Images of the Countable Ordinals

by

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Abstract We study spaces that are continuous images of the usual space $[0, \omega_1)$ of countable ordinals. We begin by showing that if Y is such a space and is T_3 then Y has a monotonically normal compactification, and is monotonically normal, locally compact and scattered. Examples show that regularity is needed in these results. We investigate when a regular continuous image of the countable ordinals must be compact, paracompact, and metrizable. For example we show that metrizability of such a Y is equivalent to each of the following: Y has a G_δ-diagonal, Y is perfect, Y has a point-countable base, Y has a small diagonal in the sense of Hušek, and Y has a σ-minimal base. Along the way we obtain an absolute version of the Juhasz-Szentmiklossy theorem for small spaces, proving that if Y is any compact Hausdorff space having $|Y| \leq \aleph_1$ and having a small diagonal, then Y is metrizable, and we deduce a recent result of Gruenhage from work of Mrowka, Rajagopalan, and Soundararajan.

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1 Introduction

The set of countable ordinals is the unique uncountable, well-ordered set in which each element has only countably many predecessors. If we endow the set with the usual open interval topology of the well-ordering[1], we obtain the usual space of countable ordinals, a space that provides a rich source of strange examples in introductory topology courses. In this paper we will use an old-fashioned notation, $[0, \omega_1)$, for this space with its usual topology. If we need to consider the set of countable ordinals with some other topology τ, we will write $([0, \omega_1), \tau)$.

Our first goal in this paper is to explore what kinds of spaces can be obtained by forming continuous and quotient images of $[0, \omega_1)$, with particular emphasis on paracompactness and metrization of such spaces. Our second goal is to continue the study of spaces that have monotonically normal

[1]In this topology, the initial point 0 is isolated and basic neighborhoods of a point $\beta > 0$ have the form $(\alpha, \beta] = (\alpha, \beta + 1)$ where $\alpha < \beta$ and $\beta + 1$ is the first element of the set larger than β.

compactifications, and as we will show, T_3-spaces that are continuous images of $[0,\omega_1)$ are among the most simple examples of this type.

Monotone normality was introduced by Borges [5] and studied in [14]. For any space X, let $Pair(X)$ be the collection of all pairs (A,U) where A is closed, U is open, and $A \subseteq U$. A space is *monotonically normal* if it has a function G which, for each $(A,U) \in Pair(X)$, produces an open set $G(A,U)$ with $A \subseteq G(A,U) \subseteq \mathrm{cl}_X(G(A,U)) \subseteq U$ and which is monotonic in the sense that if $(A_1,U_1),(A_2,U_2) \in Pair(X)$ have $A_1 \subseteq A_2$ and $U_1 \subseteq U_2$, then $G(A_1,U_1) \subseteq G(A_2,U_2)$. All metric spaces and all linearly ordered topological spaces (with the open interval topology) are monotonically normal. Monotone normality is a very strong property: it is a hereditary property, implies collectionwise normality, and is preserved by closed mappings. One problem of current research interest is to study spaces that have monotonically normal compactifications. Locally separable metric spaces have such compactifications, but (surprisingly) metric spaces in general do not [16]. Because the linearly ordered space $[0,\omega_1]$ is a compactification of $[0,\omega_1)$, we see that $[0,\omega_1)$ has a monotonically normal compactification.

In order to make this paper widely readable, in the first three sections we have tried to give proofs of an elementary type, even when our results could be deduced from more general theorems in the literature. There are two notable exceptions to that rule. The first exception is our use of the Pressing Down Lemma (see below). The second, and more major, exception to the simple-proofs-rule occurs when, in Section 3, we invoke the Balogh-Rudin theorem (see Theorem 3.6) that characterizes paracompactness in monotonically normal spaces. (As noted in Section 3, van Douwen's paper [9] provides another non-elementary approach.) Looking for more elementary proofs that do not invoke these theorems could be an interesting project for students.

Wherever possible we follow notations and conventions from Engelking's book [10]. Definitions of some of the more esoteric topics studied in Sections 3 and 4 can be found in the survey papers by Gruenhage [12] and Burke [7], [8]. A word of warning about our terminology: we know that the word "perfect" is over-used in mathematics, but we don't see any way to avoid it. A *topological space is perfect* if each closed set is a G_δ-set, and a *mapping* $f : X \to Y$ *is perfect* if it is a closed mapping and each fiber $f^{-1}[y]$ of f is a compact subset of X.

The cardinality of a set S is denoted by $|S|$. Because we want to use the symbol ω_1 as the right endpoint of the linearly ordered set $[0,\omega_1)$, and ω for the first point in $[0,\omega_1)$ with infinitely many predecessors, we will use $\aleph_0$ and $\aleph_1$ for the cardinality of the sets $[0,\omega)$ and $[0,\omega_1)$ respectively.

A closed, unbounded subset of $[0,\omega_1)$ is called a *CUB*-set. A subset $S \subseteq [0,\omega_1)$ is *stationary* if $S \cap C \neq \emptyset$ for each CUB-set C. Obviously, any set that contains a CUB-set is stationary, but there are stationary sets that do not contain any CUB-set. The most important fact about stationary sets is given by the *Pressing Down Lemma* which asserts that if $S \subseteq [0,\omega_1)$ is a stationary set, and if $g : S \to [0,\omega_1)$ is a regressive function (i.e., has the property that $g(\sigma) < \sigma$ for all $\sigma \in S - \{0\}$), then there is some $\beta \in [0,\omega_1)$ and some stationary $T \subseteq S$ with the property that $g(\sigma) = \beta$ for each $\sigma \in T$. (A more general version of the Pressing Down Lemma can be proved for regressive functions defined on stationary subsets of any regular initial ordinal [18].) For most of our applications in $[0,\omega_1)$, it will be enough to know that the set $T \subseteq S$ is uncountable. Almost any set theory book includes this lemma, e.g., [18].

We want to thank Gary Gruenhage for his valuable consultations that led to the main result in Section 4. That result, which may be of interest in other contexts, is an absolute version (i.e., one

that holds in any model of ZFC) for spaces of cardinality $\aleph_1$ of a famous consistency result proved by Juhasz and Szentmiklossy [15]. We show that any compact Hausdorff space of cardinality $\aleph_1$ is metrizable if and only if it has a small diagonal. Subsequently Gruenhage [13] proved that any compact scattered Hausdorff space with a small diagonal must be metrizable. In addition, we want to thank the referee whose suggestions simplified the paper's exposition.

Unless otherwise specified, all spaces in this paper are at least T_1 so that, for example, regular and T_3 mean the same thing, as do normal and T_4.

2 General properties of continuous images of $[0, \omega_1)$

Suppose that $f : [0, \omega_1) \to Y$ is continuous and onto. Two things follow immediately from continuity of the function f, namely that the space Y is countably compact and sequentially compact, meaning that any countable open cover of Y has a finite subcover, and any sequence of points of Y has a convergent subsequence.

Separation properties like $T_1, T_2, T_3, \cdots$ are another matter. The fact that $[0, \omega_1)$ has such strong separation properties (hereditary normality, hereditary collectionwise normality, hereditary monotone normality) might make one suspect that the image space Y must also have strong separation properties. That is not the case, as can be seen from the continuous mapping $f(x) = x$ from $[0, \omega_1)$ with the order topology onto the set $[0, \omega_1)$ with the indiscrete topology. An example where the image space Y is T_1 but not T_2 can be obtained if $Y = ([0, \omega_1), \tau)$ where τ is the co-finite topology[2] on $[0, \omega_1)$ and $f(x) = x$. The next example, which will be needed several times in this paper, shows that a continuous 1-1 image of $[0, \omega_1)$ can be Hausdorff but not regular.

Example 2.1 *There is a Hausdorff space Y that is not regular and that is a continuous 1-1 image of $[0, \omega_1)$. This space Y is not locally compact and is not sequential, and there are two points of Y that do not have neighborhoods with disjoint closures.*

Proof: We define neighborhood bases at points of the set $[0, \omega_1)$, starting with $\alpha = 0$. For each limit ordinal $\lambda \geq \omega$ let $U(\lambda) = \{0\} \cup \{\nu + 1 : \nu$ is a limit ordinal with $\nu \geq \lambda\}$ and let $\mathcal{N}(0) = \{U(\lambda) : \lambda < \omega_1$ is a limit ordinal$\}$. Notice that for any finite collection λ_i of limit ordinals, we have $\bigcap\{U(\lambda_i) : 1 \leq i \leq n\} = U(\max\{\lambda_i : 1 \leq i \leq n\})$. In a similar way we define $V(\lambda) = \{1\} \cup \{\nu + 2 : \nu$ is a limit ordinal with $\nu \geq \lambda\}$ and let $\mathcal{N}(1) = \{V(\lambda) : \lambda < \omega_1$ is a limit ordinal $\}$. Finally, for each $\alpha \in [0, \omega_1)$ with $\alpha > 1$, let $\mathcal{N}(\alpha)$ be the usual neighborhood system of α in the usual topology of ω_1. Let τ be the topology on the set $[0, \omega_1)$ that has the collection $\mathcal{S} = \bigcup\{\mathcal{N}(\alpha) : \alpha \in \omega_1\}$ as a subbase. Because each member of $\mathcal{S}$ is open in the usual topology of ω_1, we see that the function $f(x) = x$ from $[0, \omega_1)$ onto $([0, \omega_1), \tau)$ is continuous and 1-1.

Clearly, $Y = ([0, \omega_1), \tau)$ is Hausdorff. Consider neighborhoods $U(\lambda)$ and $V(\mu)$ of the points 0 and 1. Note that the limit ordinal $\max(\lambda, \mu) + \omega^2$ is a limit point of both $U(\lambda)$ and $V(\mu)$. Therefore 0 and 1 do not have neighborhoods with disjoint closures. Consequently, the space $Y = ([0, \omega_1), \tau)$ is not regular. Because any locally compact Hausdorff space is regular, we see that Y is not locally compact. $\square$

[2] A set belongs to the co-finite topology if it is empty or if its complement is finite.

However, as we shall prove in the next few propositions, if we know that the image space Y is T_3, then Y has many of the same properties as $[0, \omega_1)$ and as other linearly ordered and generalized ordered spaces[3]. Our results might lead one to ask whether all regular continuous images of $[0, \omega_1)$ must be orderable, so we start with an example of a compact continuous image of $[0, \omega_1)$ that is neither orderable nor generalized orderable.

Example 2.2 *There is a continuous image of $[0, \omega_1)$ that is compact T_2 and is neither orderable nor generalized orderable.*

Proof: Let C be a CUB-set in $[0, \omega_1)$ such that $[0, \omega_1) - C$ is uncountable. Then $[0, \omega_1) - C$ is a disjoint union of open convex subspaces that we will call "arms," each of which is countable. Let $Y = [0, \omega_1)/C$ be the quotient space obtained by identifying C to a single point C^*.

The space Y is compact. For suppose $\mathcal{U}$ is any open cover of Y. Choose $U_0 \in \mathcal{U}$ with $C^* \in U_0$. Let $q : [0, \omega_1) \to Y$ be the quotient mapping. Then the set $V_0 = q^{-1}[U_0]$ is an open subset of $[0, \omega_1)$ containing the set C and $[0, \omega_1) - V_0$ is also closed. Because $[0, \omega_1)$ cannot contain disjoint CUB-sets, $[0, \omega_1) - V_0$ must be countable and therefore compact so that finitely many members of $\mathcal{U}$ cover all of Y.

Note that the space Y cannot be first-countable at the point C^* because that would force the set C to be a G_δ-subset of $[0, \omega_1)$ so that $[0, \omega_1) - C$ would be countable.

For contradiction, suppose Y is orderable using some linear order $<$. Because Y is not first-countable at the point C^*, either $(\leftarrow, C^*)$ or $(C^*, \rightarrow)$ is not closed and has uncountable cofinality (or co-initiality). The two cases are analogous so we consider only the first. Write $J = (\leftarrow, C^*)$ and choose any $a_1 \in J$. Let A_1 be the the arm (see above) of $[0, \omega_1) - C$ that contains a_1. Then A_1 is countable. Suppose we have chosen points $a_1 < a_2 < \cdots < a_n$ of J and arms A_i containing a_i in such a way that for each $i \leq n, a_i \notin \bigcup\{A_j : j < i\}$. The set $\bigcup\{A_i : i \leq n\}$ is countable, so there is a point $a_{n+1} \in J$ with $a_n < a_{n+1}$ and $a_{n+1} \notin \bigcup\{A_i : i \leq n\}$.

Because the points a_n were chosen from distinct arms of $[0, \omega_1) - C$ we see that no point of Y except possibly C^* can be a limit of the infinite set $S = \{a_n : n < \omega\}$. Because J has uncountable cofinality, there is a point $b \in J$ with $a_n < b$ for all n. Then $(b, \rightarrow)$ is a neighborhood of C^* containing no point a_n so that C^* is not a limit point of the infinite set $S = \{a_n : n \geq 1\}$. Therefore S has no limit point in Y and that is impossible because Y is compact. Consequently Y is not orderable (or generalized orderable). $\square$

Remark 2.3 *Because any countable metric space embeds in the usual space of rational numbers, the argument in Example 2.2 shows that if C is a CUB-set in $[0, \omega_1)$, then the quotient space $Y = [0, \omega_1)/C$ is orderable if and only if Y is countable. (That every subspace of the rationals is orderable is a consequence of Herrlich's theorem given in Problem 6.3.2-f of [10].)*

Question 2.4 *Among T_3 spaces that are continuous images of $[0, \omega_1)$, characterize those that are orderable or generalized orderable. Our Example 3.2 is of this type. In addition, in [9], van Douwen*

[3] A space is *linearly orderable* if its underlying set carries a linear order and its topology is the usual open-interval topology of that ordering. A space is *generalized orderable* if it is homeomorphic to a subspace of a linearly ordered space. There are other characterizations of generalized ordered spaces, e.g., in [19].

proved the surprising theorem that if Y is a regular continuous image of $[0,\omega_1)$ and Y is not compact, then Y is orderable; thus our question concerns only those compact Hausdorff spaces that are continuous images of $[0,\omega_1)$.

Proposition 2.5 *Suppose $f : [0,\omega_1) \to Y$ is continuous and onto. If Y is T_3, then Y is (hereditarily) monotonically normal and has a monotonically normal compactification.*

Proof: First we prove that Y must be normal. Suppose C and D are disjoint closed sets in Y. Then $f^{-1}[C]$ and $f^{-1}[D]$ are disjoint closed sets in $[0,\omega_1)$ so that one of them must be countable. Suppose $f^{-1}[C]$ is countable. Then the closed set C is also countable. For each point $x \in C$, regularity of Y gives open sets $V(x)$ and $W(x)$ having $x \in V(x), D \subseteq W(x)$ and $V(x) \cap W(x) = \emptyset$. Because C is countably compact (being a closed subset of the countably compact space Y), finitely many sets $V(x_1), \cdots V(x_n)$ cover all of C. Let $V = \bigcup\{V(x_i) : 1 \le i \le n\}$ and $W = \bigcap\{W(x_i) : 1 \le i \le n\}$. Then V and W are the two disjoint open sets needed to show normality of Y.

Because Y is T_4, Y has a Čech-Stone compactification $\beta(Y)$ and there is a continuous extension $\beta(f) : \beta([0,\omega_1)) \to \beta(Y)$. We know that $\beta([0,\omega_1))$ is the usual linearly ordered space $[0,\omega_1]$ and therefore $\beta([0,\omega_1))$ is monotonically normal. Because $\beta(f)$ is a closed mapping, it follows that $\beta(Y)$ is also monotonically normal. But monotone normality is a hereditary property [14], showing that Y is (hereditarily) monotonically normal.

Because the mapping $f : [0,\omega_1) \to Y$ gives $\beta(f) : [0,\omega_1] \to \beta(Y)$ we see that $\beta(Y)$ is the desired monotonically normal compactification of Y. $\square$

The proof of Proposition 2.5 would be much easier if we knew that whenever Y is T_3, the continuous mapping $f : [0,\omega_1) \to Y$ is a closed mapping. Unfortunately, that is not the case, as the next example shows. See also Proposition 3.1 below.

Example 2.6 *There is a compact Hausdorff space Y and a 1-1 continuous mapping f from $[0,\omega_1)$ onto Y that is not a closed mapping or a quotient mapping.*

Proof: Let $Y = [1,\omega_1]$ with its usual topology and define $f : [0,\omega_1) \to Y$ by $f(0) = \omega_1$ and $f(\alpha) = \alpha$ for each $\alpha \in [1,\omega_1)$. Then f is 1-1 and continuous but is not a quotient mapping because $f^{-1}[\{\omega_1\}] = \{0\}$ is an open set in $[0,\omega_1)$, but $\{\omega_1\}$ is not open in Y. Therefore f is not a closed mapping. $\square$

Proposition 2.7 *If $f : [0,\omega_1) \to Y$ is continuous and onto, and if Y is T_3 and not compact, then f is a perfect (and therefore closed) mapping and $|\beta(Y) - Y| = 1$ so that Y is locally compact.*

Proof: As in the proof of Proposition 2.5, consider the function $\beta(f) : \beta([0,\omega_1)) \to \beta(Y)$. Because its domain is compact and its range is T_2, we know that $\beta(f)$ is a perfect mapping. We also know that $\beta([0,\omega_1)) = [0,\omega_1]$. Because Y is not compact, we cannot have $\beta(f)(\omega_1) \in Y$, showing that $Y = \beta(Y) - \{\beta(f)(\omega_1)\}$. Therefore $|\beta(Y) - Y| = 1$ and $\beta(f)^{-1}[Y] = [0,\omega_1)$. But f is $\beta(f)$ restricted to $\beta(f)^{-1}[Y]$ so that f is also a perfect mapping (Proposition 2.1.4 in [10]).

To verify that Y is locally compact, note that either Y is compact or else $\beta(Y) - Y$ is a singleton, showing that Y is open in $\beta(Y)$ and therefore is locally compact. $\square$

The space Y in Example 2.1 is not locally compact, showing that the hypothesis of regularity is needed in Proposition 2.7.

Proposition 2.8 *Suppose $f : [0,\omega_1) \to Y$ is a continuous, onto function, and suppose Y is T_3. Then Y is scattered.*

Proof: Consider $\beta(f) : \beta([0,\omega_1)) \to \beta(Y)$. We know that $\beta([0,\omega_1)) = [0,\omega_1]$. Suppose $\emptyset \neq A \subseteq Y$. We will show that A has a relatively isolated point. Let $B = \beta(f)^{-1}[A]$. Because $\beta(f)|_B$ is a perfect mapping, there is a B-closed set $C \subseteq B$ and an irreducible perfect onto mapping $g : C \to A$. The space C is scattered, so there is a relatively isolated point $x \in C$. If the set $g^{-1}[g(x)]$ has more than one point, then g cannot be irreducible. Therefore $g^{-1}[g(x)] = \{x\}$ is relatively isolated in C so that $g(x)$ must be a relatively isolated point of A, as required. □

3 Special properties of continuous images of $[0,\omega_1)$

Suppose Y is a continuous image of $[0,\omega_1)$. The goal of this section is to study three properties that Y might, or might not, have, namely compactness, paracompactness, and metrizability. We begin with a result that distinguishes sharply between spaces that are quotient images of $[0,\omega_1)$ and spaces that are merely continuous images of $[0,\omega_1)$.

Proposition 3.1 *Suppose $f : [0,\omega_1) \to Y$ is continuous and onto, and suppose that Y is T_2. The following are equivalent:*

a) $f : [0,\omega_1) \to Y$ is a quotient mapping;

b) Y is sequential [4];

c) every continuous function $g : [0,\omega_1) \to Y$ is a closed mapping;

d) Y is Frechet [5].

Proof: Quotient mappings preserve the property of being sequential (Problem 2.4.G in [10]) so that a) implies b). Every closed mapping is a quotient mapping so that when c) is applied to $f : [0,\omega_1) \to Y$, we see that a) holds. Therefore c) implies a) and it remains to prove that $b) \Rightarrow c) \Rightarrow d) \Rightarrow c)$.

To see that b) implies c) suppose Y is sequential and g is any continuous function from $[0,\omega_1)$ onto Y. Suppose C is a closed subset of $[0,\omega_1)$ and consider $D = g[C]$. Hypothesis b) gives us that Y is sequential so that it is enough to show that D is sequentially closed. To that end, suppose $d_n \in D$ has $\lim d_n = e \in Y$. For each n, choose $c_n \in C$ with $g(c_n) = d_n$. Because C is a closed subset of the sequentially compact space $[0,\omega_1)$, we know there is a subsequence c_{n_k} that converges to some point $c \in C$. But then $g(c_{n_k})$ converges to $g(c)$ because g is continuous, and because $g(c_{n_k}) = d_{n_k}$ is a subsequence of the convergent sequence $d_n \to e$, $g(c_{n_k})$ also converges to e. Therefore $e = g(c) \in g[C] = D$, so that D is sequentially closed. Because Y is a sequential space, D is a closed set. Therefore, the mapping g is closed.

[4] A set S in a space Y is *sequentially closed* provided that if $s_n \in S$ and $s_n \to t$, then $t \in S$. Clearly, any closed set is sequentially closed. A topological space is *sequential* if each sequentially closed set is closed [10]

[5] A space Y is *Frechet* if for each $S \subseteq Y$ and each $p \in \mathrm{cl}(S)$ there is a sequence $s_n \in S$ having $s_n \to p$ [10]

To see that c) implies d), apply c) to the given mapping $f : [0, \omega_1) \to Y$. Then the mapping f is a closed map, and we know that closed mappings preserve the Frechet property (Problem 2.4.G in [10]). Hence Y is a Frechet space.

To see that d) implies c), note that if C is a closed subset of $[0, \omega_1)$, then C is sequentially compact. Therefore for any continuous $g : [0, \omega_1) \to Y$, the set $g[C]$ is sequentially compact. But in any Frechet space, any sequentially compact set is closed, so that c) holds. (Another proof for T_3-spaces follows from Proposition 2.5 and a result of Kannan [17] who proved that any hereditarily normal, countably compact sequential space must be Frechet (see also 3.10.E of [10]). □

It was announced in [22] that any perfect image of $[0, \omega_1)$ must be homeomorphic to $[0, \omega_1)$, an error that was corrected by an example given on pages 310-11 of [23]. Some arguments in [23] can be simplified, and so we re-describe their example.

Example 3.2 *There is a perfect mapping f from $[0, \omega_1)$ onto a space Y where Y is regular (and hence monotonically normal) and yet Y is not homeomorphic to $[0, \omega_1)$.*

Proof: Let L be the second derived set of $[0, \omega_1)$ and let f be the quotient mapping that identifies each point $\lambda \in L$ with the point $\lambda + \omega$. Because f is a quotient mapping, Proposition 3.1 shows that f is a closed mapping, and because each fiber of f has at most two points, we see that f is a perfect mapping.

Suppose Y is homeomorphic to $[0, \omega_1)$ under the mapping $g : Y \to [0, \omega_1)$. Let $h = g \circ f$. Then h is continuous and at most 2-to-1, so that the set $L' = h[L]$ is uncountable and closed (because L, and therefore L', is a sequentially compact subset of $[0, \omega_1)$). Therefore L' is a CUB-set in $[0, \omega_1)$. For each $h(\lambda) \in L'$ (where $\lambda \in L$), the sequence $\{h(\lambda + n) : n < \omega\}$ converges to $h(\lambda)$ so there is some n_λ with $h(\lambda + n_\lambda) < h(\lambda)$, where $<$ is the usual ordering of $[0, \omega_1)$. Write $\alpha(h(\lambda)) = h(\lambda + n_\lambda)$. Then α is a regressive function on the stationary set $h[L]$ so there is an uncountable set $S \subseteq L$ with the property that $\alpha(\sigma) = \alpha(\tau)$ for every $\sigma, \tau \in S$. But that is impossible because h is at most a 2-to-1 function.□

3-a) Compactness in images of $[0, \omega_1)$

Proposition 3.1 allows us to understand the structure of compact, sequential spaces that are continuous images of $[0, \omega_1)$.

Proposition 3.3 *Suppose $f : [0, \omega_1) \to Y$ is continuous and onto, and that Y is compact and T_2. If Y is sequential, then either Y is countable, compact, and metrizable, or else Y is the one-point compactification of a locally compact metric space. Therefore there is at most one point of Y at which Y is not first-countable.*

Proof: Because Y is sequential, Proposition 3.1 shows that the mapping f is closed. If each fiber of f is countable, then each fiber is compact, so that f is a perfect mapping. Then $[0, \omega_1) = f^{-1}[Y]$ is also compact (because perfect mappings inversely preserve compactness (Theorem 3.7.2 in [10]) and that is false. Therefore for some $y_0 \in Y$ the fiber $f^{-1}[y_0]$ is uncountable and therefore is a CUB set.

Let $Z = Y - \{y_0\}$ and let $W = f^{-1}[Z]$. Because W is a complete inverse set, the restriction of the mapping f to W is a closed mapping. In addition, for each $z \in Z$, the fiber $f^{-1}[z]$ is countable because otherwise the CUB sets $f^{-1}[z]$ and $f^{-1}[y_0]$ would intersect. Therefore the fiber $f^{-1}[z]$ is compact . Consequently, the restriction mapping $f|_W : W \to Z$ is a perfect mapping (Proposition 3.7.6 in [10]). Now consider W. Because W is the complement of a CUB set in $[0, \omega_1)$, W is the disjoint union of open, metrizable subintervals of $[0, \omega_1)$, so that W is metrizable. But then so is its perfect image Z (Theorem 4.4.15 in [10]).

There are two remaining possibilities. In the first, the metrizable subspace $Z \subseteq Y$ is not compact. But then $Y = Z \cup \{y_0\}$ is its one-point-compactification, and Z, being open in Y, is locally compact. In the second, the metrizable subspace Z is compact. But then the point y_0 is an isolated point of Y so that $f^{-1}[y_0]$ is both a CUB set and an open set in $[0, \omega_1)$. The Pressing Down Lemma gives us some $\alpha < \omega_1$ with the property that $[\alpha, \omega_1) \subseteq f^{-1}[y_0]$. But then $Y \subseteq \{f(\beta) : 0 \leq \beta \leq \alpha\}$, showing that Y is countable. Because Y is given as being compact and Hausdorff, it follows that, in this second case, Y is metrizable, as claimed. $\square$

Example 3.4 Here are two examples illustrating the previous proposition. In the first, let L be the set of all limit ordinals in ω_1 and let Y be the quotient space of $[0, \omega_1)$ in which L is collapsed to a single point by the quotient map f. Then Y is uncountable and compact (because L is an uncountable fiber of f), and is the one-point compactification of the uncountable discrete subspace $\{f(\alpha) : \alpha \in [0, \omega_1) - L\}$. We note that by Proposition 3.3, the space Y is Frechet. In the second, consider the quotient space obtained from $[0, \omega_1)$ by collapsing $[\omega, \omega_1)$ to a single point. $\square$

Example 2.6 shows that a continuous image of $[0, \omega_1)$ can fail to be sequential, so the classification of compact images of $[0, \omega_1)$ is not complete. The paper [23] contains characterizations of various kinds of images of various ordinal spaces.

3-b) Paracompactness in images of $[0, \omega_1)$

We already know that any continuous image Y of $[0, \omega_1)$ is countably compact so that if Y is paracompact, then Y must be compact. That allows us to re-write Proposition 3.3 as follows:

Proposition 3.5 *Suppose $f : [0, \omega_1) \to Y$ is continuous and onto, and that Y is paracompact and T_2. If Y is sequential, then either Y is countable, compact, and metrizable, or else Y is the one-point compactification of a locally compact metric space. Therefore there is at most one point of Y at which Y is not first-countable.*

Which continuous images of $[0, \omega_1)$ are paracompact? The famous Balogh-Rudin characterization of paracompactness in monotonically normal spaces [1] is the key.

Theorem 3.6 *(Balogh-Rudin Theorem) A monotonically normal space Y is paracompact if and only if no closed subspace of Y is homeomorphic to a stationary subset of a uncountable regular cardinal*[6]

As a corollary we characterize paracompactness in regular, continuous images of $[0, \omega_1)$. Except for the D-space property, the covering conditions in our next result are, in general spaces, strictly weaker than paracompactness and assert that every open cover has a refinement of a certain special type. Reproducing all of the definitions would consume too much space; see [8] for details.

[6] An ordinal number λ is a regular cardinal if $\mu < \lambda$ implies that $|\mu| < |\lambda|$. For example, ω_1 is a regular uncountable cardinal.

Proposition 3.7 *Suppose $f : [0, \omega_1) \to Y$ is continuous and onto, where Y is T_3. Then Y is compact if and only if Y is paracompact. Furthermore, Y is paracompact if and only if any of the following hold:*

a) Y is metacompact;

b) Y θ-refinable (= submetacompact), weakly θ-refinable, weakly $\delta\theta$-refinable;

c) Y is metaLindelöf;

d) Y is subparacompact;

e) Y is a D-space[7].

Proof: We know that Y is countably compact. Therefore, in Y, compactness and paracompactness are equivalent.

Each of properties a), b), c), and d) is a well-known consequence of paracompactness. For the converses, note that no stationary subset of a regular uncountable cardinal can have one of these covering properties (a) - (d) so that the Balogh-Rudin theorem shows that each of (a) through (d) implies that Y is paracompact.

Now suppose Y has the D-space property. Suppose $\mathcal{U}$ is an open cover of Y. For each $y \in Y$ choose some $U(y) \in \mathcal{U}$ having $y \in U(y)$. (Repetitions are allowed). The D-space property gives a closed, discrete subset $S \subseteq Y$ for which $\bigcup\{U(y) : y \in S\} = Y$. Because Y is countably compact, the closed discrete set S must be finite, so we see that Y is compact (and hence paracompact). To prove the converse, note that if Y is paracompact, then as noted above, Y is compact, and recall that any compact space is a D-space. $\square$

Remark 3.8 *There is an alternate approach to Proposition 3.7. As noted above, van Douwen has proved that if Y is a regular continuous image of $[0, \omega_1)$ then Y is either compact (and therefore paracompact) or else Y is orderable. In the latter case, known theorems about paracompactness in ordered spaces [20] show that each of the properties listed in Proposition 3.7 make Y paracompact.*

In general spaces, there is no relation between compactness, separability, and countable cellularity. However, for regular spaces that are continuous images of $[0, \omega_1)$ we have the following consequence of Proposition 3.12 below.

Proposition 3.9 *Suppose Y is T_3 and is a continuous image of $[0, \omega_1)$. If Y is separable or if Y has countable cellularity, then Y is compact and hence paracompact.*

The next proposition lists many components of metrizability that have been studied by researchers, and as we will see in Proposition 3.13, with the exception of c)[8], each is equivalent to metrizability in spaces that are regular continuous images of $[0, \omega_1)$. Once again, reproducing all of the definitions would consume too much space; see the survey by Gruenhage [12] for details.

[7] Y is a D-space if for each $y \in Y$ if $U(y)$ is an open set containing y, then there is a closed, discrete set $S \subseteq Y$ for which $Y = \bigcup\{U(y) : y \in S\}$.

[8] See Example 3.15

Proposition 3.10 *Suppose $f : [0, \omega_1) \to Y$ is continuous and onto, where Y is T_3. Then Y is compact if and only if Y is paracompact. Furthermore, Y is paracompact if any one of the following holds:*

a) Y has a G_δ-diagonal, or a G-Souslin diagonal [11], [6], or a quasi-G_δ-diagonal;

b) Y is perfect (i.e., closed sets are G_δ-sets);

c) every subspace of Y is a p-space in the sense of Arhangelskii, or an M-space in the sense of Morita, or a β-space in the sense of Hodel [12];

d) Y has a point-countable base or is a member of the class MOBI;

e) Y is quasi-developable;

f) Y has a small diagonal in the sense of Hušek [12];

g) Y has a σ-minimal base.

Proof: Proposition 2.5 shows that any T_3 image of $[0, \omega_1)$ is monotonically normal so that if P is a closed-hereditary topological property that no stationary subset of $[0, \omega_1)$ can have, then standard proofs give that if Y is a regular, continuous image of $[0, \omega_1)$ that has property P, then Y is paracompact. See [20] for an extensive list of such properties. Each of the properties listed in this theorem is of that type. A proof that f) implies paracompactness of our space Y appears in Corollary 4.4 in the next section and the proof that g) implies paracompactness appears in Proposition 5.1. $\square$

Remark 3.11 The properties listed in Proposition 3.10 are sufficient, but not necessary, conditions for paracompactness in a space Y that is a continuous image of $[0, \omega_1)$, as can be seen from the space of Example 2.6 which is paracompact but has none of properties (a)-(f). In fact, using Proposition 3.10, we show in Proposition 3.13 that each of a), b), d), e), f) and g) is equivalent to metrizability for any regular continuous image of $[0, \omega_1)$.

3-c) Metrizability in images of $[0, \omega_1)$

If Y is a continuous image of $[0, \omega_1)$ and happens to be metrizable, then because Y is countably compact, it must be separable and must have countable cellularity. We begin by showing that the converse is true for regular continuous images of $[0, \omega_1)$.

Proposition 3.12 *Suppose $f : [0, \omega_1) \to Y$ is a continuous onto function, where Y is T_3. Then the following are equivalent:*

a) Y is a compact metric space;

b) Y is metrizable;

c) Y is separable;

d) Y has countable cellularity

Proof: Obviously a) implies b) and the converse holds because Y is known to be countably compact. In general $a) \Rightarrow c) \Rightarrow d)$ so it only remains to prove that $d) \Rightarrow a)$.

Suppose d) holds. We will show that $|Y| = \aleph_0$, which gives a) because Y is known to be countably compact. For contradiction, suppose $|Y| > \aleph_0$. We have the mapping $f : [0, \omega_1) \to Y$ and therefore, as in Proposition 2.5, we have a mapping $\beta(f) : [0, \omega_1] \to \beta(Y)$. Because Y has countable cellularity, so does $\beta(Y)$.

The function $\beta(f)$ is a perfect mapping and consequently there is a closed subspace $C \subseteq [0, \omega_1]$ such that the restricted mapping $g = \beta(f)|_C$ is a perfect irreducible mapping from C onto $\beta(Y)$. Because Y is uncountable, so is $\beta(Y)$ and therefore so is the set C. Then, because C is an uncountable closed subset of $[0, \omega_1]$, there are uncountably many relatively isolated points of C.

Consider any relatively isolated point $\alpha \in C$ and let $y = g(\alpha)$. Then $\alpha \in g^{-1}[y]$ so that if $\{\alpha\} \neq g^{-1}[y]$ then the mapping g would not be irreducible (because $C - \{\alpha\}$ would be a proper closed subset of C that is mapped onto $\beta(Y)$). Therefore $g^{-1}[y] = \{\alpha\}$, showing that $g^{-1}[y]$ is a relatively open subset of C. Because g is a closed mapping, it follows that y is an isolated point of $\beta(Y)$.

Note that if $\beta \neq \alpha$ is also a relatively isolated point of C, then $\beta \notin g^{-1}[y]$ so that $g(\beta) \neq g(\alpha)$. Consequently, $\{g(\gamma) : \gamma$ is a relatively isolated point of $C\}$ is an uncountable set of isolated points of $\beta(Y)$ and that is impossible because Y, and therefore also $\beta(Y)$, has countable cellularity. Therefore, the space Y is countable and consequently metrizable, so that d) implies a). □

As mentioned above, with one exception (see Example 3.15 below), each of the properties listed in Proposition 3.10 as a sufficient condition for paracompactness in a continuous regular image of $[0, \omega_1)$ actually gives metrizability in such a space.

Proposition 3.13 *Suppose $f : [0, \omega_1) \to Y$ is continuous and onto, and suppose Y is T_3. Then the following are equivalent:*

a) Y is a countable, compact metric space, and f has an uncountable fiber;

b) Y is metrizable;

c) Y has a G_δ-diagonal;

d) Y has a $\mathcal{G}$-Souslin diagonal in the sense of Braude [6];

e) Y has a point-countable base or is a member of the class MOBI;

f) Y is quasi-developable;

g) Y is perfect;

h) Y is symmetrizable;

i) Y has a small diagonal;

j) Y has a σ-minimal base.

Proof: Clearly a) implies b). Now suppose b) holds. Because Y is countably compact, it follows that Y is compact. In addition, because Y is Frechet, the mapping $f : [0, \omega_1) \to Y$ is a closed mapping. If each fiber of f were countable, then f would be a perfect mapping, so that $[0, \omega_1)$

would be a perfect pre-image of a compact metric space, making $[0,\omega_1)$ compact and that is not the case. Therefore, some fiber of f, say $f^{-1}[y_0]$, is uncountable. Given that Y is metrizable, the set $\{y_0\}$ is a G_δ-set in Y, so that the fiber $f^{-1}[y_0]$ is a G_δ-set in $[0,\omega_1)$. Then the Pressing Down Lemma gives some $\alpha < \omega_1$ such that $[\alpha,\omega_1) \subseteq f^{-1}[y_0]$. Therefore, $Y = \{f(\beta) : \beta \leq \alpha\}$ showing that Y is countable. Thus b) implies a).

Clearly b) implies each of the properties in c) through f). In addition, by Proposition 3.7, each of the properties in c) through f) is enough to make Y paracompact and therefore compact, and each is sufficient to give metrizability for any compact subset of Y [12]. Hence each of the properties c) through f) implies b).

Certainly b) implies g). To show that g) implies b), note that g) makes Y paracompact by Proposition 3.10 and therefore the countably compact space Y is compact. Because Y is compact and perfect, Y is first-countable and therefore Frechet. But then the mapping $f : [0,\omega_1) \to Y$ is a closed mapping (see Proposition 3.1). The mapping f cannot be a perfect mapping because Y is compact but $[0,\omega_1)$ is not. Therefore, some fiber, say $C = f^{-1}[y_0]$, is not compact. Consequently C must be a CUB-set. Because $\{y_0\}$ is a G_δ-set in Y, C is a G_δ-set in $[0,\omega_1)$. Then the Pressing Down Lemma gives some $\beta < \omega_1$ with $[\beta,\omega_1) \subseteq f^{-1}[y_0]$ so that $Y = \{f(\alpha) : \alpha \leq \beta\}$ is countable. Because Y is compact and countable, Y is metrizable, as required by b).

Clearly b) implies h). To show that h) implies b), recall that any symmetrizable countably compact space is metrizable [12]. Alternatively, note that if Y is symmetrizable, then Y is sequential and hence (by Proposition 3.1) is Frechet. But a Frechet symmetrizable space is semi-metrizable and hence perfect, and we now invoke part g).

Finally, it is clear that b) implies i) and j). That i) implies b) for our space Y is the subject of the next section and the equivalence of b) and j) is the subject of Section 5. $\Box$

Remark 3.14 There are proofs of parts of Proposition 3.13 based on general metrization theory for countably compact spaces (e.g., that any symmetrizable countably compact space is metrizable [12] and Chaber's theorem that any regular countably compact space with a G_δ-diagonal must be metrizable [10]). Relying on those proofs is not consistent with our simple-proofs-goal and it would be interesting to give an elementary proof that h) implies b) .

In an earlier paper we asked the following general question (see Problem 3 in [3]): Suppose P is a topological property and that any GO-space with property P must be metrizable. Is it true that any space X that has property P and has a monotonically normal compactification must be metrizable? One such property P is the F_{pp} property, namely that each subspace of X is a paracompact p-space in the sense of Arhangelskii. It is known that any generalized ordered space with property F_{pp} must be metrizable [2], and we can now show that for this property $P = F_{pp}$, the answer to our Problem 3 is negative.

Example 3.15 *There is a compact monotonically normal space Y that is a continuous image of $[0,\omega_1)$ and is not metrizable, and yet every subspace of Y is a paracompact p-space in the sense of Arhangelskii.*

Proof: As in Example 2.2, let L be the set of all limit ordinals in $[0,\omega_1)$ and let Y be the quotient space obtained from $[0,\omega_1)$ by identifying L to a single point. The space Y is monotonically normal and compact, and every subspace of Y is either compact or metrizable, depending upon whether the subspace contains the unique limit point of Y, so that every subspace of Y is a paracompact p-space, i.e., Y has property F_{pp}. However, Y is not metrizable. $\square$

4 Small diagonals

In this section, we discuss the small-diagonal property mentioned in Propositions 3.10 and 3.13 and prove a general theorem that might be of interest in other contexts.

The diagonal of a space Y is the set $\Delta = \{(y, y) : y \in Y\}$. Recall that a space Y has a *small diagonal* if whenever S is an uncountable subset of $Y^2 - \Delta$, there is an open set $W \subseteq Y^2$ with $\Delta \subseteq W$ for which the set $S - W$ is uncountable. This is a generalization of the G_δ-diagonal property because if $\Delta = \bigcap\{W_n : n < \omega\}$ where each W_n is open, then $S = S - \Delta = \bigcup\{S - W_n : n < \omega\}$ so that one of the sets $S - W_n$ must be uncountable. Juhasz and Szentmiklossy [15] proved the following theorem:

Theorem 4.1 *Suppose the Continuum Hypothesis holds. Then any compact Hausdorff space with a small diagonal is metrizable.*

In an early draft of this paper, we proved that any compact Hausdorff space having cardinality $\leq \aleph_1$ and having a small diagonal must be metrizable by combining Theorem 4.1 with a separate proof that covered the situation where the Continuum Hypothesis is false. The key step in our argument was to prove that (assuming $\aleph_1 < 2^{\aleph_0}$) any compact scattered space of cardinality $\leq \aleph_1$ must be first-countable, which is equivalent to metrizability for compact scattered spaces. Then Gary Gruenhage [13] sent us a more straightforward and more general proof that any compact scattered space (without cardinality restrictions) with a small diagonal must be metrizable. At the same time we discovered that Gruenhage's result follows from older work of Mrowka, Rajagopalan, and Soundararajan [21], as we explain below.

Theorem 4.2 *(Gruenhage) Suppose X is a scattered compact Hausdorff space with a small diagonal. Then X is metrizable.*

Proof: Theorem 4 of [21] includes a proof that if X is any scattered compact Hausdorff space and if $\{x(\alpha) : \alpha \in [0, \kappa)\}$ is any net in X indexed by an ordinal, then there is a cofinal subset $S \subseteq [0, \kappa)$ such that the subnet $\{x(\alpha) : \alpha \in S\}$ converges to some point of X.[9] If the space X of Gruenhage's theorem is uncountable, then for each $\alpha \in [0, \omega_1)$ we can choose points $x(\alpha) \in X$ without repetition. Using Theorem 4 from [21], we can find a cofinal subset $S \subseteq [0, \omega_1)$ such that the subnet $\{x(\alpha) : \alpha \in S\}$ converges to some point $p \in X$. For each $\alpha \in S$. let α^+ be the first point of S that is greater than α. Then we see that the uncountable set $U = \{(x(\alpha), x(\alpha^+)) : \alpha \in S\} \subseteq X^2 - \Delta$ violates the small diagonal condition. Therefore X must be countable. But any compact countable Hausdorff space is metrizable. $\square$

[9]Because [21] is somewhat difficult to find, here is an outline of the proof: For each $\alpha < \kappa$, let $Z(\alpha)$ be the closure of $\{x(\beta) : \beta \geq \alpha\}$. Because X is compact, the set $Z = \bigcap\{Z(\alpha) : \alpha < \kappa\}$ is nonempty. Because X is scattered, there is a relatively isolated point $z_0 \in Z$, so there is an open set U_0 with $z_0 \in U_0$ and $\mathrm{cl}(U_0) \cap Z = \{z_0\}$. Let $S = \{\alpha < \kappa : x(\alpha) \in U_0\}$. Then the cofinal subset $\{x(\alpha) : \alpha \in S\}$ converges to z_0.

Theorem 4.3 *Suppose Y is a compact Hausdorff space with cardinality $\aleph_1$ and that Y has a small diagonal. Then Y is metrizable.*

Proof: We show that our theorem is true in any model of ZFC. In a model where the Continuum Hypothesis holds, the Juhasz-Szentmiklossy Theorem 4.1 shows that Y is metrizable. Therefore, it remains to consider what happens if the Continuum Hypothesis is false, i.e., if $\aleph_1$ is less than $2^{\aleph_0}$.

We begin by showing that because $|Y| \leq \aleph_1 < 2^{\aleph_0}$, the space Y must be scattered, i.e., every subspace S of Y has an isolated point. Let $T = \mathrm{cl}_Y(S)$. If T has an isolated point, then so does S, so we focus on the compact subspace T. If T is finite, there is nothing to prove, so assume T is infinite. If T has no isolated points, then we can choose infinite open sets $U(0)$ and $U(1)$ with $\mathrm{cl}(U(0)) \cap \mathrm{cl}(U(1)) = \emptyset$. Using the assumption that T has no isolated points over and over, we use induction to find a collection $\{U(i_0, i_1, \cdots, i_n) : n < \omega, i_j \in \{0,1\}\}$ with the property that for each $(i_0, \cdots, i_n) \in \{0,1\}^n$

a) $U(i_0, \cdots, i_n, 0) \cap U(i_0, \cdots, i_n, 1) = \emptyset$ and

b) $\mathrm{cl}(U(i_0, \cdots, i_n, j)) \subseteq U(i_0, \cdots, i_n)$ for $j = 0, 1$.

Each infinite sequence $(i_0, i_1, \cdots) \in \{0,1\}^\omega$ gies a nonempty set $V(i_0, i_1, \cdots) = \bigcap\{\mathrm{cl}(U(i_0, i_i, \cdots, i_n)) : n < \omega\}$ and if $(i_0, i_1, \cdots) \neq (j_0, j_1, \cdots)$ then $V(i_0, i_1, \cdots) \cap V(j_0, j_1, \cdots) = \emptyset$. Therefore, $|Y| \geq 2^{\aleph_0}$ and that is impossible because, by assumption in this case, $|Y| \leq \aleph_1 < 2^{\aleph_0}$.

Therefore, if the Continuum Hypothesis fails, then Y is scattered so that by Theorem 4.2, Y is metrizable. $\square$

Corollary 4.4 *Suppose $f : [0, \omega_1) \to Y$ is continuous and that Y is T_3. If Y has a small diagonal, then Y is metrizable.*

Proof: First we show that Y is paracompact by using the Balogh-Rudin theorem; this is part of Proposition 3.10 above whose proof was that was deferred until now. Because $|Y| \leq \aleph_1$, it is enough to show that no stationary subset S of $[0, \omega_1)$ can have a small diagonal. Let T be the set of all limit points of S. Then T is stationary. For each $\sigma \in S$, let σ^+ be the first member of S that is larger than σ. Then $\{(\sigma, \sigma^+) : \sigma \in S\}$ is an uncountable subset of $S^2 - \Delta$. Assuming that S has a small diagonal, there is an open subset $U \subseteq [0, \omega_1)^2$ such that if we consider the relatively open subset $U \cap S^2$ of S^2, we have uncountably many points $(\sigma, \sigma^+) \notin U$. Because $\Delta \subseteq U$, for each limit ordinal $\lambda \in T \subseteq S$ we have some $g(\lambda) < \lambda$ with $(g(\lambda), \lambda]^2 \subseteq U$. The Pressing Down Lemma gives an ordinal β with the property that $g(\lambda) = \beta$ for each λ in some stationary subset of T. But then $(\beta, \lambda]^2 \subseteq U$ for cofinally many λ, and therefore $(\beta, \omega_1)^2 \subseteq U$. Now choose any $\sigma \in S$ with $\beta < \sigma$. Then $(\sigma, \sigma^+) \in (\beta, \omega_1)^2 \subseteq U$. This contradicts the fact that there are uncountably many $(\sigma, \sigma^+) \notin U$. Therefore, no stationary subset of ω_1 can have a small diagonal. The Balogh-Rudin theorem shows that the space Y is paracompact. Because Y is countably compact and paracompact, Y is compact. Now we invoke Theorem 4.3 to conclude that Y is metrizable. $\square$

5 Images of $[0, \omega_1)$ with a σ-minimal base

Recall that a collection $\mathcal{B}$ of subsets of a set Y is *minimal* if each $B \in \mathcal{B}$ contains a point (which we will call a "minimality point") that is not in any other member of $\mathcal{B}$, and that a collection is

σ*-minimal* if it is the union of countably many minimal collections. It is known that an generalized ordered space with a σ-minimal base for its topology must be paracompact [4] and that the lexicographic square has a σ-minimal base but its subspace $[0,1] \times \{0,1\}$ does not. It is an open problem whether a space with a monotonically normal compactification must be paracompact if it has a σ-minimal base [3]. In this section we characterize regular continuous images of $[0,\omega_1)$ that have a σ-minimal base by proving:

Proposition 5.1 *Suppose $f : [0,\omega_1) \to Y$ is continuous and onto, and that Y is T_3. Then Y has a σ-minimal base if and only if Y is metrizable.*

Proof: Any metric space as a σ-discrete (and hence σ-minimal) base. To prove the converse for a space $Y = f[[0,\omega_1)]$ that has a σ-minimal base $\mathcal{B} = \bigcup\{\mathcal{B}(n) : n \geq 1\}$, observe that either f has an uncountable fiber, or every fiber of f is countable and compact.

Consider the case where f has an uncountable fiber, say $C_0 = f^{-1}[z_0]$, and consider any n for which some $B_n \in \mathcal{B}(n)$ has $z_0 \in B_n$. Then $f^{-1}[B_n]$ is an open set that contains the CUB set C_0. The Pressing Down Lemma gives some β_n such that $[\beta_n,\omega_1) \subseteq f^{-1}[B_n]$. Any other $B' \in \mathcal{B}(n)$ that contains z_0 must contain a minimality point $y(B')$ which is not in B_n. Then $f^{-1}[y(B')]$ is a subset of $[0,\omega_1)$ that cannot meet $f^{-1}[B_n]$ so that $f^{-1}[y(B')]$ is a subset of the countable set $[0,\beta_n]$. Therefore, at most countably many members of $\mathcal{B}(n)$ can contain z_0. Therefore, the space Y is first-countable at z_0. As a result, the CUB set $C_0 = f^{-1}[z_0]$ is a G_δ-set in $[0,\omega_1)$ so that there is some $\gamma < \omega_1$ with $[\gamma,\omega_1) \subseteq C_0$. Therefore the set $Y - \{z_0\}$ must be countable. But then Y is countable and countably compact, so that Y is metrizable.

Now consider the case where every fiber of the mapping f is countable. Under the assumptions that Y has a σ-minimal base, we will obtain a contradiction, so that this second case cannot occur.

There is at most one point $y_1 \in Y$ for which every neighborhood is uncountable. For suppose y_1, y_2 are two such points. Because Y is regular, we may find neighborhoods U_i of y_i having $\mathrm{cl}(U_1) \cap \mathrm{cl}(U_2) = \emptyset$. If both U_i are uncountable, then $f^{-1}[\mathrm{cl}(U_i)]$ for $i = 1,2$ are two uncountable disjoint closed sets in $[0,\omega_1)$ and that is impossible.

Consider the case where one such point y_1 exists. Let $Y_0 = Y - \{y_1\}$. Then Y_0 is a dense open subset of Y and $S = f^{-1}[Y_0]$ is a subset of $[0,\omega_1)$ whose complement is the compact set $f^{-1}[y_1]$. Therefore S is a stationary subset of $[0,\omega_1)$.

If Y has a σ-minimal base, then its open subspace Y_0 also has a σ-minimal base $\mathcal{B}' = \bigcup\{\mathcal{B}'(n) : n < \omega\}$ where each member of $\mathcal{B}'$ is a countable open set (use $\mathcal{B}' = \{B \in \mathcal{B} : B \subseteq Y_0\}$). Let $Z_n = \bigcup \mathcal{B}(n) \subseteq Y_0$. Then $S = \bigcup\{f^{-1}[Z_n] : n < \omega\}$ so that one of the sets, say $S(N) = f^{-1}[Z_N]$ is stationary in $[0,\omega_1)$. Then the set $S^d(N)$ consisting of all limit points of $S(N)$ is also stationary.

For each $\lambda \in S^d(N)$ fix some $B(\lambda) \in \mathcal{B}(N)$ with $\lambda \in f^{-1}[B(\lambda)]$. Then because λ is in the open set $f^{-1}[B(\lambda)]$ there is some $\alpha(\lambda) < \lambda$ having $[\alpha(\lambda),\lambda] \subseteq f^{-1}[B(\lambda)]$. Because α is a regressive function on the stationary set $S^d(N)$, there is some β and some stationary set $T \subseteq S^d(N)$ with the property that $\alpha(\lambda) = \beta$ for each $\lambda \in T$. Consequently, $[\beta,\lambda] \subseteq f^{-1}[B(\lambda)]$ for each $\lambda \in T$.

Let λ_0 be the first element of T. Suppose $\alpha < \omega_1$ and suppose λ_β is defined for all $\beta < \alpha$ Note that the set $\bigcup\{f^{-1}[B(\lambda_\beta)] : \beta < \alpha\}$ is countable, so there is some $\lambda_\alpha \in T$ with $\bigcup\{f^{-1}[B(\lambda_\beta)] : \beta < \alpha\} \subseteq [0,\lambda_\alpha)$.

Fix any λ_α from the points found above. The set $B(\lambda_\alpha)$ has a minimality point $p(\alpha) \in B(\lambda_\alpha)$, i.e., a point not in any other member of the collection $\mathcal{B}(N)$. Choose any $q(\alpha) \in f^{-1}[p(\alpha)]$. Then $q(\alpha) \in f^{-1}[B(\lambda_\alpha]$ but $q(\alpha)$ is not in any other set $f^{-1}[B(\lambda_\gamma)]$.

We claim that $q(\alpha) < \beta$. For contradiction, suppose $\beta \leq q(\alpha)$. Choose $\gamma > \alpha$. Then λ_γ is larger than every point of $f^{-1}[B(\lambda_\alpha)]$ and, because $q(\alpha) \in f^{-1}[B(\lambda_\alpha)]$ we have $q(\alpha) \in [\beta, \lambda_\gamma] \subseteq f^{-1}[B(\lambda_\gamma)]$ so that $q(\alpha)$ belongs to at least two of the sets $f^{-1}[B(\lambda_\delta)]$ and that is impossible. Therefore each $\alpha, q(\alpha) \in [0, \beta)$. But that is impossible because there are uncountably many points $q(\alpha)$ and the set $[0, \beta)$ is countable.

At this point, we have completed the case where there is a point $y_1 \in Y$ whose every neighborhood is uncountable. In case no such y_1 exists, repeat the above proof with $Y_0 = Y$, and now the proof is complete. □

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