Lexicographically Ordered Trees

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Abstract: We characterize trees whose lexicographic ordering produces an order isomorphic copy of some sets of real numbers, or an order isomorphic copy of some set of ordinal numbers. We characterize trees whose lexicographic ordering is order complete, and we investigate lexicographically ordered ω-splitting trees that, under the open interval topology of their lexicographic orders, are of the first Baire category. Finally we collect together some folklore results about the relation between Aronszajn trees and Aronszajn lines, and use earlier results of the paper to deduce some topological properties of Aronszajn lines.

Key Words and Phrases: tree, splitting tree, lexicographic ordering, linear orders, tree representation of lines, compact, order complete, first Baire category, Aronszajn tree, Aronszajn line

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1 Introduction

In this paper we characterize trees whose lexicographic orderings give (up to order isomorphism) sets of real numbers and sets of ordinals. We then characterize trees whose lexicographic orderings are order complete (or equivalently, that are compact in the usual open interval topology of the lexicographic ordering). For a broad class of trees, we also characterize those trees that are of the first Baire category when equipped with the open interval topology of their lexicographic orderings. Finally, we collect together some known results about Aronszajn trees and lines. We prove the harder half of the folklore characterization of Aronszajn lines as being the lexicographic orderings of Aronszajn trees and then we use earlier results in the paper to establish certain topological facts about Aronszajn lines.

We generally follow [4] in our terminology and notation for trees. By a *tree* we mean a partially ordered set $(T, \leq_T)$ with the property that for each $t \in T$, the set $T_t = \{s \in T : s \leq_T t \text{ and } s \neq t\}$ is well ordered by $\leq_T$. The order type of T_t is denoted by $\mathrm{lv}(t)$ and for each ordinal α, $T_\alpha = \{t \in T : \mathrm{lv}(t) = \alpha\}$ is the α^{th} level of T. For some α, $T_\alpha = \emptyset$ and the height of T (denoted $\mathrm{ht}(T)$) is the first ordinal α with $T_\alpha = \emptyset$. For any $t \in T$ and any $\alpha < \mathrm{lv}(t)$ let $t(\alpha)$ be the unique point of $T_t \cap T_\alpha$, i.e. the unique predecessor of t that lies at level α of the tree, and for $\alpha = \mathrm{lv}(t)$, let $t(\alpha) = t$.

For each $t \in T$, the *node of T containing t* is defined to be $\mathrm{Node}_T(t) = \{s \in T : T_s = T_t\}$. Let $\mathcal{N}_T$ be the set of all nodes of T. Given a node N of T, there is some α with $N \subseteq T_\alpha$ and we write $\alpha = \mathrm{lv}_T(N)$. Let $\rho(N) = T_t$ where t is any element of N. This set $\rho(N)$ is called the *path of predecessors* of the node N. It is clear that any two members of a given node of T are incomparable with respect to the partial ordering $\leq_T$. For each node N of T, let $<_N$ be a linear ordering of N. There is no necessary relation

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between the orderings of different nodes of T. Given a set $\{(N, <_N) : N \in \mathcal{N}_T\}$ of node orderings for T, we define a new ordering, called the *lexicographic ordering*, on the set T by the rule that $t_1 \preceq t_2$ if and only if either

(i) $t_1 \leq_T t_2$; or

(ii) t_1 and t_2 are incomparable in the partial ordering $\leq_T$ and if $\delta = \Delta(t_1, t_2)$ is the first ordinal such that $t_1(\delta) \neq t_2(\delta)$, then in the node N to which both $t_1(\delta)$ and $t_2(\delta)$ belong, we have $t_1(\delta) <_N t_2(\delta)$.

It is easy to verify that $\preceq$ is a linear ordering of the set T.

From time to time we will contrast the theory of lexicographic orderings of trees with the related, but quite different, theory of branch spaces of trees. (See [1].) By a *branch* of a tree $(T, \leq_T)$ we mean a maximal (with respect to containment) totally ordered subset $b \subseteq T$. Each branch b of T is well ordered and its order type is denoted by $\mathrm{ht}(b)$. For $\alpha < \mathrm{ht}(b)$ let $b(\alpha)$ be the unique member of the set $b \cap T_\alpha$. Given a set of node orderings $\{(N, <_N) : N \in \mathcal{N}_T\}$ as above, the set of all branches of T (denoted by $\mathcal{B}_T$) is linearly ordered by a rule that is reminiscent of lexicographic ordering, namely that two branches $b_1,\ b_2 \in \mathcal{B}_T$ have $b_1 \leq_{\mathcal{B}_T} b_2$ if and only if either $b_1 = b_2$ or $b_1(\delta) <_N b_2(\delta)$ where $\delta = \Delta(b_1, b_2)$ is the first ordinal such that $b_1(\delta) \neq b_2(\delta)$ and N is the node of T that contains both $b_1(\delta)$ and $b_2(\delta)$.

In this paper we reserve the symbols $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{R}$ for the usual sets of rational, irrational, and real numbers, respectively, The set of all integers is denoted by $\mathbb{Z}$. If S is a subset of a linearly ordered set L, then a set C is a *convex component* of S if $C \subseteq S$ and C is order-convex in L and no strictly larger convex subset of L is contained in S. Throughout the paper, we will use the term *line* to mean any linearly ordered set. No topology is assumed unless specifically mentioned.

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2 Representing lines by lexicographically ordered trees

In this section, we will focus on representing some classical linearly ordered sets (namely, subsets of $\mathbb{R}$ and ordinal lines) as lexicographic orderings of trees. We begin by recalling an observation due to Todorčevic [4] showing that we must place restrictions on the trees used if we are to obtain non-trivial representations of lines via lexicographic orderings of trees.

Example 2.1 *: Any linearly ordered set is order isomorphic to a lexicographically ordered tree.*

Proof: Consider any linearly ordered set $(X, <)$. Let $T = T_0 = X$ and let $\leq_T$ be equality. Then T_0 is the unique node of T and we linearly order it to make it a copy of $(X, <)$. Using the tree T and the chosen node ordering, it is clear that $(X, <)$ is exactly the lexicographic ordering of T. $\Box$

The problem with the tree in Example 2.1 is that the the original linearly ordered set $(X, <)$ appears as a node of the tree, and the lexicographic ordering gets all of its structure from that node. Because the tree in Example 2.1 is just as complicated as the original line $(X, <)$, it is not surprising that such a tree-representation gives no additional insight into the structure of $(X, <)$. The literature contains many kinds of restrictions that one might impose on a tree, e.g., restrictions on the height of T, or restrictions on the cardinality of the nodes of T, or of the levels of T, or of the anti-chains of T. (Recall that an *anti-chain* is a subset $A \subseteq T$ such that no pair of distinct elements of A are comparable in the partial order of T.) We introduce a new kind of restriction called L-non-degeneracy that seems particularly natural if one wants to have a representation theory for a linearly ordered set $(X, <)$ using trees that are more simple than $(X, <)$ itself. For a linearly ordered set $(L, <_L)$ we say that the node orderings of a tree T are *L-non-degenerate* provided for each $N \in \mathcal{N}_T$, the set $(L, <_L)$ is not order isomorphic to any subset of $(N, <_N)$.

Example 2.2 *: The set $\mathbb{Q}$ of rational numbers is order isomorphic to a lexicographically ordered tree with $\mathbb{Q}$-non-degenerate node orderings.*

Proof: Let $T = \bigcup\{{}^n\mathbb{Z} : n \geq 1\}$, i.e., T is the set of all non-empty finite sequences of integers. Partially order T by end-extension. Each node of T is countably infinite and in its natural order is a copy of $\mathbb{Z}$. Hence the node orderings are $\mathbb{Q}$-non-degenerate. With the resulting lexicographic order $\preceq$, T is a countable densely ordered set without end points (because T has no root) and so $(T, \preceq)$ is order isomorphic to $\mathbb{Q}$. $\square$

Remark 2.3 *:* By way of contrast with Example 2.2, we show in [1] that if $\mathbb{Q}$ is order isomorphic to the branch space of some tree T, then some node of T must contain an order isomorphic copy of $\mathbb{Q}$.

Having obtained $\mathbb{Q}$ as a non-trivial lexicographically ordered tree, it is natural to wonder whether interesting uncountable sets of real numbers could be obtained in a similar way. The answer is "Yes," as can be seen from the next example.

Example 2.4 *: For any set X with $\mathbb{Q} \subseteq X \subseteq \mathbb{R}$, there is a tree V_X with countable height and countable nodes whose lexicographic ordering is order isomorphic to X.*

Proof: We begin by considering the case where $X = \mathbb{R}$. Let T be the height ω tree used above to give a lexicographic representation of $\mathbb{Q}$. Let $\mathcal{B}_T$ be the set of all branches of T and let $U = T \cup \mathcal{B}_T$. As in [4] we extend the partial order of T to a partial ordering of U as follows. For $t \in T$ and $b \in \mathcal{B}_T$ we define $t \leq_U b$ if and only if $t \in b$. Distinct members of $\mathcal{B}_T$ are not comparable in the partial ordering of U. It is straightforward to check that if $\preceq_U$ is the resulting lexicographic ordering of U, then $(U, \preceq_U)$ is densely ordered, has no endpoints, has a countable order-dense set, and satisfies the least upper bound property for non-empty subsets that have upper bounds. But that list of properties characterizes the ordered set $\mathbb{R}$. Fix an order isomorphism F from $(U, \preceq_U)$ onto $\mathbb{R}$.

Now consider the case where $\mathbb{Q} \subseteq X \subseteq \mathbb{R}$. With U and F as in the previous paragraph, let $\mathcal{B}_X$ be the set of branches $b \in \mathcal{B}_T$ with the property the $F(b) \in X - \mathbb{Q}$. Then $V_X = T \cup \mathcal{B}_X$ is a subtree of U and the restriction of F to the lexicographically ordered tree V_X is an order isomorphism from $(V_X, \preceq_{V_X})$ onto X.$\square$

The construction in Example 2.4 is somewhat unsatisfying because, while every node of the tree V is either finite or a copy of $\mathbb{Z}$, the ω^{th} level of V is a very large anti-chain that makes V look somewhat like the trivial tree mentioned in Example 2.1 in the sense that almost all of the structure grows out of a single level. One might wonder whether it is possible to find a tree T and a choice of node orderings whose lexicographic ordering represents X without including the set $X - \mathbb{Q}$ as a maximal anti-chain. The next two results answer that question in the negative and show that to a great extent, Example 2.4 is typical of what must happen when uncountable subsets of $\mathbb{R}$ are represented as lexicographic orderings of trees. We begin with a lemma that describes certain intervals in the lexicographic ordering $\preceq_T$.

Lemma 2.5 *: Suppose $(T, \leq)$ is a tree and suppose $\preceq$ is the lexicographic ordering of T associated with some choice of node orderings. Then:*

i) if $a <_T b$ are comparable elements of T and if $(a,b)_{\preceq} = \{x \in T : a \prec x \prec b\}$ then

$$(a,b)_{\preceq} = \{c \in T : a <_T c <_T b\} \cup \left(\bigcup\{W(c) : a <_T c \leq_T b\}\right)$$

where $W(c) = \bigcup\{T^x : x \in \mathrm{Node}(c) \text{ and } x <_{\mathrm{Node}(c)} c\}$ and $T^x = \{t \in T : x \leq_T t\}$.

ii) if $<_M$ is the linear order chosen for the node M of T and if $a, b \in M$ have $a <_M b$ then

$$(a,b)_{\preceq} = \bigcup\{T^x : x \in M \text{ and } a \leq_M x <_M b\} - \{a\}.$$

Proof: Let $L = \{x \in T : a \prec x \prec b\}$ and let R be the set

$$\{c \in T : a <_T c <_T b\} \cup \left(\bigcup\{\bigcup\{T^x : x \in \mathrm{Node}(c) \ \&\ x <_{\mathrm{Node}(c)} c\} : a <_T c \leq_T b\}\right).$$

We first show $R \subseteq L$. Let $t \in R$. If $a <_T t <_T b$ than $a \prec t \prec b$ is automatic, so assume there is some c with $a <_T c \leq_T b$ and some $x \in \mathrm{Node}(c)$ with $x <_{\mathrm{Node}(c)} c$ and $t \in T^x$. Then $a <_T t$ and a case-by-case analysis shows that $t \prec b$. Hence $t \in L$.

Conversely, suppose $t \in L$. If $a <_T t <_T b$, then $t \in R$ so assume that $a <_T t <_T b$ is false, i.e., that either $a <_T t$ or $t <_T b$ fails. It cannot happen that $t \leq_T a$ or $b \leq_T t$ because each of these options would force $t \notin L$. Hence if $a <_T t$ fails, then a and t are incomparable in the partially ordered set $(T, \leq_T)$. Now compute $\delta = \Delta_T(a,t)$ and conclude from $a \prec t$ that in the node M of T that contains both $a(\delta)$ and $t(\delta)$, we have $a(\delta) <_M t(\delta)$. But then we have $b \prec t$ because $a <_T b$ yields $b(\delta) = a(\delta) <_M t(\delta)$. Therefore, $a <_T t$ must occur, so that t and b are incomparable in $(T, \leq_T)$. Compute $\sigma = \Delta_T(t,b)$. Then $\sigma \leq \mathrm{lv}(b)$ and in the node N of T that contains both $t(\sigma)$ and $b(\sigma)$ we have $t(\sigma) <_N b(\sigma)$. If $\sigma \leq \mathrm{lv}(a)$, then $a <_T b$ would give $t(\sigma) <_N b(\sigma) = a(\sigma)$ and that would yield $t \prec a$, which is false. Hence $\mathrm{lv}(a) < \sigma \leq \mathrm{lv}(b)$. Then $b(\sigma)$ is the point c mentioned in the definition of R and $N = \mathrm{Node}(c)$ and $x = t(\sigma) <_N c$, showing that $t \in R$, as required.

The Lemma's second assertion is proved in a similar way. $\square$

Theorem 2.6 *: Let $(T, \leq_T)$ be a tree and let $\{(N, <_N) : N \in \mathcal{N}_T\}$ be a set of node orderings for T. Let $\preceq$ be the associated lexicographic ordering of T. Then $(T, \preceq)$ is order isomorphic to a subset of $\mathbb{R}$ if and only if there are subsets C and A of T such that:*

a) C is countable;

b) A is the anti-chain of all maximal elements of $(T, \leq_T)$;

c) if $t \in T - C$ then some $a \in A$ has $t \leq_T a$;

d) $T - A$ is countable;

e) if $|T| > \omega$, then $|A| = |T|$;

f) there are only countably many nodes of T having more than one point;

g) for each node N of T, the linearly ordered set $(N, <_N)$ is order isomorphic to some subset of $\mathbb{R}$.

Proof: Suppose $(T, \preceq)$ is order isomorphic to some subset of $\mathbb{R}$. If T is countable, then (a) through (g) are immediate and there is nothing to prove. Hence suppose $|T| > \omega$. Being order isomorphic to a subset of $\mathbb{R}$, $(T, \preceq)$ has a countable order dense set D, i.e., if $x \prec y$ in T, then some $d \in D$ has $x \preceq d \preceq y$. The existence of D guarantees that any family of non-degenerate (= having more than one point) pairwise disjoint convex subsets of $(T, \preceq)$ is countable.

Suppose b is any branch of T, say $b = \{t_\alpha : \alpha < \mathrm{ht}(b)\}$. If $\mathrm{ht}(b) \geq \omega_1$, then for each limit ordinal $\lambda < \omega_1$ let I_λ be the $\preceq$-interval $(t_\lambda, t_{\lambda+3})_\preceq$, i.e., $I_\lambda = \{s \in T : t_\lambda \prec s \prec t_{\lambda+3}\}$. Then $\{I_\lambda : \lambda < \omega_1$ and λ is a limit ordinal $\}$ is an uncountable collection of pairwise disjoint, non-degenerate convex sets in $(T, \preceq)$ and that is impossible. Hence each branch of T has countable height. Therefore $\mathrm{ht}(T) \leq \omega_1$.

Let $S = \{x \in T : 1 < |T^x|\}$, partially ordered by restricting $\leq_T$. Consider S_α, the α^{th}-level of S. If x and y are distinct members of S_α then x and y are incomparable in S and hence also in T. Therefore the sets $T^x = \{t \in T : x \leq_T t\}$ and the analogously defined T^y are disjoint non-degenerate convex subsets of $(T, \leq_T)$ so that S_α must be countable. In addition, any branch of S extends to a branch of T, so each branch of S has countable height and $\mathrm{ht}(S) \leq \omega_1$. If $\mathrm{ht}(S) = \omega_1$ then S is an Aronszajn tree. But $(S, \preceq_S)$ is order isomorphic to a subset of $\mathbb{R}$ and that is impossible by Corollary 4.2, below. Hence $\mathrm{ht}(S) < \omega_1$. Having countable levels and countable height, S must be a countable set.

For $s, t \in T$ define that $s \sim t$ if and only if the convex hull of $\{s, t\}$ in $(T, \preceq)$ is countable, i.e. the interval of $(T, \preceq)$ from $\min_\preceq(s, t)$ to $\max_\preceq(s, t)$ is countable. Then $\sim$ is an equivalence relation on T. Because $(T, \preceq)$ order-embeds in $\mathbb{R}$, the cofinality and coinitiality of each equivalence class $\mathrm{cls}(t)$ must be countable. Hence $|\,\mathrm{cls}(t)| \leq \omega$ for each $t \in T$. Furthermore the collection $\{\mathrm{cls}(t) : t \in T$ and $|\,\mathrm{cls}(t)| > 1\}$ is countable, being a pairwise disjoint collection of non-degenerate convex sets in $(T, \preceq)$. Hence the set $C = \bigcup\{\mathrm{cls}(t) : t \in T,\ |\,\mathrm{cls}(t)| > 1\}$ is also a countable set, so (a) holds.

Let $A = T - S$. For any $a \in A$ we know that $|T^a| = 1$ so that a must be a maximal element of $(T, \leq_T)$. Furthermore, because S is countable, we know that $|A| = |T|$ so that (b) and (e) hold. Assertion (d) holds because $T - A = S$.

Suppose that $t \in T - C$ and that $t \notin A$. Then T^t has at least two points. If T^t were a countable set, then $T^t \subseteq C$ contrary to $t \in T - C$. Hence T^t is uncountable. Observe that each level of the subtree T^t is contained in a level of T, and therefore T^t has only a countable number of levels. Therefore, there is a level of $(T^t)_\alpha$ that is uncountable. Because $\{T^x : x \in (T^t)_\alpha$ and $|T^x| \geq 2\}$ is a pairwise disjoint collection of non-degenerate convex subsets of $(T, \preceq)$, the collection must be countable. Hence there are (uncountably many) points $x \in (T^t)_\alpha$ with $|T^x| = 1$ and any such x must belong to A and have $t \leq_T x$. This proves assertion (c).

To prove assertion (f), fix $\alpha < \mathrm{ht}(T)$. For each node M of T at level α with $|M| \geq 2$, choose $c_M, d_M \in M$ with $c_M <_M d_M$ (where $<_M$ is the linear ordering chosen for the node M). Then the second part of Lemma 2.5 shows that the intervals $[c_M, d_M]_\preceq$ of $(T, \preceq)$ are pairwise disjoint non-degenerate convex sets, so that there are only countably many such nodes at level α. But T has only countably many levels, so that all together T has only countably many non-degenerate nodes.

Assertion (g) must hold because the order-embedding of $(T, \preceq)$ into $\mathbb{R}$ also embeds $(N, <_N)$ into $\mathbb{R}$. Therefore, if $(T, \preceq)$ is order isomorphic to a subset of $\mathbb{R}$ then assertions (a) through (g) must hold.

Conversely, we will prove that if conditions (a) through (g) hold, then the linearly ordered set $(T, \preceq)$ has a countable order dense subset. That will be enough to show that $(T, \preceq)$ order-embeds in $\mathbb{R}$. Let $\mathcal{N}_2$ be the family of non-degenerate nodes of T. Each $N \in \mathcal{N}_2$ order-embeds in $\mathbb{R}$ and therefore contains a countable set $D(N)$ that is order dense in the linearly ordered set $(N, <_N)$. Because the set $(N, <_N)$ can have at most countably many jumps, we may assume that $D(N)$ contains both points of any jump in the set $(N, <_N)$, i.e., if $u, v \in N$ are distinct and no point of N lies strictly between u and v, then $u, v \in D(N)$.

Let $D = C \cup (T - A) \cup (\bigcup\{D(N) : N \in \mathcal{N}_2\})$. Then D is a countable subset of T. We claim that D is order dense in $(T, \preceq)$. Suppose $x \prec y$ are points of T. There are two cases to consider. First suppose that x and y are comparable in the partial order $\leq_T$. Then $x <_T y$ so that $x \in T - A \subseteq D$ and hence $D \cap [x, y]_\preceq \neq \emptyset$. Next suppose that x and y are incomparable in $\leq_T$. Then compute $\delta = \Delta_T(x, y)$, obtaining $\delta \leq \min(\mathrm{lv}(x), \mathrm{lv}(y))$ and $x(\delta) <_M y(\delta)$ where M is the node of T containing both $x(\delta)$ and $y(\delta)$. Then $M \in \mathcal{N}_2$. If some point u of M has $x(\delta) <_M u <_M y(\delta)$ then there is a point $v \in D(M)$ with $x(\delta) <_M v \leq_M y(\delta)$ and then $v \in D \cap [x, y]_\preceq$. If there is no such point $u \in M$, then the points $x(\delta)$ and $y(\delta)$ constitute a jump in $(M, <_M)$ and therefore $y(\delta) \in D(M) \subseteq D$ has $y(\delta) \in D \cap [x, y]_\preceq$. Therefore, D is a countable order dense subset of $(T, \preceq)$ and hence $(T, \preceq)$ is order isomorphic to some subset of $\mathbb{R}$. $\square$

Remark 2.7 *:* Theorem 2.6, an order-theoretic result, has a topological partial analog. With notation as in (2.6), suppose $\mathcal{I}$ is the usual open interval topology of the lexicographic ordering $\preceq$, and suppose there is a topological embedding (not necessarily order-preserving) of the linearly ordered space $(T, \mathcal{I})$ into the usual space of real numbers. Then $(T, \mathcal{I})$ is a second countable space and this allows us to prove that $(T, \mathcal{I})$ has a countable topologically dense subset and also has at most countably many jumps, so that $(T, \preceq)$ has a countable order dense set. At one point we need to know that for the subtree S of T, S is not an Aronszajn tree, and it is possible to prove that if $(T, \preceq)$ embeds topologically in $\mathbb{R}$, then no subtree of T can be an Aronszajn tree. Consequently, properties (a) through (g) still hold. The problem is (potentially) with the converse. Give (a) through (g), there is an order isomorphism from $(T, \preceq)$ onto a subset $\hat{T}$ of $\mathbb{R}$, but the topology that $\hat{T}$ inherits from $\mathbb{R}$ might not be the same as the open interval topology generated by the linear order that $\hat{T}$ inherits from $\mathbb{R}$.

The lexicographic representation theory for ordinal lines, i.e., sets of the form $[0, \alpha)$ where α is an ordinal number, is more simple than the corresponding theory for subsets of $\mathbb{R}$. We need to recall the idea of a *partition tree* of a linearly ordered set $(X, <)$. For any non-degenerate (= having more than one point) convex subset $I \subseteq X$, let P(I) be a pairwise disjoint collection of (possibly degenerate) convex subsets of I that covers I. Now define a tree recursively by:

- $T_0 = \{X\}$
- if $\alpha = \beta + 1$ and T_β is defined, let $T_\alpha = \bigcup\{P(I) : I \in T_\beta \text{ and } |I| > 1\}$
- if α is a limit ordinal and T_β is defined for all $\beta < \alpha$, then $T_\alpha = \{D = \bigcap\{C_\beta : \beta < \alpha\} : C_\beta \in T_\beta \text{ and } |D| \geq 2\}$.

Because X is a set, there must be some α with $T_\alpha = \emptyset$. Partially order $T = \bigcup\{T_\alpha : T_\alpha \neq \emptyset\}$ by reverse inclusion. Then T is a tree and the α^{th} level of T is T_α. Any node N of T is a collection of pairwise disjoint convex subsets of X, so that for distinct $C_1, C_2 \in N$ we may define $C_1 <_N C_2$ if and only if each point of C_1 precedes each point of C_2 in the original ordering given for X. This node ordering is called the *precedence ordering from* X.

The central issue in the next theorem is that for a limit ordinal λ, *any* partition tree of $[0,\lambda)$ can be used to represent $[0,\lambda)$ as a lexicographic tree.

Theorem 2.8 *: Let λ be any limit ordinal and let T be any partition tree of $X = [0,\lambda)$. Order the nodes of T using the precedence order from X and let $\preceq$ be the associated lexicographic ordering of T. Then $(T,\preceq)$ is order isomorphic to X.*

Proof: We will recursively construct a strictly increasing function ψ from $(T,\preceq)$ onto $[0,\lambda)$. For each $\alpha < \lambda$, let $b_\alpha = \{t \in T : \alpha \in t\}$. Then b_α is a branch of T and $T = \bigcup\{b_\alpha : \alpha < \lambda\}$.

Claim 1 : If $I \in T$ and $I \notin \bigcup\{b_\beta : \beta < \alpha\}$ and if some $J \in b_\alpha$ has $I \prec J$, then $I \in b_\alpha - \bigcup\{b_\beta : \beta < \alpha\}$. To prove Claim 1, we note that $I \cap [0,\alpha) = \emptyset$, and $\alpha \in J$. There are two ways that $I \prec J$ can occur. In the first, $I \leq_T J$, i.e., $J \subseteq I$, and then $\alpha \in J \subseteq I$ as claimed. The second is where I and J are incomparable members of T, and in that case $I \cap J = \emptyset$ and if $\delta = \Delta_T(I,J)$, then in the node N of T that contains both $I(\delta)$ and $J(\delta)$ we have $I(\delta) <_N J(\delta)$, so that every point of the convex set $I(\delta)$ precedes every point of $J(\delta)$ in $[0,\lambda)$. But $\alpha \in J \subseteq J(\delta)$ so that every point of $I(\delta)$ precedes α and therefore $I \subseteq I(\delta) \subseteq [0,\alpha)$ contradicting $I \cap [0,\alpha) = \emptyset$. Hence Claim 1 holds.

Claim 2: The height of the branch b_α is less than $\alpha + \omega$. Write $\mu = \alpha + \omega$ and for contradiction suppose that the height of b_α is greater than or equal to μ. Then we can find members $I_\gamma \in b_\alpha$ for each $\gamma < \mu$ such that if $\gamma_1 < \gamma_2 < \mu$ then $I_{\gamma_1} <_T I_{\gamma_2}$, i.e., $I_{\gamma_2} \subset I_{\gamma_1}$. For $\gamma < \mu$ let $f(\gamma) = \sup(I_\gamma)$. The function f cannot have infinite range because there is no strictly decreasing infinite set of ordinals. Hence there is a finite $n_0 < \mu$ such that $f(\beta) = f(\gamma)$ whenever $n_0 < \beta < \gamma < \lambda$. Define $g(\gamma) = \inf(I_\gamma)$ whenever $n_0 < \gamma < \lambda$; then g is strictly increasing. However, because $\alpha \in I_\gamma$ for each γ, we see that $f(\gamma) \leq \alpha$ for each γ. Thus we have an order isomorphism from $[n_0+1, \alpha+\omega)$ into $[0,\alpha)$ and that is impossible. Therefore, $\mathrm{ht}(b_\alpha) < \alpha + \omega$.

Claim 3: For each $\alpha < \lambda$, the set $S = b_\alpha - \bigcup\{b_\beta : \beta < \alpha\}$ is finite. For each $\beta < \mathrm{ht}(b_\alpha)$, let $b_\alpha(\beta)$ be the unique member of $b_\alpha \cap T_\beta$. If there is some $\beta < \mathrm{ht}(b_\alpha)$ such that $\min(b_\alpha(\beta)) = \alpha$, let β_0 be the least such β. If $\gamma < \beta_0$ then $\min(b_\alpha(\gamma)) < \alpha$ so that $b_\alpha(\gamma)$ contains some point less than α and therefore belongs to $\bigcup\{b_\beta : \beta < \alpha\}$. Therefore, any member of S has the form $b_\alpha(\gamma)$ where $\beta_0 < \gamma < \mathrm{ht}(b_\alpha)$, and each such set contains α and is contained in $b_\alpha(\beta_0) \subseteq [\alpha,\lambda)$. For $\beta_0 < \gamma < \mathrm{ht}(b_\alpha)$ define $h(\gamma) = \sup(b_\alpha(\gamma))$. We thereby obtain a strictly decreasing function. But there are no infinite strictly decreasing sequences of ordinals, so that the domain of h must be finite. Therefore, if β_0 is defined, then the set S is finite, as claimed. The remaining case is where for every $\beta < \mathrm{ht}(b_\alpha)$, the minimum of the set $b_\alpha(\beta)$ is less than α. But then every member of b_α contains a point less than α and therefore belongs to $\bigcup\{b_\gamma : \gamma < \alpha\}$, so that the set S is empty. In any case, therefore, S is finite.

We will now recursively define a collection of functions $\{\phi_\alpha : \alpha < \lambda\}$. By claim (3) we know that the branch b_0 of the tree T is finite. Let $|b_0| = n_0$. Then there is a unique strictly increasing function $\phi_0 : b_0 \to [0,n_0)$. Now suppose $0 < \alpha < \lambda$ and that we have defined a family of functions $\{\phi_\beta : \beta < \alpha\}$ satisfying the following five assertions that we collectively call $IH(\alpha)$.

1) if $\beta < \alpha$ then $\phi_\beta : \bigcup\{b_\gamma : \gamma < \beta\} \to [0,\lambda)$ is a strictly increasing function whose range is an initial segment of $[0,\lambda)$;

2) if $\beta < \alpha$ is not a limit ordinal, then $\phi_\beta\left(\bigcup\{b_\gamma : \gamma < \beta\}\right)$ is a proper initial segment of $[0, \beta+\omega)$;

3) if $\beta < \alpha$ is a limit ordinal, then $\phi_\beta\left(\bigcup\{b_\gamma : \gamma < \beta\}\right) = [0, \beta)$;

4) if $\beta < \alpha$ and $I, J \in \bigcup\{b_\gamma : \gamma < \beta\}$ have $I \prec J$ in the lexicographic ordering of T, then $\phi_\beta(I) < \phi_\beta(J)$ in $[0, \lambda)$;

5) if $0 \leq \gamma < \beta < \alpha$, then ϕ_β extends ϕ_γ.

If α is a limit ordinal, define $\phi_\alpha = \bigcup\{\phi_\beta : \beta < \alpha\}$. Clearly assertions 1) ,2), 4), and 5) of $IH(\alpha+1)$ hold. To verify assertion (3) we must consider two cases separately, If α is a limit of smaller limit ordinals, assertion 3) clearly holds, so consider the case where $\alpha = \mu+\omega$ for some limit ordinal μ. Applying $IH(\alpha)$ to the ordinals $\beta = \mu + n$ shows that the range of ϕ_α is $[0, \mu+\omega) = [\mu, \alpha)$ as required.

Finally consider the case where α is not a limit ordinal. Write $\alpha = \mu + k$ where μ is a limit ordinal and $0 < k < \omega$. We know that the range of $\phi_{\mu+(k-1)}$ is a proper initial segment of $[0, \mu+\omega)$ so that finiteness of the set $b_\alpha - \bigcup\{b_\gamma : \gamma < \alpha\}$ allows us to uniquely extend $\phi_{\mu+(k-1)}$ to a function ϕ_α on $\bigcup\{b_\gamma : \gamma < \alpha\}$ in such a way that the five assertions of $IH(\alpha+1)$ all hold.

The above recursion produces a chain $\{\phi_\alpha : \alpha < \lambda\}$ of partial isomorphisms, and then the function $\psi = \bigcup\{\phi_\alpha : \alpha < \lambda\}$ is the order isomorphism needed to prove the theorem. $\square$

Remark 2.9 *:* Theorem 2.8 is another illustration of the marked difference between lexicographic representation theory and branch space representation theory for linearly ordered sets. In [1] we show that if λ is a regular cardinal (such as ω_1), then $[0, \lambda)$ is not isomorphic to a branch space of any tree T, unless some node of T already contains a copy of $[0, \lambda)$ or $[0, \lambda)^*$ where $[0, \lambda)^*$ indicates $[0, \lambda)$ with the reverse ordering.

Corollary 2.10 *: Every ordinal line* $[0, \alpha)$ *is order-isomorphic to a lexicographic tree whose levels and nodes are finite.*

Proof: In case α is a limit ordinal, use any binary partition tree of $[0, \alpha)$ and apply Theorem 2.8 above. In case α is not a limit, write $\alpha = \lambda + n$ where $1 \leq n < \omega$ and λ is a limit ordinal. The zeroth level T_0 of the tree is $\{[0, \lambda), \{\lambda\}, \{\lambda+1\}, \cdots, \{\lambda+n-1\}\}$, ordered naturally. The elements $\{\lambda+i\}$ are maximal in the tree, and above the element $[0, \lambda) \in T_0$ construct any binary partition tree of $[0, \lambda)$. According to Theorem 2.8, the resulting lexicographically ordered tree is exactly $[0, \alpha)$. $\square$

Example 2.11 *: There is a partition tree of* $[0, \omega+1]$ *whose lexicographic ordering is not isomorphic to* $[0, \omega+1]$*. Thus Theorem 2.8 fails for non-limit ordinals.*

Proof: For each finite height $n \geq 0$, let $T_n = \{\{n\}, [n+1, \omega+1]\}$. Let $T_\omega = \{\{\omega, \omega+1\}\}$ and $T_{\omega+2} = \{\{\omega\}, \{\omega+1\}\}$. Order each node naturally. The resulting lexicographic tree is order isomorphic to $[0, \omega+2]$, not $[0, \omega+1]$. $\square$

3 Some topology for lexicographically ordered trees

Let T be a tree and let $\{(N, <_N) : N \in \mathcal{N}_T\}$ be a fixed family of node orderings. When endowed with the open interval topology of its lexicographic ordering, T is a linearly ordered topological space and therefore has very strong separation properties (e.g., monotonic normality [3]). We begin by characterizing compactness of a tree T with the open interval topology of $\preceq$ (equivalently, we are characterizing completeness of the order $\preceq$) in terms of the properties of $(T, \leq_T)$ and of its node orderings. That will involve showing that certain subsets of T have suprema in $(T, \preceq)$ and we will need several preliminary lemmas.

Lemma 3.1 *: Let b be a branch of a tree $(T, \leq_T)$ and let $\preceq$ be the lexicographic order of T associated with a family of node orderings. If b has a maximum element s^* in $(T, \leq_T)$, then $s^* = \sup_{(T,\preceq)}(b)$. If b does not have a maximum element in $\leq_T$ (i.e., if $\mathrm{ht}(b)$ is a limit ordinal) then b has a supremum in $(T, \preceq)$ if and only if there exist an element $s \in T$ and an ordinal $\mu < \mathrm{ht}(b)$ such that*

1) $\mathrm{lv}(s) = \mu$;

2) if $\gamma < \mu$ then $s(\gamma) = b(\gamma)$ where $b(\gamma)$ is the unique point of $b \cap T_\gamma$;

3) the point s is the immediate successor of $b(\mu)$ in the node of T to which both belong; and

4) if $\mu < \alpha < \mathrm{ht}(b)$, then $b(\alpha)$ is the maximum element of the node to which it belongs.

Proof: It is clear that any branch that has a maximum element in the partially ordered set $(T, \leq_T)$ will have that maximum element as its supremum in the linearly ordered set $(T, \preceq)$.

Next, suppose that the branch b has a supremum s in $(T, \preceq)$ and that the branch b has no maximum element in the partially ordered set $(T, \leq_T)$. Then the branch has limit height and we can write $b = \{t_\beta : \beta < \mathrm{ht}(b)\}$. Because $s \notin b$, there must be some $t_\beta \in b$ that is not comparable to s in the partial order $\leq_T$. Compute $\delta = \Delta_T(s, t_\beta) \leq \mathrm{lv}(s)$. Then in the node M of T that contains both $t_\beta(\delta)$ and $s(\delta)$ we know that $t_\beta(\delta) <_M s(\delta)$. If $\delta < \mathrm{lv}(s)$, then $s(\delta) <_T s$ and therefore $s(\delta)$ would be an upper bound for the branch b that strictly precedes s in $(T, \preceq)$, and that is impossible because s is the supremum of b in $(T, \preceq)$. Hence $\delta = \mathrm{lv}(s)$. If there were some point $u \in M$ with $t_\beta(\delta) <_M u <_M s$ then u would be an upper bound for b that is strictly less than $s = \sup_T(b)$. Hence s is the immediate successor of $t_\beta(\delta)$ in M. Finally consider any γ with $\delta < \gamma < \mathrm{ht}(b)$. In the notation of the lemma, we have $b(\gamma) = t_\gamma$. If $b(\gamma)$ is not the maximum of the node to which it belongs, then we could choose a larger element t' in that node and thereby obtain an upper bound for b that is strictly less than $s = \sup_T(b)$, which is impossible. Therefore, if the branch b has a supremum in the linearly ordered set $(T, \preceq)$, then it must be the maximum of b in the partially ordered set $(T, \leq_T)$ or else it must be as described in this lemma.

The proof of the converse is straightforward. $\square$

Lemma 3.2 *: Let $(T, \leq_T)$ be a tree and suppose a family of node orderings has been chosen. Let $\preceq$ be the associated lexicographic order of T. A node N of T has a supremum s in $(T, \preceq)$ if and only if one of the following conditions hold:*

a) s is the maximum element of the linearly ordered set $(N, <_N)$;

b) the set $(N, <_N)$ has no maximum element and there is an ordinal $\mu < \mathrm{lv}(N)$ and a point $s \in T$ with the property that $\mathrm{lv}(s) = \mu$ and for every $t \in N,\ s$ is the immediate successor of $t(\mu)$ in the node M that contains both $t(\mu)$ and s, and if $\mu < \alpha < \mathrm{lv}(N)$ then for each $t \in N,\ t(\alpha)$ is the maximum point of the node to which it belongs.

Proof: The proof of Lemma 3.2 closely parallels the proof of Lemma 3.1. $\square$

Lemma 3.3 *; Let $\preceq$ be the lexicographic order associated with some choice of node orderings for the tree T, and let A be a non-empty initial segment of $(T, \preceq)$. For any $s \in T$, the following are equivalent:*

(1) for each $a \in A$, either $a \preceq s$ or $s <_T a$;

(2) if $a \in A$ has $\mathrm{lv}(a) \leq \mathrm{lv}(s)$ then $a \preceq s$.

Proof: Clearly (1) implies (2). Suppose (2) holds and $a \in A$. If $a \preceq s$ is false, then $s \prec a$. Then either $s <_T a$ (which is the conclusion we want), or else s and a are incomparable in the partial order $\leq_T$. In the latter case, if we compute $\delta = \Delta_T(a, s)$, then $\delta \leq \mathrm{lv}(s)$, and $s(\delta) <_M a(\delta)$ in the node M of T that contains $s(\delta)$ and $a(\delta)$, and for each $\gamma < \delta,\ a(\gamma) = s(\gamma)$. Note that $a(\delta) \preceq a$ because $a(\delta) \leq_T a$. Because A is an initial segment of $(T, \preceq)$, it follows that $a(\delta) \in A$. But then we have $\mathrm{lv}(a(\delta)) \leq \mathrm{lv}(s)$ and $s \prec a(\delta)$ contradicting (2). Therefore, (1) holds. $\square$

Lemma 3.4 *: Let $\preceq$ be the lexicographic order associated with some choice of node orderings for the tree $(T, \leq_T)$, and let A be a non-empty initial segment of the lexicographically ordered set $(T, \preceq)$. In each of the following cases, A has a supremum in $(T, \preceq)$:*

(1) There is a point s of a node N of T such that $s = \sup_N(A \cap N),\ s \notin A$, and $s(\alpha) \in A$ whenever $\alpha < \mathrm{lv}(s)$ (where $s(\alpha)$ is the unique predecessor of s in T_α).

(2) There is a point $s^* \in T - A$ and a node N of T such that $\emptyset \neq N \cap A = N$ and $(N, <_N)$ does not have a maximum element, and $s^* = \sup_T(N)$.

(3) There is a point $x \in N$ where N is a node at a successor level $\beta + 1$ with $N \cap A = \emptyset$, and every strict predecessor of x in $\leq_T$ belongs to A.

(4) There is a point x of a node N at a limit level such that $N \cap A = \emptyset$, and x is the minimum point of N in the order $<_N$, and every strict predecessor of x in the partial order $\leq_T$ belongs to A.

Proof: Suppose (1) holds. Because $s \notin A$ and A is an initial segment of $(T, \preceq)$, we know that each $a \in A$ has $a \prec s$. First consider the case in which $N \cap A \neq \emptyset$. We will show that there cannot be a point $t \in T$ with the property that $t \prec s$ and $a \preceq t$ for each $a \in A$. The relation $t \prec s$ can happen in two ways, depending upon whether s and t are comparable in the partial order $\leq_T$. If t and s are comparable, then $t <_T s$. Let $a_1 \in N \cap A$. Then $t <_T a_1$ because a_1 and s have exactly the same predecessors. But then $t \prec a_1$ contrary to the properties of t. Therefore, t and s are incomparable in $\leq_T$. Compute $\delta = \Delta_T(s, t)$. Then $\delta \leq \mathrm{lv}(s)$ and in the node M of T that contains both $s(\delta)$ and $t(\delta)$ we have $t(\delta) <_M s(\delta)$. If $\delta < \mathrm{lv}(s)$ then $s(\delta) \in A$ because $s(\delta)$ is a strict predecessor of s in $\leq_T$. But then $t \prec s(\delta) \in A$ contrary to assumed properties of t. Hence $\delta = \mathrm{lv}(s)$. But then $s(\delta) = s = \sup_N(N \cap A)$ so that $t(\delta) \in N$ and $t(\delta) <_M s(\delta) = s$ provides a point $a_2 \in A \cap N$ with $t(\delta) <_M a_2$. Then $t \prec a_2$ contrary to assumed properties of t. Therefore, no $t \in T$ has $t \prec s$ and also has $a \preceq t$ for each $a \in A$, so $s = \sup_{(T,\preceq)}(A)$.

Next consider the case where $N \cap A = \emptyset$. Then $s = \sup_N(\emptyset)$ tells us that s is the minimum element of N. We will separately consider the cases where $\mathrm{lv}(s)$ is a successor ordinal and where $\mathrm{lv}(s)$ is a limit ordinal.

If $\mathrm{lv}(s) = \beta + 1$ is not a limit ordinal, then we claim that $s(\beta)$, the unique predecessor of s in T_β, is the supremum of A in $(T, \preceq)$. The hypothesis of this lemma guarantees that $s(\beta) \in A$ so that it will be enough to show that $a \preceq s(\beta)$ for each $a \in A$. For contradiction, suppose $s(\beta) \prec a_3$ for some $a_3 \in A$. As noted above, $a_3 \prec s$ so that $s(\beta) \prec a_3 \prec s$. Therefore both $s(\beta) <_T a_3$ and $a_3 <_T s$ are impossible so that $s(\beta)$ and a_3 are incomparable in the partial order $\leq_T$. Compute $\delta = \Delta_T(s(\beta), a_3)$. Then $\delta \leq \mathrm{lv}(s(\beta)) = \beta$, and $s(\beta)(\delta)$, the unique predecessor of $s(\beta)$ in T_δ, is the same as $s(\delta)$, the unique predecessor of s at level δ. Because $s(\beta) \prec a_3$ we have $s(\delta) = s(\beta)(\delta) <_M a_3(\delta)$ where M is the node of T containing both $s(\beta)(\delta)$ and $a_3(\delta)$ while for each $\gamma < \delta$ we have $s(\gamma) = s(\beta)(\gamma) = a_3(\gamma)$. That is enough to show that $s \prec a_3$ and that is impossible because s is an upper bound for the set A. Therefore, $s(\beta) = \sup_{(T,\preceq)}(A)$ as claimed.

Now consider the case where $N \cap A = \emptyset$, $s = \sup_N(A \cap N)$ is not in A, and $\mathrm{lv}(s) = \lambda$ is a limit ordinal. As noted above, $s = \sup_N(\emptyset)$ means that s is the minimum element of $(N, <_N)$. We claim that $s = \sup_{(T,\preceq)}(A)$. For contradiction, suppose there is some $t \in T$ with the property that for each $a \in A$, $a \preceq t \prec s$. There are two possibilities for the relationship between s and t. If $t <_T s$, then because $\mathrm{lv}(s) = \lambda$ is a limit ordinal, there is some $\alpha < \lambda$ with $t <_T s(\alpha) <_T s$. But then $s(\alpha) \in A$ and hence $t \prec s(\alpha)$ shows that t is not an upper bound for A. Therefore, t and s are incomparable in the partial order $\leq_T$. Compute $\delta = \Delta_T(s, t)$. Then $\delta \leq \mathrm{lv}(s) = \lambda$ and $t(\delta) <_M s(\delta)$ where M is the node of T containing both $s(\delta)$ and $t(\delta)$ while $s(\gamma) = t(\gamma)$ whenever $\gamma < \delta$. If $\delta < \lambda$, then $s(\delta) \in A$. But then $t(\delta) <_M s(\delta)$ shows that $t \prec s(\delta) \in A$ so that t is not an upper bound for A. Hence $\delta < \lambda$ is impossible and we must have $\delta = \lambda$. But then the node M must be N so that $t(\delta) <_N s(\delta) = s$ shows that s is not the minimum element of N, and that is impossible. Therefore $s = \sup_{(T,\preceq)}(A)$ as claimed. This completes the proof of (1).

Now consider (2). Because A is an initial segment of $(T, \preceq)$ and $s^* \notin A$ we see that s^* is an upper bound for A in $(T, \preceq)$. But because s^* is the supremum in $(T, \preceq)$ of the nonempty subset $N \cap A$ of A, it follows that $s^* = \sup_{(T,\preceq)}(A)$.

Next consider (3). Because $x \in N$ and $N \cap A = \emptyset$, we know that $x \notin A$ and therefore x is an upper bound for the initial segment A of $(T, \preceq)$. Let $\alpha = \mathrm{lv}(x)$. Then $\alpha = \beta + 1$ so there is a point $y \in T_\beta$ that is the immediate predecessor of x, and we know from (3) that $y \in A$. We claim that $a \preceq y$ for each $a \in A$. If not, then consider some $a \in A$ with $y \prec a$. If $y <_T a$, then $\mathrm{lv}(a) \geq \mathrm{lv}(y) + 1 = \alpha$ so that a has a unique predecessor $a(\alpha) \in T_\alpha$. (Possibly $a(\alpha) = a$.) But then $a(\alpha) \in N \cap A = \emptyset$. Hence a and y must be incomparable. Therefore, a and x are also incomparable, because $y <_T x$. Compute $\delta = \Delta_T(y, a) \leq \mathrm{lv}(y) = \beta$. Then $y(\delta) <_M a(\delta)$ in the node M of T that contains both $a(\delta)$ and $y(\delta)$. Note that $x(\delta) = y(\delta)$ so that we have $x(\delta) = y(\delta) <_N a(\delta)$ and $\Delta_T(y, a) = \Delta_T(x, a)$. But then we are forced to conclude $x \prec a$ and that is impossible because x is an upper bound for A. Hence, if (3) holds, we see that A has a supremum in $(T, \preceq)$.

Finally, consider (4), where x is the minimum point of the set $(N, <_N)$, $N \cap A = \emptyset$, $\mathrm{lv}(x) = \alpha$ is a limit ordinal, and every strict predecessor of x in $\leq_T$ belongs to A. Because $x \notin A$ we know that

x is an upper bound for the initial segment A in $(T, \preceq)$. We show that there cannot be any $y \in T$ with $a \preceq y \prec x$ for every $a \in A$. There are two ways for $y \prec x$ to happen. In the first, $y <_T x$. But then the fact that $\mathrm{lv}(x)$ is a limit ordinal tells us that there is some $z \in T$ with $y <_T z <_T x$. Then $z \in A$ and we have $y \prec z$ contrary to the assumed properties of y. Therefore, x and y must be incomparable in $\leq_T$. Compute $\delta = \Delta_T(x, y) \leq \mathrm{lv}(x) = \alpha$. We have $y(\delta) <_M x(\delta)$ in the node M that contains both $y(\delta)$ and $x(\delta)$. If $\delta < \alpha$, then $x(\delta) \in A$. But then $y(\delta) <_M x(\delta)$ shows that $y \prec x(\delta) \in A$, contrary to the assumed properties of y. Hence $\delta = \alpha$. But then M is the node containing $x(\alpha) = x$ so that $M = N$ and then $y(\delta) <_N x(\delta) = x$ shows that x could not have been the minimum element of its node. Hence $x = \sup_{(T,\preceq)}(A)$. $\Box$

Theorem 3.5 *: Let $(T, \leq_T)$ be a tree and let $\{(N, <_N) : N \in \mathcal{N}_T\}$ be a family of node orderings. Let $\preceq$ be the resulting lexicographic ordering of T. Then with its open interval topology, the lexicographically ordered tree is compact if and only if the following four conditions hold:*

C1 For each $N \in \mathcal{N}_T,\ N$ has a least upper bound in $(T, \preceq)$ (see Lemma 3.2);

C2 If $N \in \mathcal{N}_T$ and if $\mathrm{lv}_T(N)$ is a limit ordinal, then $(N, <_N)$ has a least element. (Note: this condition also applies to the zeroth level of the tree, which is itself a node of T.)

C3 for each $N \in \mathcal{N}_T$ the linearly ordered set $(N, <_N)$ is conditionally complete, i.e., any non-empty subset of N that has an upper bound in N must have a least upper bound in N.

C4 for each branch $b \in \mathcal{B}_T$, either b has a maximal element in T or else the subset b of T has a supremum $s(b)$ in $(T, \preceq)$ (see Lemma 3.1).

Proof: In this proof we will need to consider several different partial and linear orders, namely the partial order $\leq_T$ and its strict version $<_T$, the lexicographic order $\preceq$ on T and its strict version $\prec$, and the linear ordering $\leq_N$ chosen for a node N of T, and its strict version $<_N$. For a set $S \subseteq T$, we will write $\sup_T(S)$ for the supremum of S in the linearly ordered set $(T, \preceq)$ and for a subset S of a node N, we will write $\sup_N(S)$ for the supremum of S in the linearly ordered set $(N, \leq_N)$.

First suppose that a tree $(T, \leq_T)$ has node orderings satisfying (C1) through (C4). We will show that every initial segment A of $(T, \preceq)$ has a least upper bound. If $A = \emptyset$, apply (C2) to the set T_0, which is a node of T at limit level. The minimum element of T_0 is the least upper bound for $A = \emptyset$.

Next consider the case where $A = T$. (This special case is a preview of the approach to be used later, when A is a proper initial segment.) The set T_0, the zeroth level of T, is a node of T. In the light of $C1$ applied to T_0, we know that T_0 has a supremum in $(T, \preceq)$. Let $t^* = \sup_{(T,\preceq)}(T_0)$. There is a unique $t_0 \in T_0$ with $t_0 \leq_T t^*$. Let $\leq_{T_0}$ be the linear ordering chosen for the node T_0. For any $s \in T_0$ we have $s \preceq \sup_{(T,\preceq)}(T_0) = t^*$ so that, if $s \neq t_0$ we must have $s \prec t_0$. Hence $s \leq_{T_0} t_0$ and so t_0 is the maximum element of T_0 in its node ordering. (In fact, one can see that $t_0 = t^*$.) For induction hypothesis, suppose $\alpha > 0$ is an ordinal and for each ordinal $\beta < \alpha$ we have found $t_\beta \in T_\beta$ such that t_β is the maximum element of its node and such that if $\beta_1 < \beta_2 < \alpha$ then $t_{\beta_1} <_T t_{\beta_2}$. There are two cases to consider, depending upon whether the set $\rho = \{t_\beta : \beta < \alpha\}$ is a branch of T.

If ρ is not a branch, then the set $N = \{t \in T_\alpha : T_t = \rho\}$ is non-empty and is a node of T. Apply (C1) to find $t^{**} = \sup_\preceq(N)$. We claim that for each $\beta < \alpha,\ t_\beta <_T t^{**}$. We know that for any $a \in N$ and for any $\beta < \alpha$ we have $t_\beta <_T a \preceq t^{**}$ so that $t_\beta \prec t^{**}$. If there is a $\beta < \alpha$ such that $t_\beta <_T t^{**}$ is false, then t_β and t^{**} would be incomparable in the partial order $\leq_T$ and we would compute $\delta = \Delta_T(t_\beta, t^{**})$ and find that, in the node M of T that contains both $s_\beta(\delta)$ and $t^{**}(\delta)$, we would have $t_\beta(\delta) <_M t^{**}(\delta)$. Because $t_\beta(\delta) = t_\delta,\ t_\delta <_M t^{**}(\delta)$ is impossible because t_δ is known to be the maximum element of its node. Therefore $t_\beta <_T t^{**}$ for each $\beta < \alpha$. But then $\mathrm{lv}(t^{**}) \geq \alpha$ so that t^{**} has a unique predecessor $t^{**}(\alpha)$ at level α of T, and it is immediate that if we define $t_\alpha = t^{**}(\alpha)$, then we obtain a point of N that is the maximum of the linearly ordered set $(N, <_N)$, and so the induction continues.

In the remaining case, the path ρ is a branch of T. Note that $\mathrm{ht}(\rho) = \alpha$. We claim that $\mathrm{ht}(\rho)$ is not a limit ordinal. If it were a limit, we would apply (C4) to find an ordinal $\mu = \mu(\rho) < \mathrm{ht}(\rho)$ such that $\rho(\mu)$ has an immediate successor in its node, and that is once again impossible because $\rho(\mu) = t_\mu$ is the maximum of its node. Therefore, $\alpha = \mathrm{ht}(\rho)$ must be a successor ordinal, say $\mathrm{ht}(\rho) = \alpha = \beta + 1$. Then t_β is the maximal element of the branch ρ, and hence is a maximal element of the partial order $\leq_T$ of T. We claim that $t_\beta = \sup_T(T)$. Consider any $s \in T$ and for contradiction suppose $t_\beta \prec s$. Because t_β is maximal in T, we know that $t_\beta <_T s$ is false. Hence s and t_β must be incomparable in $(T, \leq_T)$ so that if $\delta = \Delta_T(s, t_\beta)$, then in the node M to which both $s(\delta)$ and $t_\beta(\delta)$ belong, we have $t_\beta(\delta) <_M s(\delta)$. But that is impossible because $t_\beta(\delta) = t_\delta$ and t_δ is known to be the maximum point of its node. Therefore, $t_\beta = \sup_{(T,\preceq)}(T)$.

Now we consider the more complicated case where A is a nonempty, proper initial segment of $(T, \preceq)$, i.e., if $s \prec a \in A$, then $s \in A$. Because $A \neq T,\ A$ is bounded in $(T, \preceq_T)$. Consider the set

$$\Gamma = \{\alpha < \mathrm{ht}(T) : \text{ for some } x \in T_\alpha,\ a \prec x \text{ for all } a \in A\}.$$

The set Γ is non-empty because A is a bounded subset of $(T, \preceq)$. For each $\alpha \in \Gamma$, let $U_\alpha = \{t \in T_\alpha :$ for all $a \in A,\ a \prec t\}$ and let $\eta = \min(\Gamma)$. Observe that minimality of η combined with the fact that A is an initial segment of $(T, \preceq)$, guarantees that

$$(*) \quad \beta < \eta \text{ implies } T_\beta \subseteq A.$$

We claim that U_η is contained in a single node of T. If $|U_\eta| = 1$ this is clear, so suppose that $|U_\eta| > 1$. Fix $x, y \in U_\eta$ with $x \prec y$ We will show that y belongs to the node of T that contains x. Because $\mathrm{lv}(x) = \eta = \mathrm{lv}(y)$ the points x and y are incomparable in the partial order $\leq_T$. Compute $\delta = \Delta_T(x, y)$. Because $x \prec y$, we know that $x(\delta) <_M y(\delta)$ where M is the node of T that contains both $x(\delta)$ and $y(\delta)$. Note that $\delta \leq \mathrm{lv}(x) = \eta$. If $\delta < \eta$, then $\{x(\delta), y(\delta)\} \subseteq T_\delta \subseteq A$ in the light of (*). But then $x \prec y(\delta) \in A$ and that contradicts $x \in U_\eta$. Therefore $\delta = \eta$ and hence $x = x(\delta)$ and $y = y(\delta)$ showing that y belongs to the node of T that contains x, as claimed. Denote that node by $N(\eta)$.

We next show that either A has a supremum in $(T, \preceq)$ or else there is a point s_η such that $A \cap N(\eta) \neq \emptyset$ and $s_\eta = \sup_{N(\eta)}(A \cap N(\eta))$ belongs to A, and for each $a \in A$ with $\mathrm{lv}(a) \leq \eta$, either $a \preceq S_\eta$ or else $s_\eta <_T a$. There are two cases to analyze, depending upon whether or not $N_\eta \cap A = \emptyset$.

<u>Case 1</u>: First consider the case in which $N(\eta) \cap A = \emptyset$. If η is a limit ordinal, then (C2) implies that the node $N(\eta)$ has a minimum element z. Then $N(\eta) \cap A = \emptyset$ implies that $z \notin A$. In the light of (*), every strict predecessor of z in $\leq_T$ belongs to A. Now apply part (4) of Lemma 3.4 to conclude that A has a supremum in $(T, \preceq)$ as required. If η is not a limit ordinal, then part (3) of Lemma 3.4 applies to show that A has a supremum in $(T, \preceq)$.

Case 2: Now suppose that $A \cap N(\eta) \neq \emptyset$. Choose an element $x_\eta \in U_\eta$. Then the set $A \cap N(\eta)$ is a nonempty subset of $N(\eta)$ that is bounded above (by x_η). Hence (C3) provides a least upper bound s_η for $A \cap N(\eta)$ in $(N(\eta), <_{N(\eta)})$. Observe that because s_η and x_η belong to the same node of T, we have $s_\eta(\gamma) = x_\eta(\gamma)$ for each $\gamma < \eta$.

If $s_\eta \notin A$, then s_η must be the supremum of A in $(T, \preceq)$. This follows from part (1) of Lemma 3.4.

Next consider the case where $s_\eta \in A$. We claim that if $a \in A$ and $\mathrm{lv}(a) < \eta$, then $a \preceq s_\eta$. From $a \preceq x_\eta$ we conclude that either $a \leq_T x_\eta$ (in which case $a = x_\eta(\delta) = s_\eta(\delta) \preceq s_\eta$) or else a and x_eta are incomparable in $\leq_T$. In that second case, the ordinal $\delta_1 = \Delta(a, x_\eta)$ has $\delta_1 \leq \mathrm{lv}(a) = \delta < \eta$ and $a(\delta_1) <_M x_\eta(\delta_1)$ where M is the node of T containing both $a(\delta_1)$ and $x_eta(\delta_1)$. The fact that $x_\eta(\gamma) = s_\eta(\gamma)$ for all $\gamma < \eta$ yields $\delta_1 = \Delta(a, s_\eta) = \Delta(a, x_\eta)$ and $a(\delta_1) <_M s_\eta(\delta_1)$ and therefore $a \preceq s_\eta$ in case $\mathrm{lv}(a) < \eta$.

We also claim that if $\mathrm{lv}(a) = \eta$ then $a \preceq s_\eta$. If it happens that $a \in N_\eta$, then $a \preceq s_\eta$ follows from $s_\eta = \sup_{N(\eta)}(A \cap N(\eta))$. Hence we may assume that $a \in T_\eta - N(\eta)$. Then $\delta_2 = \Delta(a, x_\eta)$ has $\delta_2 < \eta$ so that from $a \preceq x_\eta$ we conclude $a(\delta_2) <_L x_\eta(\delta_2) = s_\eta(\delta_2)$ where L is the node of T containing both $a(\delta_2)$ and $x_\eta(\delta_2) = s_\eta(\delta_2)$. But then $a \preceq s_\eta$, as claimed.

Therefore we have proved that if $a \in A$ has $\mathrm{lv}(a) \leq \eta$ then $a \preceq s_\eta$. Now Lemma 3.3 applies to show that if $a \in A$ then either $a \preceq s_\eta$ or $s_\eta <_T a$.

At this point of our proof, we have either showed that A has a supremum in $(T, \preceq)$ or else we have initialized a recursive construction by finding the point s_η. To continue that recursion, suppose that $\alpha > \eta$ and for each β with $\eta \leq \beta < \alpha$ we have found a point $s_\beta \in A \cap T_\beta$ such that the following induction hypotheses $(IH)_\alpha$ are satisfied:

(1) if $\eta \leq \beta_1 < \beta_2 < \alpha$, then $s_{\beta_1} <_T s_{\beta_2}$;

(2) if $N(\beta)$ is the node of T containing s_β, then $A \cap N(\beta) \neq \emptyset$ and $s_\beta = \sup_{N(\beta)}(N(\beta) \cap A)$ belongs to A;

(3) if $a \in A$ has $\mathrm{lv}(a) \leq \beta$, then $a \preceq s_\beta$. (Note that in the light of Lemma 3.3, this is equivalent to the statement that for each $a \in A$, either $a \preceq s_\beta$ or else $s_\beta <_T a$).

We will consider a sequence of cases and in each we will show that either we have a supremum for the set A in $(T, \preceq)$, or else we see how to define s_α in such a way that $(IH)_{\alpha+1}$ holds and the induction continues.

Case 3: Suppose $\alpha = \beta + 1$ is a successor ordinal and $N(\alpha) \cap A = \emptyset$ where $N(\alpha)$ is the node of all immediate successors of the already-defined point s_β. We claim that in this case, A has a supremum in $(T, \preceq)$. If $N(\alpha) \neq \emptyset$, choose $x \in N(\alpha)$ and apply part (3) of Lemma 3.4 to conclude that A has a supremum in $(T, \preceq)$. If $N(\alpha) = \emptyset$, then s_β is a maximal point of $(T, \leq_T)$ so that $s_\beta <_T a$ never happens for $a \in A$. Applying part (3) of the induction hypothesis, we see that $a \preceq s_\beta$ for each $a \in A$, as claimed. Therefore, in Case 3, the set A has a supremum in $(T, \preceq)$.

Case 4: Suppose $\alpha = \beta + 1$ and $\emptyset \neq N(\alpha) \cap A = N(\alpha)$ where $N(\alpha)$, the node of immediate successors of s_β, has a maximum element in the chosen linear ordering $<_{N(\alpha)}$. Define s_α to be that maximum element. Then $s_\alpha \in A$ and because $s_\alpha \in N(\beta + 1)$, we know that $s_\beta <_T s_\alpha$ so that the first part of $(IH)_{\alpha+1}$ holds. Clearly $s_\alpha = \sup_T(A \cap N(\alpha))$ so the second part of $(IH)_{\alpha+1}$ also holds. To verify the third part of $(IH)_{\alpha+1}$, consider any $a \in A$ with $\mathrm{lv}(a) \leq \alpha = \beta + 1$. We must show $a \preceq s_\alpha$. In case $\mathrm{lv}(a) \leq \beta$ then we know that $a \preceq s_\beta \prec s_\alpha$, so consider the case where $\mathrm{lv}(a) = \alpha$. For contradiction suppose $s_\alpha \prec a$.

Because $\mathrm{lv}(s_\alpha) = \mathrm{lv}(a) = \alpha$ we cannot have $s_\alpha <_T a$, so a and s_α must be incomparable in the partial order $\leq_T$. Compute $\delta = \Delta_T(s_{\beta+1}, a)$. Then $\delta \leq \mathrm{lv}(a) = \alpha$ and $s_\alpha(\delta) <_M a(\delta)$ where M is the node of T containing both $a(\delta)$ and $s_\alpha(\delta)$. If $\delta < \alpha$, then $a(\delta)$ and $s_\alpha(\delta)$ both belong to the same node M at level δ of the tree, But $s_\alpha(\delta) = s_\delta$, which is known to be the maximum of its node in the chosen node ordering, so $a(\delta) <_M s_\alpha(\delta)$ and hence $a \prec s_\alpha$. In case $\delta = \alpha$, then a and s_α belong to the same node $N(\alpha)$ of T so that, s_α being the maximum of that node, we have $a <_{N(\alpha)} s_\alpha$ whence $a \prec s_\alpha$. Therefore the third part of $(IH)_{\alpha+1}$ holds in case 4, and the recursion continues.

Case 5: Suppose $\alpha = \beta + 1$ and $\emptyset \neq N(\alpha) \cap A = N(\alpha)$ where $N(\alpha)$, the node of immediate successors of s_β, does not have a maximum element in the chosen linear ordering $<_{N(\alpha)}$. However, (C1) guarantees that $N(\alpha)$ has a supremum t^* in $(T, \preceq)$. Then $t^* \notin N(\alpha)$. We will show that t^* is the supremum of A in $(T, \preceq)$. Choose any $a \in N(\alpha) \cap A$. Then we have $s_\beta <_T a \preceq t^*$ so that $s_\beta \prec t^*$. We claim that t^* and s_β are incomparable in the partial order $\leq_T$. If that is not the case, then $s_\beta \prec t^*$ would yield $s_\beta <_T t^*$ so that $\mathrm{lv}(t^*) \geq \alpha$ and hence $t^*(\alpha)$ exists and belongs to $N(\alpha)$. Let u be any element of $N = N(\alpha)$. If $t^*(\alpha) <_N u$ then $t^* \prec u \preceq \sup_T(N) = t^*$ which is impossible. Therefore, each $u \in N$ has $u \leq_N t^*(\alpha)$ showing that $t^*(\alpha)$ is the maximum point of $(N, <_N)$ and that is impossible in Case 5. Therefore s_β and t^* are incomparable in the partial order $\leq_T$.

Now let $a \in A$. According to the induction hypothesis, we know that either $a \prec s_\beta$ (in which case $a \prec s_\beta \prec t^*$) or else $s_\beta <_T a$. Consider the case where $s_\beta <_T a$. Then $\mathrm{lv}(a) \geq \alpha$. In case $\mathrm{lv}(a) = \alpha$, then $s_\beta <_T a$ yields $a \in N(\alpha)$ so that $a \prec t^*$. In case $\mathrm{lv}(a) > \alpha$, then $s_\beta <_T a$ yields $a(\alpha) \in N(\alpha)$ so that $a(\alpha) \preceq t^*$. But $a(\alpha) \in N(\alpha)$ while $t^* \notin N(\alpha)$ so that $a(\alpha) \prec t^*$. This could happen in two ways: either $a(\alpha) <_T t^*$ or else $a(\alpha)$ and t^* are not comparable in the partially ordered set $(T, \leq_T)$. The first option would yield $t^*(\alpha) = a(\alpha) \in N(\alpha)$ and hence that $t^*(\alpha)$ is the maximum element of $N(\alpha)$ in the ordering chosen for $N(\alpha)$, and in Case 5 that cannot happen. Hence t^* and $a(\alpha)$ are not comparable in $(T, <_T)$. Let $\delta = \Delta(a(\alpha), t^*)$. Then in the node M that contains both $a(\alpha)(\delta) = a(\delta)$ and $t^*(\delta)$ we have $a(\delta) <_M t^*(\delta)$. Because $\Delta(a(\alpha), t^*) = \Delta(a, t^*)$ we obtain $a \prec t^*$ as claimed.

At this point in Case 5, we know that $a \preceq t^*$ for each $a \in A$. To complete the proof of Case 5, recall that $N(\alpha) \cap A = N(\alpha)$ and suppose $t' \prec t^* = \sup_{(T,\preceq)}(N(\alpha) \cap A)$. Then there is some $a' \in N(\alpha) \cap A$ with $t' \prec a' \preceq t^*$ and that is enough to show that $t^* = \sup_T(A)$. Therefore in Case 5, the set A has a supremum (namely t^*) and the induction stops.

Case 6: Suppose $\alpha = \beta + 1$ and $\emptyset \neq N(\beta+1) \cap A \neq N(\beta+1)$. Choose $u \in N(\beta+1) - A$. Because A is an initial segment of $(T, \preceq)$ it must be true that $a \prec u$ for each $a \in A$. Then $N(\beta+1) \cap A$ is a non-empty bounded set in $N(\beta+1)$ so that (C3) provides a point $s = \sup_{N(\beta+1)}(N(\beta+1) \cap A)$. In case $s \notin A$, then part (1) of Lemma 3.4 shows that $s = \sup_T(A)$. If $s \in A$ then we define $s_{\beta+1} = s$. Clearly the first two parts of $(IH)_{\alpha+1}$ are satisfied, so we verify the third part. Let $a \in A$ have $\mathrm{lv}(a) \leq \beta+1$. If $\mathrm{lv}(a) \leq \beta$, then the induction hypothesis gives $a \preceq s_\beta \prec s_{\beta+1}$ so suppose $\mathrm{lv}(a) = \beta + 1$. Then the induction hypothesis gives $s_\beta <_T a$ so that $a \in A \cap N(\beta + 1)$. But then $a \leq_{N(\beta+1)} \sup_{N(\beta+1)}(A \cap N(\beta + 1)) = s_{\beta+1}$ and therefore $a \preceq s_{\beta+1}$ as required.

Cases 3 through 6 show that in case $\alpha = \beta + 1$ is a successor ordinal and $(IH)_\alpha$ holds, then either we can construct the supremum of A in $(T, \preceq)$, or else the induction continues and $(IH)_{\alpha+1}$ holds. It remains to consider the case where α is a limit ordinal and $(IH)_\alpha$ holds. Let $S = \{t \in T :$ for some $\beta < \alpha,\ t \leq s_\beta\}$. Then S is a linearly ordered subset of $(T, \leq_T)$ with the property that $t <_T s \in S$ guarantees that $t \in S$. The set S might, or might not, be a branch of T and that leads to our next cases.

Case 7: Suppose α is a limit ordinal and S is a branch of T. Then $\mathrm{ht}(S) = \alpha$. Because α is a limit ordinal, S does not have a maximal element. Apply (C4) to find a supremum s^* for S in $(T, \preceq)$. Observe that there cannot be an $a \in A$ such that $s_\beta <_T a$ for each $\beta \in [\eta, \alpha)$ because in that case, S would not be a branch of T. However, for a fixed $a \in A$ and a fixed β we know that either $a \preceq s_\beta$ or else $s_\beta <_T a$ by the last part of the induction hypothesis. Therefore, given $a \in A$ some s_β has $a \preceq s_\beta \preceq s^*$ showing that s^* is an upper bound for A in $(T, \preceq)$. But $s_\beta \in A$ for $\eta \leq \beta < \alpha$ and s^* is the supremum in $(T, \preceq)$ of the set $\{s_\beta : \eta \leq \beta < \alpha\}$. Hence $s^* = \sup_{(T,\preceq)}(A)$.

Case 8: Suppose α is a limit ordinal and S is not a branch of T, and the node $N(S)$ of immediate successors of S has $N(S) \cap A = \emptyset$. Because S is not a branch of T, we know that $N(S) \neq \emptyset$. Because $N(S)$ is a nonempty node at a limit level of T, (C2) guarantees the existence of a least element s^* of $N(S)$ with respect to the linear ordering $<_{N(S)}$ chosen for $N(S)$. Note that $s_\beta <_T s^*$ for each $\beta < \alpha$. We claim that s^* is the supremum for A in $(T, \preceq)$. If there were some $a \in A$ with $s_\beta <_T a$ for each $\beta \in [\eta, \alpha)$ then $a \in N(S) \cap A = \emptyset$. Therefore, the final part of $(IH)_\alpha$ shows that the points s_β of A are cofinal in A so that s^* is an upper bound for the set A. We claim that s^* is the least upper bound for A in $(T, \preceq)$.

For contradiction, suppose that some $t \in T$ has $a \preceq t \prec s^*$ for each $a \in A$. Then $s_\beta \preceq t \prec s^*$ for each $\beta \in [\eta, \alpha)$. The points t and s^* must be incomparable in the partially ordered set $(T, \leq_T)$, because otherwise $t <_T s^*$ so that $t \in S$ and then we could choose an $s_\beta \in S$ with $t <_T s_\beta$ and that would give $t <_T s_\beta \preceq t$ which is impossible. Compute $\delta = \Delta_T(s^*, t) \leq \mathrm{lv}(s^*) = \alpha$. If $\gamma < \delta$ we have $t(\gamma) = s^*(\gamma)$, and in the node M of T that contains both $t(\delta)$ and $s^*(\delta)$ we have $t(\delta) <_M s^*(\delta)$. Because s^* is the least member of the node $N(S)$ we know that $\delta < \alpha$. But then $s^*(\delta) \in S$ so we can choose some s_β with $s^*(\delta) < s_\beta$. Then $s_\beta(\delta) = s^*(\delta)$ so that $t(\delta) <_M s_\beta(\delta)$. Furthermore, if $\gamma < \delta$, then $t(\gamma) = s^*(\gamma) = s_\beta(\gamma)$. That is enough to show that $t \prec s_\beta$. But we know that $s_\beta \preceq t$ so that $t \prec t$ which is impossible. Therefore, s^* is the least upper bound for A in $(T, \preceq)$, as claimed.

Case 9: Suppose α is a limit ordinal and S is not a branch of T and $N(S)$, the node of immediate successors of S, has $\emptyset \neq N(S) \cap A = N(S)$, and $N(S)$ has a largest element in the linear ordering $<_{N(S)}$ chosen for it. Let s_α be that largest element. Then $s_\alpha \in A \cap N(S)$ and $s_\alpha = \sup_{N(S)}(N(S))$ so that the first two parts of $(IH)_{\alpha+1}$ are satisfied. Observe that because $\emptyset \neq A \cap N(S) \subseteq N(S)$ the point s_α must also be the supremum of $A \cap N(S)$ in the set $(T, \preceq)$.

We now verify the third part of $(IH)_{\alpha+1}$. Suppose $a \in A$ has $\mathrm{lv}(a) \leq \alpha$. In the light of Lemma 3.3 we need to show $a \preceq s_\alpha$. If $\mathrm{lv}(a) < \alpha$ then there must be some β with $\mathrm{lv}(a) \leq \beta < \alpha$ so that $(IH)_\alpha$ tells us that either $a \preceq s_\beta$ or else $s_\beta <_T a$. Because $\mathrm{lv}(a) \leq \beta$, the second option cannot occur, so we have $a \preceq s_\beta <_T s_\alpha$ and hence $a \prec s_\alpha$. If $\mathrm{lv}(a) = \alpha$. Then for each $\beta < \alpha$, the last part of $(IH)_\alpha$ yields $s_\beta <_T a$ so that $a \in N(S) \cap A$. But then $a \preceq \sup_T(N(S) \cap a) = s_\alpha$, as required. Hence $(IH)_{\alpha+1}$ holds and the induction continues.

Case 10: Suppose α is a limit ordinal and S is not a branch of T and $N(S)$, the node of immediate successors of S, has $\emptyset \neq N(S) \cap A = N(S)$, and the node $N(S)$ has no largest element in the linear ordering $<_{N(S)}$ chosen for it. Nevertheless, $N(S)$ has a supremum s^* in $(T, \preceq)$ according to (C1). We claim that s^* must be the supremum of A in $(T, \preceq)$. Because s^* is the supremum in $(T, \preceq)$ of the nonempty subset $N(S)$ of A, in order to show that $s^* = \sup_{(T,\preceq)}(A)$ it will be enough to show that $a \preceq s^*$ for each $a \in A$.

We claim that for some $\beta < \alpha$, the points s^* and s_β are incomparable in $\leq_T$. Otherwise $\mathrm{lv}(s^*) \geq \alpha$ and $s^*(\alpha) \in N(S)$ would be the maximum element of $S(N)$, and in Case 10 there is no such maximum element. Hence there is a $\beta < \alpha$ such that s_β and s^* are incomparable in $\leq_T$.

Compute $\delta = \Delta_T(s_\beta, s^*)$. Then $\delta \leq \beta < \alpha$ and if $\gamma < \delta$ then $s_\beta(\gamma) = s^*(\gamma)$ while $s_\beta(\delta) <_M s^*(\delta)$ in the node M of T that contains both $s_\beta(\delta)$ and $s^*(\delta)$.

Now consider any $a \in A$. If $a \preceq s_\beta$ then for any $a_0 \in N(S) \cap A$ we have $a \preceq s_\beta <_T a_0 \preceq s^*$ whence $a \preceq s^*$. Hence suppose $a \preceq s_\beta$ is false. Then by the last part of $(IH)_\alpha$ we know that $s_\beta <_T a$. Because s_β and s^* are incomparable in the partial order $\leq_T$, so are a and s^*. Furthermore, $a(\gamma) = s_\beta(\gamma) = s^*(\gamma)$ whenever $\gamma < \delta$ and $a(\delta) = s_\beta(\delta) <_M s^*(\delta)$. But that is enough to show that $a \prec s^*$ as required. Hence s^* is the supremum of A in $(T, \preceq)$, as claimed.

Case 11: Suppose α is a limit ordinal and S is not a branch of T, and the node $N(S)$ of immediate successors of S, has $\emptyset \neq N(S) \cap A \neq N(S)$. Write $N = N(S)$ and $<_N$ for $<_{N(S)}$. Choose $v \in N - A$. Because A is an initial segment of $(T, \preceq)$, v is an upper bound for the non-empty set $N \cap A$ in $N(S)$. According to (C3), there is a point $u = \sup_{<_N}(N \cap A)$. If $u \notin A$, then Lemma 3.4 shows that u is the desired supremum of A in $(T, \preceq)$. If $u \in A$, then we define $s_\alpha = u$. Because $s_\alpha \in N(S)$ we know that the first part of $(IH)_{\alpha+1}$ is satisfied, and the second part holds by construction of s_α. It remains to verify the third part, i.e., that for each $a \in A$, either $a \preceq s_\alpha$ or else $s_\alpha <_T a$.

Let $a \in A$. Suppose $a \preceq s_\alpha$ is false. Then for each $\beta < \alpha$, $a \preceq s_\beta$ is false. According to the induction hypothesis, $s_\beta <_T a$ must hold for every $\beta < \alpha$. Therefore $\mathrm{lv}(a) \geq \alpha$ so that $a(\alpha)$ is defined and $a(\alpha) \in N(S) \cap A$. Therefore $a(\alpha) \leq_N s_\alpha$. However, it cannot happen that $a(\alpha) <_N s_\alpha$ because that would yield $a \preceq s_\alpha$, so we must have $a(\alpha) = s_\alpha$. But then $s_\alpha = a$ or else $s_\alpha <_T a$, as required.

Let us summarize what has happened so far: either at some stage $\alpha < \mathrm{ht}(T)$ we have found a point of T that is the supremum of A in $(T, \preceq)$ or else we have constructed a set $B = \{s_\alpha : \alpha < \mathrm{ht}(T)\}$ of points that satisfy $(IH)_\alpha$ for each $\alpha < \mathrm{ht}(T)$. The set B is cofinal in a branch $b^* = \{t \in T :$ for some $\alpha < \mathrm{ht}(T),\ t \leq_T s_\alpha\}$.

Apply (C4) to the branch b^*. If $\mathrm{ht}(b^*)$ is a successor ordinal, then b^* has a maximal element $s_\alpha \in A$. Then s_α is also maximal in the partial order $\leq_T$ and has the property that for every $a \in A$, either $a \preceq s_\alpha$ or else $s_\alpha <_T a$. But the second option cannot happen because s_α is maximal in T, so we see that s_α is the supremum (actually, the maximum) of A in $(T, \preceq)$. Hence assume that b^* has limit height. According to (C4) there is a supremum $s(b^*)$ in $(T, \preceq)$ for the subset b^* of T and there is an ordinal $\mu < \mathrm{ht}(b^*)$ and if $t \in b^*$ has $\mu < \mathrm{lv}(t)$ then t is the maximum element of the node to which it belongs, and $s(b^*)$ is the immediate successor of $b^*(\mu)$ in the node to which $b^*(\mu)$ belongs (where $b^*(\mu)$ is the unique point of $b* \cap T_\mu$.) But then we see that $s(b^*)$ is the supremum of A in $(T, \preceq)$, because $s(b^*) \notin A$.

We have now completed the proof that conditions (C1) through (C4) are sufficient for $(T, \preceq)$ to be order-complete. It remains to verify necessity. Suppose $(T, \preceq)$ is known to be order complete. Then every subset of T has both a supremum and an infimum in $(T, \preceq)$ so that C1 and C4 are automatic.

To verify C2, suppose $N_0 \neq \emptyset$ is a node of T at a limit level λ. Then there is a point $x_0 \in T$ satisfying $x_0 = \inf_T(N_0)$. If $x_0 \in N_0$ we have our minimum point for $(N_0, <_{N_0})$, so assume $x_0 \notin N_0$.

Fix $y_0 \in N_0$. Then $x_0 \prec y_0$. If x_0 and y_0 were comparable in the partial order $\leq_T$, then $x_0 <_T y_0$ so x_0 is a strict predecessor of y_0 in T. Because λ is a limit, there would be some $\alpha < \lambda$ with $x_0 <_T y_0(\alpha) <_T y_0$. But all points of the node N_0 have the same strict predecessors in $(T, \leq_T)$ and so $y_0(\alpha) <_T y$ for each $y \in N_0$ showing that $y_0(\alpha) \prec y$ for each $y \in N_0$. But that is impossible because $x_0 \prec y_0(\alpha)$ and $x_0 = \inf_{(T,\preceq)}(N_0)$. Therefore, the points y_0 and x_0 are incomparable in the partial order $\leq_T$.

Compute $\delta_0 = \Delta_T(y_0, x_0)$. Then $\delta_0 \leq \mathrm{lv}(y_0) = \lambda$ and in the node M of T that contains both $y_0(\delta_0)$ and $x_0(\delta_0)$ we have $x_0(\delta_0) <_M y_0(\delta_0)$. If $\delta_0 < \lambda$, then because all members of N_0 have the same strict predecessors, we know that $y(\delta_0) = y_0(\delta_0)$ for all $y \in N_0$. But then $x_0 \prec y_0(\delta_0) \prec y$ for all $y \in N_0$ and that is impossible because $x_0 = \inf_T(N_0)$. Therefore, $\delta_0 < \lambda$ is impossible, so we have $\delta_0 = \lambda$. From $\lambda = \delta_0 = \Delta_T(y_0, x_0) \leq \mathrm{lv}(x_0)$ we know that $x_0(\lambda)$ is defined and belongs to the same node of T that contains $y_0(\lambda) = y_0$, i.e. the node N_0, and hence $x_0(\lambda) \in N_0$. Let $y \in N_0$. If $y <_{N_0} x_0(\lambda)$ then $y \prec x_0$ which is impossible because $x_0 = \inf_T(N_0)$ and $y \in N_0$. Therefore, $x_0(\lambda) \leq_{N_0} y$ for each $y \in N_0$ and that is enough to show that $(N_0, <_{N_0})$ has a minimum element, as required in C2.

Finally we verify that C3 holds provided $(T, \preceq)$ is order-complete. Let N be any node of T. Let α be the level of N and suppose $\emptyset \neq B \subseteq N$ is bounded above in $(N, <_N)$ by $v_0 \in N$. Choose any $b_0 \in B$. For contradiction, suppose

$$\sup_{(N,<_N)}(B) \text{ does not exist.} \qquad (****)$$

Because $(T, \preceq)$ is order complete, there is a point $u_0 \in T$ with $u_0 = \sup_{(T,\preceq)}(B)$. Then in the lexicographic order $\preceq$ of T, $b_0 \preceq u_0 \preceq v_0$.

Claim 1: $u_0 \notin N$ because if $u_0 \in N$ then u_0 would be the supremum in $(N, <_N)$ of B, contrary to $(****)$.

Claim 2: No $x_0 \in N$ can have $x_0 \leq_T u_0$. For if such an $x_0 \in N$ existed, then by Claim 1 $\mathrm{lv}(x_0) = \alpha < \mathrm{lv}(u_0)$. But then $x_0 \prec u_0$ and for each $b \in B$ with $b \neq x_0$, if $x_0 <_N b$ then $u_0 \prec b$ contrary to $u_0 = \sup_{(T,\prec)}(B)$. But then for each $b \in B$ we have $b \preceq x_0 \prec u_0$ and that is impossible because u_0 is the supremum of B in $(T, \prec)$. This establishes Claim 2.

Claim 3: No $x_1 \in N$ can have $u_0 <_T x_1$ because all members of N, including $b_0 \in B$, have exactly the same predecessors in $(T, \leq_T)$ as does x_1, and that would force $u_0 <_T b_0$, contrary to the fact that u_0 is the supremum of B in $(T, \preceq)$. Hence Claim 3 holds.

Both u_0 and v_0 are upper bounds for B in $(T, \preceq)$ so that because $\sup_{(T,\preceq)}(B) = u_0 \neq v_0$ we must have $u_0 \prec v_0$. Claims 2 and 3 show that u_0 and v_0 must be incomparable elements of the tree $(T, \leq_)$ so that, if we compute $\delta = \Delta_T(u_0, v_0)$ we have $\delta \leq \min(\mathrm{lv}(u_0), \mathrm{lv}(v_0))$ and $u_0(\delta) <_M v_0(\delta)$ where M is the node of T that contains both $u_0(\delta)$ and $v_0(\delta)$.

Claim 4: $\delta < \mathrm{lv}(v_0)$. If not, then $\delta = \mathrm{lv}(v_0)$ so that $v_0(\delta) = v_0$ and the node M of T containing $v_0(\delta)$ and $u_0(\delta)$ must be identical with N. But then some member of N, namely $u_0(\delta)$ is a predecessor of u_0 in the tree $(T, \leq_T)$ and that is impossible in the light of Claim 3. Hence Claim 4 holds.

Claim 5: $\delta < \mathrm{lv}(u_0)$ is impossible, because if $\delta = \mathrm{lv}(u_0)$ then $u_0 = u_0(\delta) <_M v_0(\delta)$ in the node M of T that contains both $u_0(\delta)$ and $v_0(\delta)$. But b_0 and v_0 belong to the same node N of T and therefore have exactly the same strict predecessors. By Claim 4, $\delta < \mathrm{lv}(v_0)$ so that $u_0 = u_0(\delta) <_M v_0(\delta) = b_0(\delta)$. Because u_0 and b_0 are incomparable in T, that inequality in M yields $u_0 \prec b_0$ contrary to $u_0 = \sup_{(T,\prec)}(B)$. Hence Claim 5 holds.

At this stage, we know that $\delta < \min(\mathrm{lv}(u_0, \mathrm{lv}(v_0))$ and by Claims 2 and 3, we know that b_0 and u_0 are incomparable in $(T, \leq_T)$. Furthermore, we know that v_0 and b_0 have exactly the same strict predecessors in $(T, \leq_T)$ and that gives $u_0(\delta) <_M v_0(\delta) = b_0(\delta)$ from which we conclude that $u_0 \prec b_0$, contrary to $u_0 = \sup_{(T,\prec)}(B)$. Therefore, $(* * **)$ cannot hold, so that property C3 is established. $\square$

Example 3.6 *: Theorem 3.5 gives an easy way to describe ordered compactifications of lexicographic trees with order complete nodes by adding suprema to branches of the tree.*

Construction: Let $(T, \leq_T)$ be a tree whose node orderings are complete. Let $\mathcal{B}_0$ be the set of all branches of T that have limit height (i.e., that have no supremum in the partial order $\leq_T$). Consider the disjoint union $T^* = T \cup \mathcal{B}_0$. The partial ordering $\leq^*$ of T^* is an extension of $\leq_T$, with all additional relations defined as follows. If $t \in T$ and $b \in \mathcal{B}_0$ then we define $t \leq^* b$ if and only if $t \in b$. For $t \in T$, the node of T^* to which t belongs is exactly the same as the node of T to which t belongs, and for $b \in \mathcal{B}_0$ the node of T^* containing b is a singleton. Hence every node of T^* is order complete, and it is clear that each branch of T^* has a maximum point. Therefore T^* satisfies C1 through C4 of Theorem 3.5 so that in its lexicographic ordering, T^* is order complete. $\square$

We now turn our attention from compactness to Baire category. In our next theorem we give necessary and sufficient conditions for a broad class of lexicographically ordered trees (namely, the splitting trees) to be of the first Baire category when equipped with their open interval topologies. Recall that a topological space is of the *first Baire category* if and only if it is the union of countably many closed nowhere dense subsets. The corresponding tree property is that the tree T is *semi-special*, i.e., there are countably many anti-chains A_n in T such that for each $t \in T$, there is an $n \geq 1$ and some $a \in A_n$ such that $t \leq_T a$. We chose that name because if it happens that $T = \bigcup\{A_n : n \geq 1\}$, then T is said to be a *special tree*. Being special is a property of trees that appears frequently in the literature.

Lemma 3.7 *: Suppose that $(T, \leq_T)$ is a splitting tree, and that no node N of T at a non-limit level has a least element in its chosen linear ordering $(N, <_N)$. Let $\preceq$ be the associated lexicographic ordering and let $\mathcal{T}$ be the open interval topology associated with $\preceq$. For any closed nowhere dense set D of $(T, \mathcal{T})$ there is an anti-chain A of T with the property that for each $d \in D$ some $a(d) \in A$ has $d \leq_T a(d)$.*

Proof: Recall that for any $t \in T$, the set $T^t = \{s \in T : t \leq_T s\}$ is a convex subset of $(T, \preceq)$ whose minimum point in $\preceq$ is t. Because T is a splitting tree, each T^t has at least three points and $T^t - \{t\}$ is a non-void convex open subset of $(T, \preceq)$.

Fix $d \in D$. We claim that some $e(d) \in T$ has $d \leq_T e(d)$ and $T^{e(d)} \cap D \subseteq \{e(d)\}$. (Notice that this allows the case where $T^{e(d)} \cap D = \emptyset$.) If that is not the case, fix any $t \in T^d - \{d\}$ and any pair $u \prec v$ with $t \in (u, v)_{\preceq} \subseteq T^d$. Then by Lemma 2.5 there is a point c with $u <_T c \leq_T v$ and some $x \in \mathrm{Node}(c)$ with $x <_{\mathrm{Node}(c)} c$ such that $T^x \subseteq (u, v)_{\preceq}$. Note that $x \in T^d$ and therefore must have $T^x \cap D \not\subseteq \{x\}$. Therefore $\emptyset \neq T^x \cap D \subseteq (u, v)_{\preceq}$ showing that every point of the nonempty open set $T^d - \{d\}$ is a limit point of D. But that is impossible because D is closed and nowhere dense. Hence some $e(d) \in T$ has $d \leq e(d)$ and $T^{e(d)} \cap D \subseteq \{e(d)\}$.

Let $E = \{e(d) : d \in D\}$ and let A be the set of minimal elements of E (in the partial order $\leq_T$). Then A is an anti-chain of $(T, \leq_T)$. Now fix any $d \in D$ and its associated $e(d) \in E$. If $e(d) \in A$ the proof is complete. If $e(d) \notin A$ then there is some $e(d') \in A$ with $d' \in D$ and $e(d') <_T e(d)$. Then the elements $d, d', e(d')$ are all predecessors of $e(d)$ so that d and $e(d')$ are comparable. If it were true that $e(d') <_T d$, then $d \in T^{e(d')} \cap D \subseteq \{e(d')\}$ and this is impossible. Therefore $d \leq_T e(d') \in A$ as claimed. $\square$

Theorem 3.8 *: Suppose T is an ω- splitting tree whose nodes are ordered in such a way that no node at a non-limit level has a first point in its chosen linear ordering. Then in the open interval topology of $\preceq$, T is of the first Baire category if and only if T is semi-special.*

Proof: First suppose that the tree T is semi-special. Then we have a sequence A_i of anti-chains with the property that for each $t \in T$ there is some i and some $a \in A_i$ with $t \leq_T a$. Without loss of generality we may assume that each A_n is a maximal anti-chain. For each $i \geq 1$ let $B_i = \{t \in T :$ for some $a \in A_i,\ t \leq_T a\}$. Let $\mathcal{T}$ be the usual open interval topology of $\preceq$.

We claim that each B_i is $\mathcal{T}$-closed. Let $t \in T - B_i$. Because A_i is maximal, some $a_t \in A_i$ is comparable to t, and because $t \notin B_i$ it cannot happen that $t \leq_T a_t$. Hence $a_t <_T t$. Because no member of T is maximal, there is some $b \in T$ with $t <_T b$. Then $(a_t, b)_\preceq$ is a $\mathcal{T}$-open neighborhood of t and we claim that $(a_t, b)_\preceq \cap B_i = \emptyset$. If that is not true then let $y \in (a_t, b)_\preceq$ with $y \in B_i$. As noted in the proof of Lemma 2.5, $a_t <_T b$ forces $a_t <_T y$. Because $y \in B_i$ there is some $a_y \in A_i$ with $y \leq_T a_y$. But then $a_t <_T y \leq_T a_y$ so that $a_t <_T a_y$ and that is impossible because A_i is an anti-chain. Therefore $(a_t, b) \cap B_i = \emptyset$. Hence B_i is closed.

We claim that B_i is nowhere dense in $(T, \mathcal{T})$. It will be enough to show that B_i contains no non-empty open intervals of $\preceq$. Suppose $x \preceq z$ and suppose that the nonempty open interval $(x, z)_\preceq$ is contained in B_i. Either x and z are incomparable in the partially ordered set $(T, \leq_T)$ or else $x <_T z$. In the latter case, because T is a splitting tree and nodes at non-limit levels have no first points, there is an immediate successor u of x with $u \leq_T z$ and a predecessor w of u in the ordering of $Node(u)$ such that the nonempty set T^w is contained in $(x, z)_\preceq$. Replacing x with w if necessary, we may assume that x and z are incomparable in the partial order $\leq_T$. Note that $\emptyset \neq T^x - \{x\} \subseteq (x, z)_\preceq \subseteq B_i$.

Because A_i is a maximal anti-chain, there is some $a_x \in A_i$ that is comparable to x in the partial order $\leq_T$. We claim that $a_x \leq_T x$ is not possible. For if $a_x \leq_T x$, then because x is not maximal in $(T, \leq_T)$ we may choose a point $y \in T$ with $x <_T y$. Then $y \in (x, z)_\prec \subseteq B_i$ and so some $a_y \in A_i$ has $y \leq_T a_y$ and therefore we would have $a_x <_T x <_T y \leq_T a_y$, something that is impossible because A_i is an anti-chain. Therefore we have $x <_T a_x$ Because no member of T is maximal in $\leq_T$ we may choose $w \in T$ with $a_x <_T w$ Then $w \in T^{a_x} \subseteq T^x \subseteq B_i$ so that some $a_w \in A_i$ has $w \leq_T a_w$. But then we have $a_x <_T w <_T a_w$ and that is impossible in the anti-chain A_i. Therefore, B_i contains no nonempty interval $(x, z)_\preceq$ and hence B_i is closed and nowhere dense in $(T, \mathcal{T})$.

Because T is semi-special, we see that $T = \bigcup\{B_i : i \geq 1\}$ and therefore $(T, \mathcal{T})$ is of the first Baire category.

Conversely, suppose that $T = \bigcup\{D_i : i \geq 1\}$ where each D_i is a closed, nowhere dense subset of $(T, \mathcal{T})$. for each $i \geq 1$ apply Lemma 3.7 to find an anti-chain A_i of $(T, \leq_T)$ such that for any $d \in D_i$, some $a(d, i) \in A_i$ has $d \leq a(d, i)$. Therefore, $(T, \leq_T)$ is semi-special. □

Recall that a *Souslin tree* is a tree of height ω_1 such that each anti-chain is countable. Whether such tree exist is undecidable in ZFC.

Corollary 3.9 *: Let $(T, \leq_T)$ be a splitting Souslin tree such that no node N of T at a non-limit level has a least element in its chosen linear ordering $<_N$. Then in the open interval topology $\mathcal{T}$ of the lexicographic ordering $\preceq$, the space $(T, \mathcal{T})$ is not of the first Baire category.*

Proof: Let $\mathcal{T}$ be the open interval topology of $\preceq$. If $(T, \mathcal{T})$ were of the first category, then each of the anti-chains A_i found in the proof of Lemma 3.7 would be countable. For each $t \in T$, the set of predecessors of t is countable (because Souslin trees have no uncountable branches) and hence $\{t \in T : t \leq_T a$ for some $a \in A_i\}$ is countable. Hence so is T and that is impossible. □

4 Aronszajn lines and trees

The results in this section are part of the folklore of the subject, but we have not been able to find a proof of the hard part of Theorem 4.1 in the literature. Furthermore, we needed some of this material in the proof of Theorem 2.6.

An *Aronszajn tree* is a tree $(T, \leq_T)$ of height ω_1 that has countable levels and countable branches. Aronszajn trees exist in ZFC [4]. By an *Aronszajn line* we mean a linearly ordered set $(X, <)$ that has cardinality ω_1, contains no order isomorphic copy of ω_1 or of ω_1^*, and contains no order isomorphic copy of any uncountable set of real numbers. Aronszajn lines also exist in ZFC.

It is important to understand that being an Aronszajn line is an order-theoretic issue, and not a topological property. It is easy to show that if $(X, <)$ is an Aronszajn line, then so is the lexicographic product $Y = X \times \mathbb{Z}$ and the latter set, when endowed with its open interval topology, is a discrete metric space. Other Aronszajn lines are certainly not metrizable. Aronszajn lines and trees are intimately linked, as our next result shows. The result is known, but we have not been able to find a proof of half of it in the literature.

Theorem 4.1 *: Every Aronszajn line is order isomorphic to a lexicographically ordered Aronszajn tree, and any lexicographic ordering of an Aronszajn tree is an Aronszajn line.*

Proof: The proof that any lexicographic ordering of an Aronszajn tree gives an Aronszajn line appears in [4]. We have not been able to find the converse in the literature.

For the converse, let $(L, <_L)$ be any Aronszajn line. By recursion over $\alpha < \omega_1$, we will define two related families $\{L(\alpha) : \alpha < \omega_1\}$ and $\{\mathcal{U}(\alpha) : \alpha < \omega_1\}$ and in the end the desired tree T will be $T = \bigcup\{L(\alpha) : \alpha < \omega_1\}$.

As a set, T will coincide with L, so we cannot use a partition tree construction. Instead we begin with a standard way to choose cofinal and coinitial subsets of convex subsets of L. For any singleton set I, let $S(I)$ be the unique point of I. For any non-degenerate convex subset $I \subset L$ we know that $\mathrm{cf}(I)$ is either finite (in which case I has a right end point) or else $\mathrm{cf}(I) = \omega$ because L contains no copy of ω_1. An analogous assertion holds for coinitialities. Therefore we can find a subset $S(I) \subseteq I$ that is both coinitial and cofinal in I, and is an order-copy of $\{0, 1\},\ \omega,\ \omega^*$ or $\omega^* + \omega$ when ordered using $<_L$. (We will later use these linear orderings of $S(I)$ as node orderings for a tree.) We may assume that if I is an infinite set, then for any distinct $x, y \in S(I)$, some point of I lies strictly between x and y.

In the following recursive construction it will be convenient to think of partial orders as being sets of ordered pairs. To initialize our recursion, we let $\mathcal{U}(0) = \{L\}$ and $L(0) = S(L)$. Define the partial order $\leq_0$ on $L(0)$ to be equality. Let $\mathcal{U}(1)$ be the collection of all convex components of $L - L(0)$. Now let $L(1) = \bigcup\{S(I) : I \in \mathcal{U}(1)\}$. We will say that an ordered pair (x, y) is 1-acceptable if $x \in L(0)$ and $y \in L(1)$ and if J is the unique member of $\mathcal{U}(1)$ containing y, then $x <_L J$ (meaning that x precedes every point of J in the linear ordering of L) and $\{x\} \cup J$ is a convex subset of L. Now define a partial order on $T(1) = L(0) \cup L(1)$ by the rule that

$$\leq_1 = \leq_0 \cup \{(z, z) : z \in L(1)\} \cup \{(x, y) : (x, y) \text{ is a } 1-\text{acceptable pair}\}.$$

For our induction hypothesis suppose that $\alpha < \omega_1$ and that the following is satisfied:

$(IH)_\alpha$: for each $\beta < \alpha$ we have

1) $\mathcal{U}(\beta)$ is the family of all convex components of the set $L - \bigcup\{L(\gamma) : \gamma < \beta\}$;

2) $L(\beta) = \bigcup\{S(I) : I \in \mathcal{U}(\beta)\} \subseteq \bigcup\mathcal{U}(\beta) \subseteq L - \bigcup\{L(\gamma) : \gamma < \beta\}$;

3) $\leq_\beta$ is a partial order on the set $T(\beta) = \bigcup\{L(\gamma) : \gamma \leq \beta\}$ and $\leq_\gamma \subseteq \leq_\beta$ whenever $\gamma < \beta$;

4) $\leq_\beta = \bigcup\{\leq_\gamma : \gamma < \beta\} \cup \{(z,z) : z \in L(\beta)\} \cup \{(x,y) : (x,y) \text{ is a } \beta\text{−acceptable pair}\}$ where a pair (x,y) is said to be β-acceptable if and only if $y \in L(\beta)$ and $x \in L(\gamma)$ for some $\gamma < \beta$ and if J is the unique member of $\mathcal{U}(\gamma+1)$ that contains y, then $x <_L J$ and $\{x\} \cup J$ is convex in $(L, <_L)$.

Given $(IH)_\alpha$ we define $\mathcal{U}(\alpha)$ to be the family of all convex components of $L - \bigcup\{L(\beta) : \beta < \alpha\}$ and $L(\alpha) = \bigcup\{S(I) : I \in \mathcal{U}(\alpha)\}$. We define $\leq_\alpha = \bigcup\{\leq_\beta : \beta < \alpha\} \cup \{(z,z) : z \in L(\alpha)\} \cup \{(x,y) : (x,y)$ is an α-acceptable pair$\}$ where an α-acceptable pair is defined to fit the pattern in part (4) of the recursion hypothesis. Clearly $(IH)_{\alpha+1}$ holds and the recursion continues. Let $T = \bigcup\{L(\alpha) : \alpha < \omega_1\}$ and let $\leq_T = \bigcup\{\leq_\alpha : \alpha < \omega_1\}$. Then $(T, \leq_T)$ is a tree.

Claim 1: $L = \bigcup\{L(\alpha) : \alpha < \omega_1\}$. It is enough to verify that $L \subseteq \bigcup\{L(\alpha) : \alpha < \omega_1\}$. Let $x \in L$. If $x \notin L(\alpha)$ for each $\alpha < \omega_1$, then for each α there is a member $I(\alpha) \in \mathcal{U}(\alpha)$ with $x \in I(\alpha)$. But then $\{I(\alpha) : \alpha < \omega\}$ is a strictly decreasing collection of convex subsets of L and that allows us to find an order copy of either ω_1 or ω_1^* in L, which is impossible. Hence Claim 1 holds.

Claim 2: Each branch of $(T, \leq_T)$ is countable because otherwise $(L, <_L)$ would contain an order copy of ω_1.

Claim 3: Each level of T is countable. The levels of T are the sets $L(\alpha)$. If Claim 3 is false, let $L(\alpha)$ be the first uncountable level of T. Because $T(\alpha) = \bigcup\{S(I) : I \in \mathcal{U}(\alpha)\}$ and each $S(I)$ is countable, the collection $\mathcal{U}(\alpha)$ must be uncountable. For each $I \in \mathcal{U}(\alpha)$ choose a point $p(I) \in I$. Let $D = \bigcup\{L(\beta) : \beta < \alpha\}$. Minimality of α insures that D is countable. Let $M = D \cup \{p(I) : I \in \mathcal{U}(\alpha)\}$. Then D is a countable order-dense subset of M, so that M is order isomorphic to some uncountable subset of $\mathbb{R}$. But that is impossible because $M \subseteq L$ and L contains no order isomorphic copies of any uncountable subset of $\mathbb{R}$. Hence Claim 3 holds.

Claims 1,2, and 3 combine to prove that $(T, \leq_T)$ is an Aronszajn tree. Let $\preceq$ be the lexicographic ordering of T associated with the node orderings given by the chosen sets $S(I)$. We claim that the function $f : (L, <_L) \to (T, \preceq)$ given by $f(x) = x$ is an order isomorphism. It is enough to show that if $x <_L y$ in L then $x \prec y$ in T. For contradiction, suppose $x \prec y$ is false. Because $x \neq y$ it follows that $y \prec x$. This can happen in two different ways, depending upon whether y and x are comparable in the partial order $\leq_T$. In case y and x are comparable in $\leq$, then $y \prec x$ forces $y <_T x$. Then there are ordinals $\beta < \alpha$ such that $y \in L(\beta)$ and $x \in L(\alpha)$, and if J is the unique member of $\mathcal{U}(\beta+1)$ that contains x, then $y <_L J$ and $\{y\} \cup J$ is convex in L. But $y <_L J$ and $x \in J$ give $y <_L x$ and that is not true. Hence x and y must be incomparable in the partial order of T. Therefore we compute $\delta = \Delta_T(x,y) \leq \min(\mathrm{lv}(x), \mathrm{lv}(y))$ and we know that $y(\delta) <_M x(\delta)$ in the node M of T that contains both $x(\delta)$ and $y(\delta)$. Then in $(L, <_L)$ we have $y(\delta) <_L x(\delta)$. There are several possibilities to consider. In the first $\delta < \min(\mathrm{lv}(x), \mathrm{lv}(y))$. Then there are unique members J_y and J_x of $\mathcal{U}(\delta+1)$ with $y \in J_y$ and $x \in J_x,\ y(\delta) <_L J_y$ and $x(\delta) <_L J_x$, and having both of the sets $\{y\} \cup J_y$ and $\{x\} \cup J_x$ convex in L. Then $J_y = J_x$ is impossible because the left endpoints of the convex sets $\{x\} \cup J_x$ and $\{y\} \cup J_y$ are different. Because $x \in J_x,\ y \in J_y$ and $x <_L y$ we know that $J_x <_L J_y$. Therefore $x(\delta) <_L y(\delta)$ as required to show that $x \prec y$. The next case is where $\delta = \mathrm{lv}(x) < \mathrm{lv}(y)$. Then $x(\delta) = x$ and $y(\delta) <_T y$. Let J_y be the unique member of $\mathcal{U}(\delta + 1)$ that contains y. Because $\{y\} \cup J_y$ is convex in L and $x <_L y$ we know that $x \leq_L y(\delta)$. But we also know that $x = x(\delta) \neq y(\delta)$ so that $x <_L y(\delta)$ and that is enough to show $x \prec y$. The third case is where $\delta = \mathrm{lv}(y) < \mathrm{lv}(x)$ and that is analogous to the second case. The final case is where $\mathrm{lv}(x) = \delta = \mathrm{lv}(y)$. But then $x <_L y$ forces $x <_M y$ in the node M of T that contains both x and y so that, once again, $x \prec y$. Therefore, $f : (L, <_L) \rightarrow (T, \preceq)$ is the required order isomorphism $\Box$

Corollary 4.2 *: If $\preceq$ is the lexicographic ordering of an Aronszajn tree T, then there is no order isomorphism from $(T, \preceq)$ into $\mathbb{R}$.*

Proof: By Theorem 4.1, $(T, \preceq)$ is an Aronszajn line which, by definition, cannot contain (or be) an uncountable set of real numbers. $\Box$

Remark 4.3 *:* Theorem 4.1 allows us to point out once again the contrast between the theory of lexicographic and branch space representations of linearly ordered sets. Theorem 4.1 shows that any lexicographic ordering of an Aronszajn tree gives an Aronszajn line, while in [1] we show that the branch space of an Aronszajn tree is never an Aronszajn line (although it must contain an Aronszajn line).

Corollary 4.4 *: In its open-interval topology, any Aronszajn line is*

a) not separable;

b) hereditarily paracompact;

c) not compact;

d) zero-dimensional.

Proof: Let $(L, <_L)$ be an Aronszajn line. From Theorem 4.1 we know that L is order isomorphic to the lexicographic ordering of some Aronszajn tree $(T, \leq_T)$. If the Aronszajn line L is separable in its open interval topology, then so is the Aronszajn tree T in the open interval topology $\mathcal{I}$ of its lexicographic order $\preceq$. Let D be a countable dense subset of $(T, \mathcal{I})$. Then there is a countable ordinal α such that $\mathrm{lv}(d) \leq \alpha$ for each $d \in D$. Being an ω_1-tree, T has a point t with $\mathrm{lv}(t) = \alpha + \omega$. Let $s = t(\alpha + 1)$ be the unique predecessor of t at level $\alpha + 1$ of the tree. Then T^s is an infinite convex subset of $(T, \preceq)$ and therefore has non-void interior. However $T^s \cap D = \emptyset$, showing that D is not dense in $(T, \mathcal{I})$.

If the Aronszajn line $(L, <_L)$ is not hereditarily paracompact, then by a result of Engelking and Lutzer [2] there is a strictly increasing or strictly decreasing embedding into $(L, \preceq)$ of a stationary subset S of a regular uncountable cardinal κ. But that gives an order isomorphism from ω_1 or ω_1^* into L, and that is impossible.

Finally, suppose the Aronszajn line L is compact. Then so is the Aronszajn tree T with the open interval topology of the linear ordering $\preceq$. It is known [4] that any Aronszajn tree contains a complete binary tree S of height ω. For each branch b of S choose a branch b^* of T that has $b \subseteq b^*$. In the light of (C4) of Theorem 3.5, each b^* has a supremum $f(b^*)$ in $(T, \preceq)$. Observe that $b_1 <_{\mathcal{B}_S} b_2$ implies $b_1^* <_{\mathcal{B}_T} b_2^*$ and hence that $f(b_1^*) \preceq f(b_2^*)$.

For $b_1, b_2 \in \mathcal{B}_S$ define $b_1 \sim b_2$ if and only if $f(b_1^*) = f(b_2^*)$. Clearly $\sim$ is an equivalence relation on $\mathcal{B}_S$. Suppose $b_1 <_{\mathcal{B}_S} b_2$ and $b_1 \sim b_2$. If both b_1^* and b_2^* have successor height, then $\sup_{\preceq}(b_1^*) = \sup_{\preceq}(b_2^*)$

implies $b_1^* = b_2^*$ and hence $b_1 = b_2$. If both b_1^* and b_2^* have limit height, then (C4) of Theorem 3.5 forces $b_1 = b_2$. Suppose b_1^* has successor height and b_2^* has limit height. Apply (C4) of 3.5 to b_2 to find $\mu < \mathrm{ht}(b_2^*)$ with the property that $b_2^*(\mu)$ has $f(b_2) = \sup_{\preceq}(b_2^*)$ as its immediate successor in the node to which $b_2^*(\mu)$ belongs. Then $f(b_2) = f(b_1)$ would force $b_2^* <_{\mathcal{B}_T} b_1^*$ and hence $b_2 <_{\mathcal{B}_S} b_1$, which is false. Hence the only possibility for $b_1 <_{\mathcal{B}_S} b_2$ and $b_1 \sim b_2$ is where b_1^* has limit height and b_2^* has successor height. That is enough to guarantee that the function $f : \mathcal{B}_S \to T$ is at most two-to-one. We know that the branch space $\mathcal{B}_S$ is an uncountable real order (in fact, it is the Cantor set). Then $\mathrm{Im}(f)$, being the image of $\mathcal{B}_S$ under a weakly increasing function that is at most two-to-one, is also an uncountable real order, and is a subset of the Aronszajn line $(T, \preceq)$, and that is impossible.

To see than an Aronszajn line $(L, <_L)$ is zero-dimensional in its open-interval topology, note that if L contained a non-degenerate connected open interval J, then a "middle third" construction inside of J would produce a Cantor-like set in L, and that would yield an uncountable subset of L this is order-isomorphic to a subset of $\mathbb{R}$, which is impossible. □

5 Open Questions

- If an Aronszajn line has countable topological cellularity in its open interval topology, must the Aronszajn tree from which it comes (see Theorem 4.1) contain a Souslin subtree?
- Can an Aronszajn line be Lindelöf in its open interval topology without containing a Souslin line?
- Characterize properties such as paracompact, Lindelöf, and perfect in the open interval topology of the lexicographic ordering of a tree, in terms of tree and node properties.
- In terms of the partial order $\leq_T$ of a tree T and the chosen node orderings of T, characterize which lexicographically ordered trees $(T, \prec)$ are of the first Baire Category in their open interval topology. (Theorem 3.8 provides an answer, but only for certain kinds of trees.)

References

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