At several points in the rest of the paper, we will need to invoke the following fact, which was probably known to Kurepa; a proof can be found in [5].

Lemma 2.1 *Suppose that the GO-space X has a σ-closed-discrete dense subset. Then so does every subspace of X.*

Whether or not they have σ-closed-discrete dense subsets, perfect GO-spaces have a delicate special structure, as the next results show.

Lemma 2.2 *Let S be a first category subset of a perfect GO-space X. Then S contains a subset that is dense in S and σ-closed-discrete in X.*

Proof: To say that S is a first category subset of X means that there are closed, nowhere-dense subsets K_n of X having $S \subseteq \bigcup\{K_n : n < \omega\}$. If we could prove that each K_n contained a dense subset that is a σ-closed-discrete subset of X, then $\bigcup\{K_n : n < \omega\}$ would also have such a dense subset. But the existence of such a dense subset is a hereditary property in

GO-spaces (see Lemma 2.1) and therefore S would also have a dense subset of the required type.

Therefore, it is enough to show that every closed, nowhere-dense subset K of X has a dense subset that is σ-closed discrete in X. Let $\mathcal{C}$ be the collection of all convex components of $X-K$. Because X is perfect, $X-K$ is an F_σ-subset of X, say $X-K=\bigcup\{F_n : n\geq 1\}$, so that the collection $\mathcal{C}$ can be written is a countable union of collections $\mathcal{C}_n=\{C\in\mathcal{C} : C\cap F_n\neq\emptyset\}$ for $n<\omega$. Each $\mathcal{C}_n$ has the property that if $x\in X$ then some open neighborhood W of x meets at most two members of $\mathcal{C}_n$.

For each $C\in\mathcal{C}_n$, choose $p(C)\in C\cap F_n$. Then the set $L_n=\{p(C) : C\in\mathcal{C}_n\}$ is closed in X, and is disjoint from K. Let $\mathcal{W}_n$ be the collection of convex components of $X-L_n$. Each $\mathcal{W}_n$ covers K. Let $\mathcal{V}_n=\{W\cap K : W\in\mathcal{W}_n\}$. We claim that for each $p\in K$, $|\bigcap\{St(p,\mathcal{V}_n) : n<\omega\}|\leq 3$. If not, then we may choose points $a,b\in\bigcap\{St(p,\mathcal{V}_n) : n<\omega\}$ with either $a<b<p$ or else $p<a<b$.

The two cases are analogous, so we consider only the first. For each n, some member of $W_n\in\mathcal{W}_n$ contains both a and p. Then convexity forces $b\in[a,p]\subseteq W_n$. By hypothesis, the set K is nowhere dense in X, so that K cannot contain the non-void open set (a,p). Hence $\emptyset\neq(a,p)-K\subseteq X-K$ so there is an $m<\omega$ and a set $C\in\mathcal{C}_m$ with $(a,p)\cap C\neq\emptyset$. Because $a,p\in K$, neither a nor p can belong to C, so that convexity forces $C\subseteq(a,p)$. Therefore $p(C)\in C\subseteq(a,p)$. Because $(a,p)\subseteq W_m,\ p(C)\in C\subseteq(a,p)\subseteq W_m$. But $C\in\mathcal{C}_m$ implies $p(C)\in L_m$ so that we have $p(C)\in W_m\cap L_m=\emptyset$ which is impossible.

Therefore, $|\bigcap\{St(p,\mathcal{V}_n) : n<\omega\}|\leq 3$ as claimed. We now apply Theorem 2.1 of [2] to conclude that K contains a dense subset that is σ-closed-discrete in K and hence also in X. $\square$

Corollary 2.3 *If X is a perfect GO-space that is of the first category in itself, then X has a σ-closed-discrete dense set. Hence any Souslin space (if there is one) is of second category in itself and any Souslin space that has no non-empty open, separable subspaces is a Baire space.*

Proposition 2.4 *Let X be any perfect GO-space. Let G be the union of all open subsets of X that contain a dense subset that is σ-closed-discrete in X. Then:*

1) G is open in X and has a dense subset that is σ-closed-discrete in X;

2) the set $H=X-G$ is dense-in-itself and for any subset $T\subseteq H$, the following are equivalent:

- *a) T has a dense subset that is σ-closed-discrete in H;*
- *b) T is nowhere-dense in H;*
- *c) T is of the first Baire category in H.*

3) when H is topologized as a subspace of X, H is a Baire space.

Proof: Recall that in the perfect space X, every σ-relatively-discrete set is σ-closed-discrete.

To prove assertion (1), let $\mathcal{U}$ be a cover of G by open sets that, in their relative topology, each have a dense set that is σ-relatively discrete. Because any perfect GO-space is hereditarily paracompact [7], there is a relatively closed cover $\mathcal{F}$ of G that refines $\mathcal{U}$ and is a σ-discrete collection in the subspace G of X. Then each member of $\mathcal{F}$ inherits a dense set that is σ-closed-discrete from the member of $\mathcal{U}$ that contains it, by 2.1. Hence $G=\bigcup\mathcal{F}$ has a σ-closed-discrete dense set. Note that this dense set is σ-closed-discrete in the perfect space X, and not just in the subspace G.

Next consider the subspace $H = X - G$. If p is a relatively isolated point of H, then there is a convex open set $J \subseteq X$ with $J \cap H = \{p\}$. Then $J - \{p\} \subseteq G$ so that $J - \{p\}$ has a σ-relatively-discrete dense subset D. Then $D \cup \{p\}$ is a σ-relatively discrete dense subset of J, so that $J \subseteq G$, contrary to $J \cap H \neq \emptyset$. Hence H has no relatively isolated points, i.e., H is dense-in-itself.

Because (b) implies (c) in any space, to prove assertion (2) it is enough to show that (a) implies (b) and (c) implies (a).

a)$\Rightarrow$ b) Suppose T has a dense subset that is σ-closed discrete in H (and hence also in X). Then so does $\mathrm{cl}_H(T)$ so we may assume that T is closed in H (and hence also in X). We claim that $\mathrm{Int}_H(T) = \emptyset$. If not, then there is an open subset U of X with $\emptyset \neq U \cap H \subseteq T$. Because T has a dense subset that is σ-closed-discrete in X, so does its subspace $U \cap H$. Note that $U - H \subseteq G$, so that $U - H$ inherits (from G) a dense subspace that is σ-closed-discrete in X. But then U has a dense subspace of the same type, so that $U \subseteq G$, showing that $U \cap H = \emptyset$, which is impossible. Therefore, T is nowhere dense in H as claimed.

(c) $\Rightarrow$ (a): Suppose there are closed, nowhere dense subsets K_n of H with $T \subseteq \bigcup\{K_n : n < \omega\}$. Then by Lemma 2.2, each K_n has a dense subset that is σ-closed-discrete in H (and hence also in X) so that $\bigcup\{K_n : n < \omega\}$ has the same property. Hence the subspace T also has a dense subset that is σ-closed-discrete in X.

To prove assertion 3), it is enough to show that no non-empty, relatively open subset V of H is of the first category in H. In the light of (2), any first category relatively open set V would have a dense subset D that is σ-closed-discrete in X. Write $V = H \cap W$ where W is open in X. Then $W = (W \cap H) \cup (W - H) = V \cup (W \cap G)$ so that W is the union of two subsets, each having a dense set that is σ-closed-discrete in X. Hence $W \subseteq G$ so that $\emptyset = W \cap H = V$, and that is impossible. $\square$

Corollary 2.5 *If there is a perfect GO-space X having no σ-closed-discrete dense subset then X contains a subspace Y that is a dense-in-itself, perfect, non-Archimedean GO-space, a Baire space, is a LOTS, and has the property that L is a first category subset of Y if and only if L has a σ-relatively-discrete dense subset.*

Proof: Suppose there is a perfect GO-space having no σ-closed-discrete dense subset. Apply Proposition 2.4 and consider the resulting subspace H. Theorem 7 of [13] asserts that any first-countable LOTS has a dense non-Archimedean subspace. Essentially the same proof shows that the same is true for first-countable GO-spaces, so let Y be a dense non-Archimedean subspace of H. According to a result of Purisch [11], the subspace Y is actually a LOTS, perhaps under some order different from the one that Y inherits from X. Because Y is a perfect GO-space, every σ-relatively discrete set in Y is also σ-closed-discrete. Density of Y in H allows us to use part (2) of Corollary 2.4 to show that a subset of Y is of the first category in Y if and only if it has a dense subset that is σ-relatively-discrete. The fact that no relatively open subset of Y can have such a dense subspace shows that Y is a Baire space. $\square$

Proposition 2.4 and Corollary 2.5 show that in order to study whether a perfect GO-space X has a σ-closed-discrete subspace, we should look only at the special subspaces H and Y, both of which are Baire spaces without isolated points.

We close this section by mentioning a ZFC decomposition theorem that is a consequence of one announced in [4] for first-countable paracompact GO-spaces. It may also be useful in studying Maurice's problem:

Proposition 2.6 *Suppose that X is a perfect GO-space. Then $X = A \cup B$ where*

a) *A is an open metrizable subspace of X;*

b) *$B = X - A$ and is dense-in-itself;*

c) *B can be written as $B = C \cup D$ where $C \cap D = \emptyset$ and where $[c_1, c_2] \cap C$ (respectively $[d_1, d_2] \cap D$) is not compact whenever $c_1 < c_2$ are points of C (respectively, whenever $d_1 < d_2$ are points of D).*

If we apply Proposition 2.6 to a perfect GO-space X, we see that X has a σ-closed-discrete dense set if and only if both of the sets C and D have σ-relatively-discrete dense subsets.

3 Some consistency results

The results in this section involve minor modifications of observations and theorems appearing in [12]. The following theorem is due to Shelah and Todorčevic [14].

Theorem 3.1 *If ZFC is consistent, then so is ZFC plus the following two statements simultaneously:*

a) *MA $+ 2^{\omega_0} = \omega_2$;*

b) *There is no non-atomic Baire space of size ω_1.*

To say that a space is *non-atomic* means that its regular-open algebra is non-atomic. A nonempty Hausdorff space without isolated points is a non-atomic space. In what follows, let $\mathcal{M}_{ST}$ be any model of the type described in Theorem 3.1.

Corollary 3.2 *If ZFC is consistent, then there is a model of set theory in which every perfect GO-space of local density ω_1 has a σ-closed-discrete dense subset.*

Proof: Suppose that some X in $\mathcal{M}_{ST}$ is a perfect GO-space of density ω_1 that has no σ-closed-discrete dense subset. Let D be a dense subset of X with cardinality ω_1. Then D is also a perfect GO-space with no σ-relatively-closed-discrete dense subset. Use Corollary 2.5 to find a dense-in-itself subspace Y of D that is a Baire space and has no σ-relatively-closed-discrete dense subset. But in the light of Theorem 3.1, such a Y cannot exist.

Now suppose that $\mathcal{M}_{ST}$ contains a perfect GO-space X that has local density ω_1. Combining the first paragraph of the proof with paracompactness of X (recall that any perfect GO-space is paracompact [7]), we see that X has a σ-discrete cover by closed subspaces that each have a σ-closed-discrete dense set in their relative topology. But then X has a σ-closed-discrete dense set, as required. $\square$

Corollary 3.3 *It is undecidable in ZFC whether any perfect GO-space of local density ω_1 must have a σ-closed-discrete dense subset.*

Proof: Any model of $V = L$ contains a Souslin space of density ω_1 so it cannot be proved in ZFC that any perfect GO-space of density ω_1 must have a σ-closed-discrete dense subset. On the other hand, in the model $\mathcal{M}_{ST}$, every perfect GO-space of local density ω_1 must have a σ-closed-discrete dense set, so that no ZFC proof can produce a perfect GO space of local density ω_1 that has no σ-closed-discrete dense subset. Hence Maurice's question for GO-spaces of local density ω_1 is undecidable in ZFC. $\Box$

Corollary 3.4 *It is undecidable in ZFC whether a perfect GO-space of local density ω_1 and having a point-countable base must be metrizable.*

Proof: Proof: For half of the proof, recall that Bennett [1] and Ponomarev [10] showed that if there is a Souslin space, then there is a Souslin space with a point-countable base. Souslin spaces exist in many models of set theory, e.g., in V=L, always have density ω_1, and are non-metrizable. Thus it is consistent with ZFC that there is a counterexample to Heath's question with density ω_1.

For the other half of the proof, consider the model $\mathcal{M}_{ST}$ and start with any perfect GO-space X with a point-countable base and local density $\leq \omega_1$. As in the proof of 3.2, X must have a σ-closed-discrete dense subset. But in that case, X must be metrizable (see [3]). Thus it is consistent with ZFC that there is no counterexample to Heath's question having density ω_1. $\Box$

Does Corollary 3.3 settle Maurice's question for all perfect GO-spaces? In other words, is there a ZFC theorem saying that if there is a perfect GO-space without a σ-closed-discrete dense subset, then there is such a space of density ω_1? The answer is "No" because Qiao proved in [12] that if one starts with the model L and does the usual ccc forcing to obtain $MA + c = \omega_2$, one can obtain a model satisfying $MA + c = \omega_2$ that contains a perfectly normal, non-metrizable GO-space of weight and size ω_2, even though the model contains no such space of size ω_1.

Theorem 3.1 also has consequences for a more specialized old question from ordered space theory, namely, "Can every perfect GO-space X be topologically embedded in a perfect LOTS $Y(X)$?" (In that question, we make no assumptions about the relation between the orderings of X and of $Y(X)$.) We have:

Corollary 3.5 *In the model $\mathcal{M}_{ST}$, any perfect GO-space with local density $\leq \omega_1$ can be embedded in a perfect LOTS.*

Proof: We know that in $\mathcal{M}_{ST}$, any perfect GO-space X with local density $\leq \omega_1$ has a σ-closed-discrete dense set. Now apply a theorem of Shi [15] to conclude that X can be embedded in a perfect LOTS. $\Box$

Remark 3.6 W-X Shi's proof that any GO-space with a σ-closed-discrete dense set can be embedded into a perfect LOTS might suggest that, in studying the question "Can any perfect GO-space be embedded in some perfect LOTS?", it might be worthwhile to work in a model of ZFC that contains a Souslin line S, and then try to build a counterexample from S by introducing some isolated points and some one-sided Sorgenfrey points in the usual way. However, that approach cannot work, because in [16] the authors show in ZFC that any GO-space constructed in the usual way on a perfect LOTS will be embeddable in some other perfect LOTS.

References

[1] Bennett, H., On quasi-developable spaces, Ph.D. Thesis, Arizona State University, Tempe, AZ 1968.

[2] Bennett, H., Heath, R., and Lutzer, D., Generalized ordered spaces with σ-closed-discrete dense subsets, *Proc. Amer. Math. Soc.* 129(2000), 931-939.

[3] Bennett, H. and Lutzer, D. Generalized ordered spaces with capacities, *Pacific J. Math.* 122(1984), 11-19.

[4] Bennett, H., and Lutzer, D. Problems in perfect ordered spaces, pp. 232-236 in *Open Problems in Topology* ed. by G.M Reed and J. van Mill, North Holland, Amsterdam, 1990.

[5] Bennett, H., Lutzer, D., and Purisch, S., On dense subspaces of generalized ordered spaces, Topology and its Applications 93 (1999), 191-205.

[6] Čech, E., *Topological Spaces*, Academia (Czechoslovak Academy of Sciences), Prague, 1966.

[7] Engelking, R., and Lutzer, D., Paracompactness in ordered spaces, *Fundamenta. Math.* 94 (1977), 49–58.

[8] Faber, M., *Metrizability in Generalized Ordered Spaces*, Math Centre Tracts 53 (1974), Mathematical Center Amsterdam.

[9] Gruenhage, G., Metrizable spaces and generalizations, pp. 201-225 in *Recent Progress in General Topology II*, ed by M. Hušek and J. van Mill, Elsevier, Amsterdam, 2002.

[10] Ponomarev, V., Metrizability of a finally compact p-space with a point-countable base, *Soviet Math. Dokl.* 8(1967), 765-768.

[11] Purisch, S., Orderability of Non-archimedean spaces, *Topology and its Applications* 16 (1983), 273-277.

[12] Qiao, Y-Q, Martin's Axiom does not imply Perfectly normal non-Archimedean spaces are metrizable, *Proc. Amer. Math. Soc.*129(2000)1179-1188.

[13] Qiao, Y-Q , and Tall. F., Perfectly normal non-metrizable non-archimedean spaces are generalized Souslin lines, *Proc. Amer. Math. Soc.*, 131 (2003), no. 12, 3929–3936.

[14] Shelah, S., and Todorčevic, S., A note on small Baire spaces, *Canad. J.Math.* 38(1986), 659-665.

[15] Shi, W-X., Extensions of perfect GO-spaces with σ-discrete dense sets, *Proc. Amer. Math. Soc.* 127(1999), 615-618.

[16] Shi, W., Miwa, T., and Gao, Y., Any perfect GO-space with the underlying LOTS satisfying local perfectness can embed in a perfect LOTS, *Topology and its Applications* 74(1996), 17-24.

[17] van Wouwe, J., *GO-spaces and Generalizations of Metrizability*, Mathematical Centre Tracts 104, Amsterdam, 1979.