Mary Ellen Rudin and Monotone Normality

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Abstract: This paper focuses on the remarkable contributions that Mary Ellen Rudin made to the study of monotonically normal spaces.

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1 Introduction

This paper is not a comprehensive survey of monotone normality. Instead, it is organized around parts of the theory that interested Mary Ellen Rudin and in which she made major contributions[1]. For more comprehensive surveys of progress in the study of monotonically normal spaces interested readers should see Mary Ellen's article in the *Encyclopedia of General Topology* [56] and the papers of Collins [8], Collins and Gartside [9], and Gruenhage [20], [21] [22].

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2 Origins of monotone normality

Although researchers never settled on the exact meaning of the phrase, the study of "generalized metric spaces" was a major research theme in the period 1950 to 1980 (and for some of us, much later than that). The goal was to find classes of spaces that are larger than the class of metric spaces, that are closed under certain reasonable topological operations, and that shared as many as possible of the desirable properties of metrizable spaces. Ceder [7] studied M_1, M_2, and M_3 spaces in the late 1950s, and Borges [3] focused on Ceder's M_3 spaces, renaming them "stratifiable spaces" in the 1960s. Stratifiable spaces are spaces in which every open set is an F_σ-set in a strong and order-preserving way. More precisely, X is *stratifiable* if for each open set $U \subseteq X$ there is a sequence $G(n,U)$ of open sets such that

a) $U = \bigcup\{G(n,U) : n \geq 1\} = \bigcup\{\mathrm{cl}(G(n,U)) : n \geq 1\}$, and

[1] As a result, we will not mention several papers on the topology of linearly ordered spaces that Mary Ellen wrote, some being gems of exposition, except to include them in the bibliography [58], [59] [60], [61], [62], [63], and [2].

b) if $U \subseteq V$ are open sets in X, then $G(n,U) \subseteq G(n,V)$ for all $n \geq 1$.

Stratifiability is a very strong property. It implies paracompactness, is preserved by countable products and closed mappings, and yields a strange new normality property. In Lemma 2.1 of [3], Borges proved

Proposition 2.1 *If X is stratifiable, then for every pair (A,U) where A is closed, U is open, and $A \subseteq U$, there is an open set $M(A,U)$ with $A \subseteq M(A,U) \subseteq \mathrm{cl}(M(A,U)) \subseteq U$ and with the property that for any (closed,open) pairs (A,U) and (B,V) with $A \subseteq B$ and $U \subseteq V$ we have $M(A,U) \subseteq M(B,V)$.*

Later, the property described in Proposition 2.1 was named "monotone normality" (by Zenor [71]) and topologists began investigating it. As described in Proposition 2.1, the property is sometimes hard to work with and other characterizations were found. Parts a), b), and c) of the next result appear in [24] and parts d) and e) are easily seen to be equivalent to the others.

Theorem 2.2 *For any T_3-space X, the following are equivalent:*

a) X is monotonically normal as described in Proposition 2.1;

b) there is a function G that assigns to each pair (S,T) of separated subsets of X an open set $G(S,T)$ in such a way that $S \subseteq G(S,T) \subseteq \mathrm{cl}(G(S,T)) \subseteq X - T$ and in such a way that if (S',T') is another pair of separated sets with $S \subseteq S'$ and $T \supseteq T'$ then $G(S,T) \subseteq G(S',T')$

c) for each pair (p,G) with G open and $p \in G$ there is an open set $\mu(p,G)$ with $p \in \mu(p,G) \subseteq G$ and such that if $\mu(p,G) \cap \mu(q,H) \neq \emptyset$, then either $p \in H$ or $q \in G$;

d) for each pair (p,G) with $p \in G$ and G open, there is an open set $V(p,G)$ with $p \in V(p,G) \subseteq G$ which satisfies $V(p,G) \subseteq V(p,G')$ if $p \in G \subseteq G'$ and $G(x, X - \{y\}) \cap G(y, X - \{x\}) = \emptyset$ if $x \neq y$;

e) for each pair (p,G) with G open and $p \in G$ there is an open set $\nu(p,G)$ containing p such that if $\nu(p,G) \cap \nu(q,H) \neq \emptyset$ then either $p \in H$ or $q \in G$.

Remark: In [72], Zhang and Shi point out that in any T_1-space, statement e) of Theorem 2.2 guarantees that the closure of $\nu(p,G)$ is a subset of G.

Corollary 2.3 *If Y is a subspace of a monotonically normal space X, then Y is monotonically normal.* [24]

Corollary 2.4 *Any monotonically normal space is hereditarily collectionwise normal.* [24]

Corollary 2.5 *Every stratifiable space and every GO-space*[2] *is monotonically normal.* [24]

[2]A generalized ordered space (GO-space) is a topological space (X,τ) equipped with a linear order $<$ such that every open interval (a,b) of the order is τ-open, and such that there is a base for τ consisting of order convex sets. It is known that the class of GO-spaces coincides with the class of subspaces of linearly ordered topological spaces. A GO-space whose topology coincides with the interval topology of the given linear order is a linearly ordered topological space (LOTS).

A natural question asks which monotonically normal spaces are stratifiable, and which monotonically normal spaces are GO-spaces. The answer to the first is well known:

Proposition 2.6 *A space X is stratifiable if and only if either one of the following holds:*

a) X is monotonically normal and semi-stratifiable in the sense of Creede [10]*;*

b) the product space $X \times S$ is monotonically normal, where $S = \{0\} \cup \{\frac{1}{n} : n \geq 1\} \subseteq \mathbb{R}$ [24].

Which monotonically normal spaces are GO-spaces is less clearly understood (beyond applying the suborderability theorems of van Dalen and Wattel [11] and the selection-theoretic orderability theorem of van Mill and Wattel [36]) and has not attracted much attention. However, the special case of the relation between compact monotonically normal spaces and compact linearly ordered spaces has been of major interest and this is discussed in Section 7 below.

Personal conversations with Mary Ellen made it clear that she saw ordered space results as guideposts showing what one should try to prove for monotonically normal spaces. Evidence that there is a close relation between monotonically normal spaces and generalized ordered spaces comes from the remarkable parallelism between the cardinal function theories for the two classes (Section 3) and from the theory of paracompactness for the two classes (Section 4). Further evidence is found in the fact that trees and tree constructions are important tools in the study of each. A paper of Williams and Zhou [70], whose stated goal was "to show how similar MN spaces are to compact orderable spaces," introduced tree-methods that turned out to be very useful in other studies of monotonic normality by Gartside [16], and Rudin [56][3]. These tree methods allowed Williams and Zhou to prove, for example, that any compact monotonically normal space has a dense orderable subspace. Here is a slight modification of the way that Gartside [16] described the Williams-Zhou tree. Suppose $V(x, U)$ is a monotone normality operator in a space X, as in part (d) of Proposition 2.2. A Williams-Zhou tree for a X is a collection $\mathcal{T}$ of open subsets of X, ordered by reverse set containment, such that:

a) the least element of $\mathcal{T}$ is X;

b) for all $T \in \mathcal{T}$, if $|T| \geq 2$ then T has at least two successors in $\mathcal{T}$;

c) for every $T \in \mathcal{T}$ with $T \neq X$, there is an open set G_T among the predecessors of T and a point $x_T \in G_T$ such that $T = V(x_T, G_T)$;

d) if $\mathcal{C}$ is a chain in $\mathcal{T}$ with $\mathrm{Int}(\bigcap \mathcal{C}) \neq \emptyset$ then $\mathrm{Int}(\bigcap \mathcal{C}) \in \mathcal{T}$;

e) for each $T \in \mathcal{T}$, the set $\{S \in \mathcal{T} : S$ and T have the same predecessors$\}$ is maximal among open collections that have pairwise disjoint closures;

f) the closure of each $T \in \mathcal{T}$ is contained in the closure of each predecessor of T.

Different monotone normality operators for X can give rise to different trees.

[3] More recently, Onal and Vural [45] used a Williams-Zhou tree argument in their proof that any monotonically normal Čech-complete space is subcompact, a result that generalizes an earlier theorem by Fleissner, Tkachuk, and Yengulalp[15] that every Čech-complete GO-space is subcompact.

3 Cardinal functions and monotone normality

In GO-spaces, the relations between various cardinal functions are well understood. To what extent do these ordered-space results generalize to the wider class of monotonically normal spaces?

Following Hodel's paper [26], in the next theorem $d(X)$ is the density of X, $c(X)$ is cellularity, $e(X)$ is the extent of X, $L(X)$ is the Lindelöf degree of X, $I(X)$ is the cardinality of the set of isolated points of X, $\Delta(X)$ is the smallest cardinal κ such that there are open sets $\{W_\alpha : \alpha < \kappa\}$ in X^2 with the property that the diagonal D of X has $D = \bigcap\{W_\alpha : \alpha < \kappa\}$, and $\tau(X)$ is tightness. Writing h in front of a cardinal function refers to the hereditary version of that cardinal function, e.g., $hL(X)$ is the hereditary Lindelöf degree of X. The results in the next theorem are due to Ostaszewski, Moody, Williams, Zhou, Gartside, and Mcintyre.

Theorem 3.1 *Suppose X is monotonically normal. Then*

a) $c(X) = hc(X) = hL(X)$ [46] [37]

b) $\tau(X) \leq c(X)$ [16]

c) $d(X) = hd(X)$ [16]

d) $d(X) \leq c(X)^+$ [69]

e) $d(X) \leq e(X) \cdot \Delta(X) \cdot I(X)$ [16]

Souslin spaces – GO-spaces that satisfy the countable chain condition but are not separable – were a long-time interest of Mary Ellen's, and were a frequent tool in her work on other problems (e.g., Dowker spaces). It was known that a Souslin space X, which by definition has $c(X) = \omega$, must have $c(X^2) = \omega_1$. Mcintyre [32] proved that an analogous result holds for monotonically normal spaces, namely:

Proposition 3.2 *If X is a non-separable monotonically normal space with $c(X) = \omega$, then $c(X^2) \geq \omega_1$.*

In fact, in Proposition 3.2 one can say $c(X^2) = \omega_1$ because Williams and Zhou (see (d) in Theorem 3.1) showed that if X is not separable and has $c(X) = \omega$, then $d(X^2) = d(X) = c(X)^+ = \omega_1$.

Whether Souslin spaces exist is undecidable in ZFC. Williams and Zhou [69] proved an analogous result for monotonically normal spaces, namely,

Proposition 3.3 *There is a Souslin space if and only if there is a non-separable monotonically normal space X that has $c(X) = \omega$.*

4 Paracompactness in monotonically normal spaces

When is a monotonically normal space paracompact? One type of answer comes from general theory, because any monotonically normal space is collectionwise normal. Consequently, a monotonically normal space X is paracompact if and only if X is subparacompact[4]. But there are other more interesting answers that grew out of the theory of GO-spaces.

Gillman and Henriksen [19] had proved that a linearly ordered space X is paracompact if and only if every gap of the underlying order is what they called a Q-gap. After an exchange of letters between Lutzer and Mary Ellen, Engelking and Lutzer [14] proved:

Proposition 4.1 *A GO-space X is paracompact if and only if no closed subspace of X is homeomorphic to a stationary set in a regular uncountable cardinal.*

Proposition 4.1 can be used to prove that each of twenty properties listed in [28] will guarantee paracompactness in a GO-space [14] [28]. Here is a sampler of the 20 properties:

Corollary 4.2 *A GO-space X is paracompact if it has any one of the following properties:*

a) X is metacompact or metalindelöf;

b) X is screenable (every open cover has a σ-disjoint open refinement);

c) X has a G_δ-diagonal or a quasi-G_δ-diagonal;

d) X is perfect;

e) X is Dieudonné complete.

Mary Ellen's doctoral student Diana Palenz studied properties that guarantee paracompactness in monotonically normal spaces. For example, she proved [47]:

Lemma 4.3 *If X is monotonically normal and metaLindelöf, then X is countably paracompact.*

That result was improved in Mary Ellen's paper [57] (see Theoren 5.1 below). Palenz [47] also proved:

Theorem 4.4 *Suppose X is monotonically normal. Then:*

a) X is paracompact if and only if X is screenable;

b) X is paracompact if and only if X is paralindelof;

c) if X has a quasi-G_δ-diagonal, then X is paracompact;

d) if X has a σ-locally countable base, then X is paracompact.

[4]A space is subparacompact provided every open cover has a σ-discrete closed refinement.

Palenz left open questions about paracompactness in monotonically normal spaces that are metalindelöf, or perfect, or have a point-countable separating open cover. Those questions, and many more, were all answered ten years years later in the definitive paper on paracompactness in monotonically normal spaces written by Zoli Balogh and Mary Ellen Rudin in [1].

According to introductory comments in that joint paper [1], Mary Ellen was trying to prove that any open cover of a monotonically normal space could be shrunk. Discussions with Balogh led to improvement after improvement, and the final result gives a surprising characterization of paracompactness in monotonically normal spaces, namely:

Theorem 4.5 *Suppose X is monotonically normal.*

> *a) Any open cover of X has a σ-disjoint open partial refinement $\mathcal{V}$ such that $X - \bigcup \mathcal{V}$ is the union of a discrete collection of closed subspaces, each homeomorphic to a stationary set in some regular uncountable cardinal;*
>
> *b) Therefore, a monotonically normal space X is paracompact if and only if X no closed subspace of X is homeomorphic to a stationary set in a regular uncountable cardinal.*

Given part a) of Theorem 4.5, part b) follows easily from Theorem 4.4 (a) because if X has no closed subspaces that are homeomorphic to a stationary set in some regular uncountable ordinal, then the partial refinement $\mathcal{V}$ given by 4.5 (a) would be a σ-disjoint open cover of X, showing that X is screenable, and now part (a) of Theorem 4.4 applies to establish paracompactness.

At the end of the Balogh-Rudin paper, the authors give examples of questions from the literature that are settled by their results.

Corollary 4.6 *Suppose X is monotonically normal. Then:*

a) every open cover of X can be shrunk;

b) any increasing open cover of X has an increasing shrinking;

c) X is paracompact if X is weakly $\delta\theta$-refinable;

d) X is paracompact if X is perfectly normal;

e) if X is a manifold of dimension ≥ 2 then X is paracompact and therefore metrizable.

In her Encyclopedia article [56], Mary Ellen comments that Theorem 4.5 shows that the twenty properties listed in [28] as giving paracompactness in a GO-space will also give paracompactness in any monotonically normal space. The proofs tend to have the following general form. Suppose X is monotonically normal and has the closed-hereditary property P. Since P is closed-hereditary, if X contained a closed subspace homeomorphic to a stationary set S in a regular uncountable cardinal κ, then S would also have property P. But a Pressing Down Lemma argument shows that this cannot happen. That general approach works in 19 of the 20 cases, the exception being the case where X is monotonically normal and has a σ-minimal base. The problem is that the existence of a σ-minimal base is not closed hereditary. In GO-spaces, there is a way around this problem, but we do not know whether the problem can be fixed in the larger class of monotonically normal spaces.

The Balogh-Rudin theorem is non-trivial to prove, and this led C. Vural [68] to give a simple direct proof of part (a) of the theorem for GO-spaces. A particularly nice application of Theorem 4.5 is the theorem of Buzyakova and Vural on paracompactness in monotonically normal paratopological groups (see Proposition 9.1 below).

5 Product theory for monotonically normal spaces

Earlier work of Sorgenfrey [65] and Michael [33] [34] established that products of normal spaces can behave very strangely. Because many of these classical examples (the Sorgenfrey and Michael lines and their subspaces) were GO-spaces, the classical examples showed that monotonically normal spaces also have a pathological product theory. For example, the Sorgenfrey line shows that the product of two monotonically normal spaces can fail to be normal, and the Michael line shows that the product of a monotonically normal space and a metric space can fail to be normal. Furthermore, Michael's paper [34] shows that there are monotonically normal spaces W and X such that: (1) W^n is paracompact for all finite n and yet W^ω is not normal; (2) under the Continuum Hypothesis (CH), X^n is Lindelöf for all finite n, but X^ω is not normal. In addition, under (CH), for each $n \geq 1$ there are monotonically normal spaces Y_n and Z_n such that (3) $(Y_n)^n$ is Lindelöf and $(Y_n)^{n+1}$ is paracompact, but is not Lindelöf; and (4) $(Z_n)^n$ is hereditarily Lindelöf but $(Z_n)^{n+1}$ is not normal.

However, there are several ways in which the product theory of monotonically normal spaces is better-behaved than the theory for T_4-spaces. For example, there are T_4 spaces whose product with the unit interval $[0, 1]$ is not normal (these are the Dowker spaces, another area in which Mary Ellen was an expert). For contrast, consider the following theorem that Mary Ellen[5] proved in [57]:

Theorem 5.1 *There is no monotonically normal Dowker space, i.e., for any monotonically normal space X, the space X is (hereditarily) countably paracompact so that the product $X \times [0, 1]$ is normal.*

Purisch proved that the product of a GO-space and a countable T_4-space must be normal. He and Mary Ellen investigated whether that result could be generalized to the class of monotonically normal spaces, and in [49] they proved:

Proposition 5.2 *There is a monotonically normal space X and a countable T_4 space Y such that $X \times Y$ is not normal. However, if both X and Y are monotonically normal with Y countable, then $X \times Y$ is normal.*

As noted above, the Sorgenfrey line shows that the square of a monotonically normal space can fail to be monotonically normal. Gartside [17] showed that monotone normality in the space X^2 has surprising consequences for X:

Theorem 5.3 *If X^2 is monotonically normal, then X^n is monotonically normal and hereditarily paracompact for each finite n, and there is a topological group Y such that each finite power Y^n is monotonically normal but Y^ω is not.*

Recall that Proposition 2.6 shows that if S is a non-trivial convergent sequence and if $X \times S$ is monotonically normal, then X is stratifiable. This can be used to show that infinite products will not be monotonically normal unless they are much more, and that uncountable products are never monotonically normal.

[5] A weaker version of the theorem had been proved by D.Palenz (Theorem 3.3 of [47]) who showed that a monotonically normal, metalindelöf space must be countably paracompact.

Proposition 5.4 *Suppose that $|X_n| \geq 2$ for each $n \geq 1$. Then the space $Y = \Pi\{X_n : n \geq 1\}$ is monotonically normal if and only if each X_n and the space Y are stratifiable. If $|X_\alpha| \geq 2$ for each $\alpha < \kappa$ (where κ is uncountable), then the space $Z = \Pi\{X_\alpha : \alpha < \kappa\}$ cannot be monotonically normal.*

Proof: Choose a two-point set $D_n \subseteq X_n$. Fix k and note that the space $Y = \Pi\{X_n : n \geq 1\}$ is homeomorphic to $X_k \times \Pi\{X_n : n \geq 1, n \neq k\}$ which contains the subspace $X_k \times \Pi\{D_n : n \geq 1, n \neq k\}$. Now $\Pi\{D_n : n \geq 1, n \neq k\}$ contains a nontrivial convergent sequence S so we see that $X_k \times S$, being a subspace of the monotonically normal space Y, is monotonically normal. Then by part (b) of Proposition 2.6, we see that X_k is stratifiable. Hence so is Y.

Now suppose we have an uncountable collection of spaces $\{X_\alpha : \alpha < \kappa\}$, each containing a two point subspace D_α, and suppose that the space $Z = \Pi\{X_\alpha : \alpha < \kappa\}$ is monotonically normal. The space $\Pi\{X_n : n < \omega\}$ contains a copy of the convergent sequence S so that the space $Z_0 = S \times \Pi\{X_\alpha : \omega \leq \alpha < \kappa\}$ is a subspace of the monotonically normal space Z, showing that Z_0 is monotonically normal. But then, again by part (b) of Proposition 2.6, the space $Z_1 = \Pi\{X_\alpha : \omega \leq \alpha < \kappa\}$ must be stratifiable, and that is impossible because no point of Z_1 is a G_δ. $\square$

The theory of product spaces $X \times Y$ where X is monotonically normal and Y is a space with special properties (e.g., compactness, orthocompactness, etc.) remains an active research area. See for example, the recent paper of Hirata, Kemoto and Yajima [25].

6 Acyclic monotone normality, K_0 and K_1 spaces, and monotonically normal compactifications

Topologists at Oxford University, led by Peter Collins and Mike Reed, came to monotone normality through their study of chain-point-networks [56]. To conserve space we skip over the definitions involved in their studies, except to say that a space has a chain-point-network if and only if it has an acyclic monotone normality operator [38], where a monotone normality operator $V(x, U)$ (as described in part (d) of Theorem 2.2) is *acyclic* provided for any finite sequence of distinct points $x_0, x_1, \cdots, x_{n-1}$, if we write $x_n = x_0$, then $\bigcap\{V(x_i, X - \{x_{i+1}\}) : 0 \leq i \leq n-1\} = \emptyset$. Moody and Roscoe [39] showed that the familiar examples of monotonically normal spaces (stratifiable spaces, elastic spaces, GO-spaces) are all acyclically monotonically normal, as is any closed continuous image of an acyclically monotonically normal space. They also showed that any acyclic monotonically normal space has the Kuratowski property[6] that van Douwen called K_0.

Recall that a space (X, τ) is K_0 if for each subspace (A, τ_A) of (X, τ) (with τ_A being the subspace topology on A), there is a function $k : \tau_A \to \tau$ with the properties that (1) $A \cap k(U) = U$ for each $U \in \tau_A$; (2) $k(U \cap V) = k(U) \cap k(V)$ for each $U, V \in \tau_A$; and (3) $k(\emptyset) = \emptyset$. A weaker property called K_1 was introduced by van Douwen in his thesis [12]: a space (X, τ) is K_1 if for each subspace $A \subseteq X$ there is a function $k : \tau_A \to \tau_X$ such that $k(U) \cap A = U$ for each $U \in \tau_A$ and such that if U_0, U_1 are disjoint members of τ_A, then $k(U_0) \cap k(U_1) = \emptyset$. Clearly any K_0 space is K_1, and in [13] van Douwen asked whether K_1 implies K_0. In 1982, Jan van Mill [35] gave a negative answer to this question

[6]In Chapter 2, Section 21 of [27], Kuratowski proved that any metric space has the properties that van Douwen, in his thesis [13], called K_0 and K_1.

using the Continuum Hypothesis, but without CH the question was still open, and in 1989, the question was repeated by Borges in [4]. There is a link between K_0-spaces, monotonically normal spaces, and K_1-spaces. In his thesis (Theorems 2.2 and 2.3.1 of [12]) van Douwen had proved that that every GO-space is K_0 and that every monotonically normal space is K_1. Later, as mentioned above, Moody and Roscoe [39] showed that any acyclically monotonically normal space is K_0.

Therefore, the obvious question was: "Does every monotonically normal space have an acyclic monotone normality operator?" and this problem had clear links to van Douwen's question "Does K_1 imply K_0?" In [54], Mary Ellen answered both questions negatively, describing a monotonically normal space with the property that every one of its monotone normality operators has a cycle of length three. In addition, Rudin's space, which has property K_1 (as do all monotonically normal spaces), does not have property K_0, thereby giving a ZFC solution to the problem posed by van Douwen. Finally, her space has no monotonically normal compactification, as can be deduced from her Theorem 7.1, below. Later, in [55], she described a locally compact monotonically normal space that has no monotonically normal compactification, thereby solving a problem posed by Purisch. Also in [55] she used a special kind of Souslin tree that exists given the set-theoretic principle $\diamondsuit$ to create a compact K_1-space that is not K_0, thereby partially answering a question of Arhangel'skii who had asked whether each compact K_1 space must be K_0 (see [8]). As Corollary 7.2 in the next section will show, Mary Ellen's consistent example of a compact K_1 space that is not K_0 could not be monotonically normal.

There is one question from [39] and [8] that, as far as we know, remains open:

Is every monotonically normal K_0-space acyclically monotonically normal?

Roscoe and Moody [39] have isolated a condition that may be the key to answering the question above. Their condition demands that there is a family $\{k_Y : Y \subseteq X\}$ of K_0 functions for a K_0-space X whose members interrelate in a special way. The interrelationship property is as follows: a space (X, τ) is a *monotone* K_0*-space* provided:

> for each subspace (Y, τ_Y) of (X, τ), there is a K_0 function $k_Y : \tau_Y \to \tau$ with the additional property that if Y and Y' are subspaces of X, and if $U \in \tau_Y$ and $U' \in \tau_{Y'}$ have $U \subseteq U'$ and $Y - U \subseteq Y' - U'$, then $k_Y(U) \subseteq k_{Y'}(U')$.

In Theorem 4.9 of [39] they prove

Theorem 6.1 *The space X is acyclically monotonically normal if and only if X is a monotone K_0-space.*

In [8], Collins suggested that the study of compactifications of monotonically normal spaces would be a fruitful area for study. It would be interesting to know when a monotonically normal space X has a monotonically normal compactification. Corollary 7.4 in the next section shows that if X has a monotonically normal compactification, then there is a GO-space K and a perfect irreducible mapping from K onto X, so that X must be acyclically monotonically normal and K_0. But acyclic monotone normality is not enough because Mary Ellen's example in [55] of a locally compact monotonically normal space X having no monotonically normal compactification is acyclically monotonically normal and K_0, as can be seen from Corollary 7.3 in the next section. In addition, Collins [8] points out that Heath's countable stratifiable space [23] is acyclically monotonically normal and hence K_0, but has no monotonically normal compactification.

As Mary Ellen points out in her paper [55], a locally compact space X has a monotonically normal compactification if and only if the one-point-compactifiaction of X is monotonically normal. Cairns, Junilla, and Nyikos[6] proved that a locally compact space X has a monotonically normal compactification if and only if X is weakly orthocompact (i.e., every directed open cover of X has an interior-preserving open refinement.

7 Compact monotonically normal spaces and Nikiel's question

Except for compact linearly ordered spaces, compact monotonically normal spaces were somewhat hard to find and this led early researchers to conjecture that the class of compact monotonically normal spaces would be very closely related to the class of compact linearly ordered spaces. For example, Purisch [48] asked whether each separable, totally disconnected, monotonically normal compact space must be orderable, a question that was answered negatively in [43]. Then in [44], Nyikos and Purisch proved that any monotonically normal scattered space is the continuous image of a compact space of ordinals. In 1986, J. Nikiel [41] asked what became the central question in the study of compact monotonically normal spaces:

> Is it true that every compact monotonically normal space is a continuous image of a compact linearly ordered topological space?

In a sequence of very hard papers [50] [51] [52], culminating in [53], Mary Ellen answered Nikiel's question affirmatively. She proved:

Theorem 7.1 *A compact Hausdorff space X is the continuous image of a compact linearly ordered space if and only if X is monotonically normal.*

Before Mary Ellen's Theorem 7.1, many special cases of Nikiel's conjecture had been proven. But her proof did not rely on any of them, with the exception that she used a "modified Williams-Zhou type scheme." [56]. In the introduction of her paper [53], Mary Ellen writes

> I have discovered that hundreds of papers have been written on the structure of monotonically normal spaces, many containing partial results on Nikiel's conjecture (I have written ten myself!)... A group of very powerful young topologists from Oxford...pushed questions about pathologies which might exist in monotonically normal spaces but could not exist in continuous images of linearly ordered compact spaces, and they were responsible for my interest in Nikiel's conjecture...But few to none of these [partial] results are used in the current paper which just gives an elementary proof of Nikiel's conjecture.

Those who have read her proof know that "elementary" does not mean "easy"! Because closed continuous mappings preserve acyclic monotone normality, we have:

Corollary 7.2 *Any compact monotonically normal space is acyclically monotonically normal and is a K_0-space.*

As reported in Collins' paper [8], Arhangel'skii had asked whether a compact monotonically normal space must be acyclically monotonically normal, and Corollary 7.2 provides an affirmative answer. It also explains why Mary Ellen's consistent example of a compact K_1-space that is not a K_0-space [55] must fail to be monotonically normal and why her monotonically normal non-K_0 space in [54] cannot have a monotonically normal compactification. In addition, a "local implies global" result of Moody and Roscoe gives:

Proposition 7.3 *Any locally compact monotonically normal space is acyclically monotonically normal and therefore* K_0.

Proof: Suppose Y is an open subspace of the monotonically normal space X whose closure $\overline{Y}$ is compact. Then $\overline{Y}$, being a compact monotonically normal space, is acyclically monotonically normal in the light of Corollary 7.2. But then the open subspace Y is also acyclically monotonically normal, so that X has a cover by open subspaces each of which is acyclically monotonically normal. Theorem 4.2 of [39] shows that any such space is acyclically monotonically normal and therefore is K_0. □

Corollary 7.4 *If a space X has a monotonically normal compactification then there is a GO-space L and a perfect irreducible mapping f from L onto X so that X is acyclically monotonically normal and* K_0

Proof: Let Y be the monotonically normal compactification of X. Use Theorem 7.1 to find a continuous mapping g from some compact linearly ordered space N onto X. Let $M = g^{-1}[X]$. Then the restricted mapping $g|_M : M \to X$ is a perfect mapping onto X, so that there is a closed subset $L \subseteq M$ such that the restricted mapping $f = g|_L$ is a perfect irreducible mapping from L onto X. Because L is a subspace of the linearly ordered space N, L is a GO-space. The space L is acyclically monotonically normal and hence K_0, and then so is $X = f[L]$. □

The next section describes a significant corollary of Theorem 7.1 that dates from the 1990s. Mary Ellen's theorem continues to have interesting consequences, as can be seen from the paper of Cairns, Junilla, and Nyikos [6] on a property called "utter normality" that appears in this special issue.

8 Solution of the Hahn-Mazurkiewicz problem

The classical Hahn-Mazurkiewicz theorem, proved independently by Hahn and Mazurkiewicz around 1914, asserts:

Theorem 8.1 *A space X is the continuous image of the unit interval $[0, 1]$ if and only if X is compact, connected, locally connected, and metrizable.*

Toward the middle of the 20th century, continua theory researchers began to ask for a characterization of spaces that are continuous images of compact, connected, linearly ordered spaces (not just images of $[0, 1]$ as in Theorem 8.1) and this became known as "the Hahn-Mazurkiewicz problem". There were many partial solutions,[7] but none had the elegance of the original Hahn-Mazurkiewicz theorem. In their survey [66] Treybig and Ward describe several folk conjectures (e.g., that among T_2-spaces, the images of connected ordered spaces coincide with locally connected continua, and that among T_2 spaces, the images of compact linearly ordered spaces coincide with the compact spaces) that were already (in 1980) known to be false, and they wrote "The question remains whether additional hypotheses can be found to provide affirmative solutions [for the folk conjectures] in such a way as to generalize the classical [Hahn-Mazurkiewicz theorem]."

Continua theorists had proved many results that begin with "Suppose X is the continuous image of a compact ordered space". For example, in [67] and [40] (see also [31]) Treybig and Nikiel proved

Theorem 8.2 *If X is a locally connected, compact, connected Hausdorff space that is a continuous image of a compact ordered space, then X is the continuous image of a compact, connected linearly ordered space.*

Combining that result with Rudin's Theorem 7.1 solves the Hahn-Mazurkiewicz problem:

Theorem 8.3 *A space X is the continuous image of a compact connected linearly ordered space if and only if X is compact, connected, locally connected, and monotonically normal.*

This issue of *Topology and its Applications* contains a detailed paper by S. Mardešić [30] on the long history of the the Hahn-Mazurkiewicz problem.

9 Monotone normality in topological algebra and analysis

At the end of her Encyclopedia article on monotone normality [56] Mary Ellen mentioned, probably with a smile, that she did not mention monotone normality in topological groups in her article. Perhaps that little circularity is sufficient justification to include three results illustrating the role that monotone normality can play in topological algebra and functional analysis.

First, Buzyakova and Vural [5] used the Balogh-Rudin theorem (see part (a) of Theorem 4.5) to improve an earlier result of Gartside (announced in [8]) by showing that:

Proposition 9.1 *Any monotonically normal paratopological group*[8] *is hereditarily paracompact.*

Second, Shkarin [64] improved an earlier theorem of Gartside [18] for locally convex topological vector spaces by proving:

Proposition 9.2 *Let E be a Hausdorff topological vector space over the real field $\mathbb{R}$. Then E is monotonically normal if and only if E is stratifiable.*

Third, Gartside [18] pinned down the role of monotone normality in the function space $C_p(X)$ of continuous real-valued functions on X, with the pointwise convergence topology:

Theorem 9.3 *If $C_p(X)$ is monotonically normal then X is countable and $C_p(X)$ is metrizable.*

[7] See the survey papers [29](up to 1966), [66](up to 1980), [31](up to 2002), and the article by Mardešić [30] in this issue for a summary up to the present.

[8] A paratoplogical group is an abstract group G with a topology under which the group multiplication is jointly continuous from $G \times G$ to G. It is not required that the mapping $x \to x^{-1}$ be continuous.

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