A shorter proof of a theorem on hereditarily orderable spaces

by

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Abstract: We give a shorter proof a result of Purisch and Hirata and Kemoto that any subspace of any space of ordinals is a LOTS (under some linear ordering).

MathSciNet classifications Primary = 54F05, Secondary= 03E04

*******draft of February 13, 2012

1 Introduction

A topological space is orderable if it is homeomorphic to some linearly ordered topological space (LOTS) $(X, <, \mathcal{L}(<))$ where $<$ is a linear ordering of X and $\mathcal{L}(<)$ is the usual open interval topology of $<$. As the subspace $[0,1] \cup (2,3)$ of the usual space $\mathbb{R}$ of real numbers shows, a subspace of a LOTS may fail to be orderable, as may a topological sum of two LOTS (no matter what linear ordering is used).

In their paper [3], Hirata and Kemoto showed that any subspace of any space of ordinal numbers must be orderable (under some ordering), a result that follows from an earlier paper by Purisch [4] [5]. In this paper we give a new proof that is shorter than the proofs given by Purisch or by Hirata and Kemoto, and we raise some questions about hereditary orderability, where we say that a space X is *hereditarily orderable* if each of its subspaces is an orderable space.

Recall that a *generalized ordered* (GO) space is a triple $(X, <, \tau)$ where $<$ is a linear ordering of X and where τ is a Hausdorff topology on X that has a basis consisting of order-convex (possibly degenerate) sets.

We want to thank the referee whose comments significantly improved this paper.

2 Ordinals are hereditarily orderable

For any linearly ordered set $(X, <)$, the symbol $(X, <)^*$ denotes the set X with the reverse ordering $<^*$. It is easy to see that the LOTS $(X, <, \mathcal{L}(<))$ is homeomorphic to the LOTS $(X, <^*, \mathcal{L}(<^*))$. For a given linearly ordered set X, we sometimes write X^* for $(X, <^*, \mathcal{L}(<^*))$.

Suppose $(X_1, <)$ and $(X_2, \prec)$ are disjoint linearly ordered sets. We use the symbol $(X_1, <) \frown (X_2, \prec)$ to mean the set $X_1 \cup X_2$ with the ordering defined by $a \ll b$ if either $a, b \in X_1$ and $a < b$, or $a \in X_1$ and $b \in X_2$, or $a, b \in X_2$ with $a \prec b$. We sometimes write $\ll = < \frown \prec$. The relation $\ll$ is always a linear ordering, but if $\mathcal{L}(<)$ and $\mathcal{L}(\prec)$ are the usual open interval topologies on X_1 and X_2 respectively, then the open interval topology $\mathcal{L}(\ll)$ might not be the topology of the topological sum $(X_1, \mathcal{L}(<)) \oplus (X_2, \mathcal{L}(\prec))$. For an example, let $X_1 = [0,1]$ and $X_2 = (2,3)$ have their usual orderings. However, there are times when the topological sum of two or more LOTS is guaranteed to be a LOTS.

Lemma 2.1 *Let $(X_1, <)$ and $(X_2, \prec)$ be disjoint linearly ordered sets and let $\ll$ be the order $< \frown \prec$.*

1) If the LOTS $(X_1, <, \mathcal{L}(<))$ contains a right end point and $(X_2, \prec, \mathcal{L}(\prec))$ contains a left end point, then the topological sum $X_1 \oplus X_2$ is a LOTS under the order $\ll$.

[1]supported by a Texas Tech University Faculty Development Leave

2) If the LOTS $(X_1, <, \mathcal{L}(<))$ contains no right endpoint and if the LOTS $(X_2, \prec, \mathcal{L}(\prec))$ contains no left end point, then the topological sum $X_1 \oplus X_2$ is a LOTS under the order $\ll$. $\square$

Lemma 2.2 *Suppose that $(X, <, \tau)$ is a GO-space having a right endpoint $b \in X$, and suppose that b is a τ-limit point of the set $Y = X - \{b\}$. Suppose that $\prec$ is a linear ordering of the set Y such that the open interval topology $\mathcal{L}(\prec)$ on Y coincides with $\tau|_Y$, and suppose that a subset $C \subseteq Y$ is cofinal* [2] *in $(Y, <)$ if and only if C is cofinal in $(Y, \prec)$. Extend the linear order $\prec$ to a linear order $\lhd$ on X by making b larger than each point of $(Y, \prec)$. Then $\mathcal{L}(\lhd) = \tau$, i.e., $(X, \lhd, \tau)$ is a LOTS.*

Proof: It is enough to show that the topologies τ and $\mathcal{L}(\lhd)$ agree at the point b because, by hypothesis, they agree at each point of the open set Y. Because b is a limit point of Y, the set $(Y, <)$ contains no right endpoint. Hence neither does $(Y, \prec)$.

Let U be a τ-neighborhood of b. We may assume U is order convex with respect to $<$. Because b is a τ-limit of Y, we may choose $a', a \in U \cap Y$ and $a' < a < b$. Then $a \in \{x \in X : a' < x < b\} \subseteq U$. We will show that there is some $c \in X$ with $c \lhd b$ and $\{x \in X : c \lhd x\} \subseteq \{x \in X : a \leq x\} \subseteq \{x \in X : a' < x\}$. If that is not true, then for each $c \in Y$ there is some $d(c) \in Y$ with $c \lhd d(c)$ and $d(c) \notin \{x \in X : a \leq x\}$. The set $\{d(c) : c \in Y\}$ is cofinal in $(Y, \prec)$, and therefore it is also cofinal in $(Y, <)$ so there is some $d(c)$ with $a \leq d(c)$, contrary to the choice of $d(c)$. An analogous argument shows that for each $a \lhd b$ there is some c with $c < b$ and $\{x \in X : c < x\} \subseteq \{x \in X : a \lhd x\}$. Consequently, the two topologies τ and $\mathcal{L}(\lhd)$ agree at b, as required. $\square$

The following example illustrates a key idea in the proof of our main result. Write $2\omega = \omega + \omega$ and $3\omega = \omega + \omega + \omega$. Form a GO-topology τ by isolating the points ω and 2ω in $[0, 3\omega)$. In the usual ordering $<$, this is not a LOTS because the non-limit points ω and 2ω have no immediate predecessors in the order $<$. However, in the linear ordering $\prec$ of

$$[0, \omega) \frown \Big([\omega, 2\omega)^* \frown [2\omega, 3\omega)\Big)$$

the set $\{n : 0 \leq n < \omega\}$ has no supremum, and the points ω and 2ω both have immediate predecessors and immediate successors. Consequently $([0, 3\omega), \prec, \tau)$ is a LOTS. Flipping the order on subsegments of a GO-space is the key to our next proof.

Theorem 2.3 *Let Δ be any ordinal with $\Delta \geq \omega$ and let T be any set of limit ordinals in $[0, \Delta)$. Let $[0, \Delta)_T$ denote the GO-space obtained from the usual ordinal space $[0, \Delta)$ by isolating every element of T. Then the GO-space $[0, \Delta)_T$ is homeomorphic to some LOTS.*

Proof: In this proof, the symbol $\cong$ means "is homeomorphic to" and we use $\leq$ and $<$ to denote the usual well-ordering of $[0, \Delta)$. Interval notation such as $[\alpha, \beta)$ will always refer to intervals in the usual ordinal ordering. For any $\alpha < \Delta$, the symbol $[0, \alpha)_T$ denotes the GO-space obtained from the usual LOTS $[0, \alpha)$ by isolating all points of $T \cap [0, \alpha)$. We will write $\mathcal{S}$ for the GO-topology of $[0, \Delta)_T$ and $\mathcal{S}_{[\alpha, \beta)}$ for the relative topology that $[\alpha, \beta)$ inherits from $[0, \Delta)_T$. We will argue by contradiction. For contradiction, suppose that

(*) $[0, \Delta)_T$ is not homeomorphic to any LOTS.

By an *acceptable pair* we mean an ordered pair $([0, \alpha), \prec_\alpha)$ where

1) $\alpha \leq \Delta$,
2) $\prec_\alpha$ is a linear ordering of the set $[0, \alpha)$,
3) 0 is the left end point of the linearly ordered set $([0, \alpha), \prec_\alpha)$,
4) a subset $C \subseteq [0, \alpha)$ is cofinal in $([0, \alpha), \prec_\alpha)$ if and only if C is cofinal in $([0, \alpha), <)$, and
5) $\mathcal{S}_{[0, \alpha)} = \mathcal{L}(\prec_\alpha)$ where $\mathcal{L}(\prec_\alpha)$ is the usual open interval topology of the linear order $\prec_\alpha$.

[2] A subset S of a linearly ordered set $(X, <)$ is *cofinal* if for each $x \in X$ there is some $s \in S$ with $x \leq s$.

Let $\mathcal{P}$ be the set of all acceptable pairs. Then $\mathcal{P}$ is not empty because $([0,2),<)$ is in $\mathcal{P}$. Partially order $\mathcal{P}$ by the rule that $([0,\alpha),\prec_\alpha) \sqsubseteq ([0,\beta),\prec_\beta)$ if and only if the following four statements hold:

a) $\alpha \leq \beta$;

b) $\prec_\beta|_{[0,\alpha)} = \prec_\alpha$ (so that $\prec_\beta$ extends $\prec_\alpha$);

c) if $\alpha < \beta$, then $\{x \in [0,\beta) : x \prec_\beta \alpha\} = [0,\alpha)$ (so $\prec_\beta$ adds no points to the domain of $\prec_\alpha$);

d) $\mathcal{L}(\prec_\alpha) \subseteq \mathcal{L}(\prec_\beta)$.

Suppose $\mathcal{C} = \{([0,\alpha),\prec_\alpha) : \alpha \in A\}$ is a chain in the partially ordered set $(\mathcal{P},\sqsubseteq)$. For some $\gamma \leq \Delta$, the set $\bigcup\{[0,\alpha) : \alpha \in A\} = [0,\gamma)$. Define $\prec_\gamma = \bigcup\{\prec_\alpha : \alpha \in A\}$. We will show that $([0,\gamma),\prec_\gamma)$ is an acceptable pair that is an upper bound for $\mathcal{C}$ in $(\mathcal{P},\sqsubseteq)$.

It is clear that $\prec_\gamma$ is a linear ordering of $[0,\gamma)$ and that its left end point is 0. If $\gamma = \alpha$ for some $\alpha \in A$, then $([0,\gamma),\prec_\gamma) = ([0,\alpha),\prec_\alpha)$ is an acceptable pair that is an upper bound for the chain $\mathcal{C}$, so assume that $\alpha < \gamma$ for all $\alpha \in A$. Consequently γ is a limit ordinal and the set A is cofinal in the usual ordering of $[0,\gamma)$.

We claim that $([0,\gamma),\prec_\gamma)$ satisfies part (4) in the definition of an acceptable pair. We first show that the set A is cofinal in the ordering $\prec_\gamma$. Let $x \in [0,\gamma)$ and choose $\alpha,\beta \in A$ with $x < \alpha < \beta$. In the light of (c) we have $x \in [0,\alpha) = \{y \in [0,\beta) : y \prec_\beta \alpha\}$ so that $x \prec_\gamma \alpha$. Hence A is cofinal in the order $\prec_\gamma$. Now suppose that C is a cofinal subset of $([0,\gamma),<)$. Fix $(\alpha,\prec_\alpha) \in \mathcal{C}$. Choose $x \in C$ with $\alpha < x$ and then choose $\beta \in A$ with $\alpha < x < \beta$. By (c) we have $x \notin [0,\alpha) = \{y \in [0,\beta) : y \prec_\beta \alpha\}$ so that $\alpha \preceq_\beta x$. Therefore $\alpha \preceq_\gamma x$, showing that C is cofinal in the ordering $\prec_\gamma$. Next suppose that C is cofinal in the ordering $\prec_\gamma$. If C is not cofinal in the usual ordering $<$ of $[0,\gamma)$ then there is some $\alpha \in A$ with $C \subseteq [0,\alpha)$. Then for each $\beta \in A$ with $\alpha < \beta$ we have $C \subseteq [0,\alpha) = \{y \in [0,\beta) : y \prec_\beta \alpha\}$ so that $x \prec_\beta \alpha$ for each $x \in C$, and therefore $x \prec_\gamma \alpha$ for each $x \in C$. But that is impossible because C is cofinal in the ordering $\prec_\gamma$.

We next show that $([0,\gamma),\prec_\gamma)$ satisfies $\mathcal{S}_{[0,\gamma)} = \mathcal{L}(\prec_\gamma)$, which is part (5) in the definition of an acceptable pair. First note that the collection $\mathcal{B} := \bigcup\{\mathcal{L}(\prec_\alpha) : \alpha \in A\}$ is a base for the topology $\mathcal{L}(\prec_\gamma)$ and that $\mathcal{B}' := \bigcup\{\mathcal{S}_{[0,\alpha)} : \alpha \in A\}$ is a base for $\mathcal{S}_{[0,\gamma)}$. Because we know that $\mathcal{L}(\prec_\alpha) = \mathcal{S}_{[0,\alpha)}$ for each $\alpha \in A$, we see that $\mathcal{B} = \mathcal{B}'$ which gives $\mathcal{L}(\prec_\gamma) = \mathcal{S}_{[0,\gamma)}$ as required.

Now that we have $([0,\gamma),\prec_\gamma) \in \mathcal{P}$, we must show that $([0,\alpha),\prec_\alpha) \sqsubseteq ([0,\gamma),\prec_\gamma)$ for each $\alpha \in A$. Clearly (a) and (b) are satisfied. For (c), note that for each $\alpha < \beta$ in the set A, we have

$$[0,\alpha) = \{y \in [0,\beta) : y \prec_\beta \alpha\} \subseteq \{y \in [0,\gamma) : y \prec_\gamma \alpha\}.$$

To prove that $\{y \in [0,\gamma) : y \prec_\gamma \alpha\} \subseteq [0,\alpha)$, suppose $y < \gamma$ satisfies $y \prec_\gamma \alpha$. Choose $\beta \in A$ so large that $\{\alpha, y\} \subseteq [0,\beta)$. Then $y \in \{z \in [0,\beta) : z \prec_\beta \alpha\} = [0,\alpha)$ as required. To verify (d) note that the collection $\bigcup\{\mathcal{L}(\prec_\alpha) : \alpha \in A\}$ is a basis for the topology $\mathcal{L}(\prec_\gamma)$.

At this stage we know that every chain in $(\mathcal{P},\sqsubseteq)$ has an upper bound in $(\mathcal{P},\sqsubseteq)$ so that Zorn's Lemma gives us a maximal element $([0,\delta),\prec_\delta)$ of $\mathcal{P}$. We have $\delta \leq \Delta$. If $\delta = \Delta$, then we have contradicted (*) because $([0,\delta),\prec_\delta)$ satisfies part (5) of the definition of acceptable pair, so we have

(**) $\delta < \Delta$.

Claim 1 We claim that δ must be a limit ordinal. Otherwise write $\delta = \lambda + n$ where λ is a limit and $n \geq 1$ is an integer. Then $[0,\delta)$ has a right endpoint (namely $\lambda + (n-1)$) in the usual ordinal ordering, so that $\{\lambda + (n-1)\}$ is a cofinal subset of $[0,\delta)$ in the usual ordering. Therefore $\{\lambda + (n-1)\}$ is a cofinal subset of $[0,\delta)$ in the linear ordering $\prec_\delta$, i.e., $([0,\delta),\prec_\delta)$ has $\lambda + (n-1)$ as its right endpoint. Because $\delta < \Delta$ by (**), we know that $\delta + 1 \leq \Delta$. Define a linear ordering $\prec_{\delta+1}$ of $[0,\delta+1)$ that agrees with $\prec_\delta$ on $[0,\delta)$ and has $\delta = \lambda + n$ as its right endpoint. Then the LOTS $([0,\delta+1),\prec_{\delta+1},\mathcal{L}(\prec_{\delta+1}))$ is homeomorphic to the

GO-space $[0,\delta+1)_T$ and it is clear that $([0,\delta+1),\prec_{\delta+1})$ belongs to $\mathcal{P}$ and is strictly larger than $([0,\delta),\prec_\delta)$ in the ordering $\sqsubseteq$, contrary to maximality of $([0,\delta),\prec_\delta)$. Therefore, Claim 1 is established and δ must be a limit ordinal.

Two possibilities remain. Either δ is an isolated point of the GO-space $[0,\Delta)_T$, or else δ is a limit point of the set $[0,\delta)$ in the space $[0,\Delta)_T$, i.e, either $\delta \in T$ or $\delta \notin T$.

Claim 2 We claim that $\delta \in T$ is impossible. For suppose $\delta \in T$. There are two subcases, depending upon whether $(\delta,\Delta)\cap T$ is, or is not, empty.

In the first subcase, we have $(\delta,\Delta)\cap T=\emptyset$, and then $[\delta,\Delta)_T$ is identical to the LOTS $[\delta,\Delta)$ with the usual ordering. Consider the linearly ordered set $X=[\delta,\Delta)^*$ obtained by reversing the usual order of $[\delta,\Delta)$, and let $<^*$ denote the reversal of the usual ordering $<$. The linearly ordered set $([\delta,\Delta)^*,<^*)$ has a final point (namely δ), and the linearly ordered set $Y=([0,\delta),\prec_\delta)$ has 0 as its first point by part (3) of the definition of acceptable pair. Consequently part (1) of Lemma 2.1 guarantees that the LOTS topology of the linear order $\lhd := <^* \frown \prec_\delta$ on the set $X\oplus Y$ is homeomorphic to the disjoint sum topology of the space $X\oplus Y$. But because $\delta\in T$, we have $[0,\delta)_T\oplus[\delta,\Delta)_T\cong[0,\Delta)_T$ so that

$$X\oplus Y\cong Y\oplus X=\Big([0,\delta),\prec_\delta,\mathcal{L}(\prec_\delta)\Big)\oplus[\delta,\Delta)^*\cong[0,\delta)_T\oplus[\delta,\Delta)=[0,\delta)_T\oplus[\delta,\Delta)_T=[0,\Delta)_T$$

showing that $[0,\Delta)_T$ is a LOTS under the linear ordering $\lhd$, contrary to (*). Therefore the first subcase cannot occur.

In the second subcase, $(\delta,\Delta)\cap T\neq\emptyset$. Let η be the first element of $(\delta,\Delta)\cap T$. Then η is a limit ordinal (because all members of T are limit ordinals) and $\eta+1\le\Delta$ because $\eta<\Delta$. The LOTS $[\delta,\eta)$ with its usual order $<$ and usual order topology is homeomorphic to the clopen subspace $[\delta,\eta)_T$ of $[0,\Delta)_T$ and hence so is the reversed LOTS $Y=([\delta,\eta)^*,<^*,\mathcal{L}(<^*))$. Observe that the LOTS $X=([0,\delta),\prec_\delta,\mathcal{L}(\prec_\delta))$ has no final point and that the LOTS $Y=([\delta,\eta)^*,<^*,\mathcal{L}(<^*))$ has no first point. According to part (2) of Lemma 2.1 the LOTS topology of the linear order $\lhd=\prec_\delta\frown<^*$ on the set $[0,\eta)$ coincides with the topology of the topological sum

$$([0,\delta),\prec_\delta)\oplus[\delta,\eta)^*\cong[0,\delta)_T\oplus[\delta,\eta)\cong[0,\eta)_T.$$

Note that the linear order $\lhd$ has a right endpoint, namely δ. Now extend the linear order $\lhd$ on $[0,\eta)$ to the set $[0,\eta]$ by making η greater than each point of $([0,\eta),\lhd)$. The set $[0,\eta+1)$ with this extension of $\lhd$ is a member of $\mathcal{P}$ that is strictly larger than $(\delta,\prec_\delta)$ in the partial order $\sqsubseteq$, and that is impossible. Therefore, Claim 2 is established.

Claim 3 We claim that $\delta\notin T$ is also impossible. For suppose $\delta\notin T$. Because δ is a limit ordinal (see Claim 1), the point δ is a limit point of the set $[0,\delta)$ in the space $[0,\Delta)_T$. Because $([0,\delta),\prec_\delta)\in\mathcal{P}$ we know that the orders $<$ and $\prec_\delta$ have exactly the same cofinal subsets of $[0,\delta)$, and then Lemma 2.2 allows us to extend the order $\prec_\delta$ to a linear order $\lhd$ of the set $[0,\delta+1)$ by making the point δ greater than all points of $([0,\delta),\prec_\delta)$ and guarantees that the LOTS topology of $([0,\delta+1),\lhd)$ coincides with the GO topology $[0,\delta+1)_T$. It is clear that $([0,\delta+1),\lhd)\in\mathcal{P}$ and that is impossible by maximality of $([0,\delta),\prec_\delta)$ in $\mathcal{P}$. Therefore, Claim 3 holds.

In summary, assumption (*) has led us to a maximal element $([0,\delta),\prec_\delta)$ of $\mathcal{P}$ and we have proved that both $\delta\in T$ and $\delta\notin T$ are impossible. Consequently, Theorem 2.3 is proved. □

The hereditary orderability theorem of Purisch, Hirata and Kemoto is an immediate corollary.

Corollary 2.4 *Let Z be an initial segment of the ordinals with the usual topology. Any subspace X of Z is homeomorphic to some LOTS.*

Proof: The set X inherits a well-ordering from Z and we have an order isomorphism h from X onto some set $[0,\Delta)$ of ordinals. Let $\mathcal{S}$ be the topology on $[0,\Delta)$ that makes h a homeomorphism from X onto

$([0,\Delta),\mathcal{S})$. The topology $\mathcal{S}$ will fail to be the open interval topology of the usual ordering $<$ of $[0,\Delta)$ if and only if there are limit ordinals $\lambda < \Delta$ such that λ is not a limit of the set $[0,\lambda)$ in the space $([0,\Delta),\mathcal{S})$. Let T be the set of all limit ordinals $\lambda < \Delta$ that are not topological limits of $[0,\lambda)$ in the topology $\mathcal{S}$. Then X is homeomorphic to the GO-space $[0,\Delta)_T$ obtained from the usual ordinal space $[0,\Delta)$ by isolating each point of T. But from Theorem 2.3 we know that $[0,\Delta)_T$ is homeomorphic to some LOTS, and that completes the proof of the corollary. □

3 Additional comments

In this section, we use dimension theory definitions from [1]. The following result is part of the folklore.

Lemma 3.1 *In any GO-space X, the following three properties are equivalent:*

a) Ind(X) = 0

b) ind(X) = 0

c) a connected subset of X has at most one point (i.e., X is totally disconnected*).* □

Herrlich's theorem ([2]; see also Problem 6.3.2 in [1]) is the key to understanding hereditary orderability in metrizable spaces.

Proposition 3.2 *Let X be a metrizable space. Then the following are equivalent:*

i) $Ind(X) = 0$;

ii) X is orderable and $Ind(X) = 0$;

iii) X is orderable and totally disconnected;

iv) X is hereditarily orderable.

Proof: Herrlich's theorem is that $(i) \Rightarrow (ii)$, and (ii) and (iii) are equivalent in light of Lemma 3.1. Because X is metrizable, for any subspace $Y \subseteq X$ we have $Ind(Y) \leq Ind(X)$ so that Herrlich's theorem shows that $(ii) \Rightarrow (iv)$. Finally, $(iv) \Rightarrow (iii)$ because if X contains a connected subset C with at least two points, then X contains an infinite connected open interval (a,b) (containing no end points of itself) and a point $c \notin [a,b]$. But then the subspace $Y = (a,b) \cup \{c\}$ is not linearly orderable by any ordering. □

However, outside the class of metrizable spaces, $Ind(X) = 0$ is not enough to make a LOTS hereditarily orderable.

Example 3.3 *Let X be the Alexandroff double arrow, i.e., $X = [0,1] \times \{0,1\}$ with the lexicographic ordering. Then X is a compact separable LOTS, and has $Ind(X) = 0$, but its subspace $S := \{(x,1) : x \in [0,1]\}$ is not a LOTS under any ordering, because S has a G_δ-diagonal but is not metrizable.* □

Question 3.4 *Characterize those LOTS that are hereditarily orderable.*

There is an important topological characterization of orderability by van Dalen and Wattel [6]. By a *nest*, van Dalen and Wattel meant a collection that is linearly ordered by set containment. A nest $\mathcal{N}$ is *interlocking* if, whenever a member $N_0 \in \mathcal{N}$ has $N_0 = \bigcap\{N \in \mathcal{N} : N \neq N_0 \text{ and } N_0 \subseteq N\}$, then N_0 also satisfies $N_0 = \bigcup\{N \in \mathcal{N} : N \neq N_0, N \subseteq N_0\}$. Van Dalen and Wattel [6] proved:

Theorem 3.5 *A T_1 space is orderable if and only if it has a sub-base that is the union of two nests, each of which is interlocking.*

That theorem ought to play a key role in studies of hereditary orderability and should give an even shorter proof of the the theorem of Purisch, Hirata, and Kemoto.

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