Compact G_δ Sets

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Abstract

In this paper we study spaces in which each compact subset is a G_δ-set and compare them to H. W. Martin's c-semi-stratifiable (CSS) spaces, i.e. spaces in which compact sets are G_δ-sets in a uniform way. We prove that a (countably) compact subset of a Hausdorff space X is metrizable and a G_δ-subset of X provided X has a $\delta\theta$-base, or a point-countable, T_1-point-separating open cover, or a quasi-G_δ-diagonal. We also show that any compact subset of a Hausdorff space X having a base of countable order must be a G_δ-subset of X and note that this result does not hold for countably compact subsets of BCO-spaces. We characterize CSS spaces in terms of certain functions $g(n,x)$ and prove a "local implies global" theorem for submetacompact spaces that are locally CSS. In addition, we give examples showing that even though every compact subset of a space with a point-countable base (respectively, of a space with a base of countable order) must be a G_δ-set, there are examples of such spaces that are not CSS. In the paper's final section, we examine the role of the CSS property in the class of generalized ordered (GO) spaces. We use a stationary set argument to show that any monotonically normal CSS space is hereditarily paracompact. We show that, among GO-spaces with σ-closed-discrete dense subsets, being CSS and having a G_δ-diagonal are equivalent properties, and we use a Souslin space example due to Heath to show that (consistently) the CSS property is not equivalent to the existence of a G_δ-diagonal in the more general class of perfect GO-spaces.

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1 Introduction

Let $\mathcal{C}$ be a collection of subsets of a topological space X. We say that members of $\mathcal{C}$ are *uniformly* G_δ*-sets* if for each $C \in \mathcal{C}$ there are open sets $G(n,C)$ in X such that:

i) $\bigcap\{G(n,C) : n \geq 1\} = C$;

ii) $G(n+1,C) \subseteq G(n,C)$ for each $n \geq 1$; and

iii) if $C \subseteq D$ are members of $\mathcal{C}$, then $G(n,C) \subseteq G(n,D)$ for each n.

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In case $\mathcal{C}$ is the collection of all closed subsets of X, one obtains the well-known class of *semi-stratifiable* spaces introduced by Creede [13]. In case $\mathcal{C}$ is the collection of all compact subsets of X, one has the class of all *c-semi-stratifiable* (CSS) spaces introduced by H. Martin [25].

In Sections 2 and 3 of this paper, we examine the class of CSS-spaces, comparing the CSS property with the weaker property "every compact subset of X is a G_δ-set." For example, in Propositions 2.1, 2.2, and 2.3 we show that any (countably) compact subset of X will be compact, metrizable, and a G_δ-subset of X provided X is a Hausdorff space with a $\delta\theta$-base, with a point-countable, T_1-point-separating open cover, or with a quasi-G_δ-diagonal. In Proposition 2.6 we note that "local implies global" for the property "every compact set is a G_δ," a result that is not true for the CSS-property in general(see Example 4.2). However, Proposition 3.5 shows that for submetacompact (= θ-refinable) spaces, locally CSS *does* imply globally CSS. The CSS property also has a role to play in metrization theory: it is essentially a result of Martin that a space X is metrizable if and only if it is paracompact, a p-space in the sense of Arhangel'skii, and is CSS.

In Section 4, we study the role of the CSS property among generalized ordered spaces. Recall that a *generalized ordered space*(GO-space) is a triple $(X,\mathcal{S},<)$ where $(X,\mathcal{S})$ is a Hausdorff space that has a base of order-convex sets. If $\mathcal{S}$ is the usual open-interval topology of the order $<$, then X is a *linearly ordered topological space*(LOTS). We show that any GO-space that is CSS must be hereditarily paracompact and that any GO-space with a quasi-G_δ-diagonal is CSS. In Theorem 4.10 we show that among GO-spaces with a σ-closed-discrete dense subset, the CSS property is equivalent to having a G_δ-diagonal, and we provide examples of GO-spaces that are or are not CSS.

Throughout this paper, all spaces are assumed to be at least Hausdorff (so that compact sets are always closed). It will be important to distinguish between subsets of a space X that are closed and discrete (to be called closed-discrete sets) and those that are merely discrete-in-themselves (to be called relatively discrete sets). We will need to distinguish between sets that are "σ-closed-discrete" and those that are "σ-relatively-discrete." Of course, among perfect spaces (= spaces in which closed sets are G_δ-sets), the last two notions are equivalent. We reserve the symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{P}$, and $\mathbb{Z}$ for the usual sets of real, rational, and irrational numbers, and for the set of all integers, respectively.

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2 Spaces in which compact sets are G_δ-sets

Some of the results in this section must be known, but neither the authors nor the referee know references for them. They probably have some independent interest, but we record them here to stand in contrast with the behavior of the CSS property, to be studied in the third section.

Our first three results show that (countably) compact sets will be G_δ-subsets of X provided X has certain base, covering, or diagonal conditions.

Recall that a $\delta\theta$-*base* for a space X is a base $\mathcal{B} = \bigcup\{\mathcal{B}(n) : n \geq 1\}$ with the additional property that if U is open and $x \in U$, then there is some $n = n(x,U)$ with the properties that

a) some $B \in \mathcal{B}(n)$ has $x \in B \subseteq U$, and

b) $\mathrm{ord}(x,\mathcal{B}(n)) \leq \omega$, i.e., x belongs to only countably many members of $\mathcal{B}(n)$.

This is a natural generalization of the notion of a σ-point-finite base and was introduced by Aull [2]. We thank the referee for pointing out how to generalize a result on quasi-developable spaces that appeared in an earlier draft of this paper.

Proposition 2.1 *Suppose X is a Hausdorff space with a $\delta\theta$-base. Then any countably compact subset of X is a compact, metrizable, G_δ-subset of X*

Proof: Let $\mathcal{B} = \bigcup\{\mathcal{B}(n) : n \geq 1\}$ be a $\delta\theta$-base for X and let $H_n = \{x \in X : 1 \leq \mathrm{ord}(x, \mathcal{B}(n)) \leq \omega\}$.

First consider the case where K is a compact subset of X. Then K must be metrizable (see Corollary 8.3(ii) of [17]) and therefore hereditarily separable. Let $\mathcal{C}(n) = \{B \in \mathcal{B}(n) : B \cap H_n \cap K \neq \emptyset\}$. Then the set $K \cap H_n$ has a countable dense subset so that the collection $\mathcal{C}(n)$ must be countable. Hence so is the collection $\mathcal{C} = \bigcup\{\mathcal{C}(n) : n \geq 1\}$, and we know that $\mathcal{C}$ contains a base of neighborhoods for each point of K. Let Φ be the collection of all finite subcollections of $\mathcal{C}$ that cover K. Then Φ is countable and non-empty, and $K \subseteq \bigcap\{\bigcup \mathcal{D} : \mathcal{D} \in \Phi\}$. Now suppose that $p \in X - K$. For each $x \in K$ there is an $n = n(x, p)$ such that some $B(x, p) \in \mathcal{B}(n)$ has $x \in B(x, p) \subseteq X - \{p\}$ and $\mathrm{ord}(x, \mathcal{B}(n)) \leq \omega$. Then $B(x, p) \in \mathcal{C}(n) \subseteq \mathcal{C}$. Because K is compact, there is some finite k and points $x_i \in K$ for $1 \leq i \leq k$ such that $\mathcal{D}(p) = \{B(x_i, p) : 1 \leq i \leq k\}$ covers K. Then $\mathcal{D}(p) \in \Phi$ and we have

$$K \subseteq \bigcap\{\bigcup \mathcal{D} : \mathcal{D} \in \Phi\} \subseteq \bigcup \mathcal{D}(p) \subseteq X - \{p\}.$$

Because p was an arbitrary element of $X - K$, we see that $K = \bigcap\{\bigcup \mathcal{D} : \mathcal{D} \in \Phi\}$.

Next consider the case where K is countably compact. We know that, in its relative topology, K inherits a $\delta\theta$-base, namely $\{K \cap B : B \in \mathcal{B}\}$. Then, given any cover $\mathcal{V}$ of K by relatively open sets, let $\mathcal{W} = \{B \cap K : B \in \mathcal{B}$ and for some $V \in \mathcal{V}, B \cap K \subseteq V\}$. According to Theorem 3.2.8 of [11], some finite subcollection of $\mathcal{W}$ covers K. Hence K is compact so that the proof's first paragraph applies. □

Proposition 2.2 *Let X be a Hausdorff space that has a point-countable, T_1-point-separating open cover, i.e., an open cover $\mathcal{U}$ such that if $x \neq y$ are points of X, then some member of $\mathcal{U}$ contains x but not y. Then each countably compact subset of X is a compact, metrizable, G_δ-subset of X.*

Proof: Let C be a countably compact subset of X. According to Theorem 7.6 in [17], C must be compact. According to Miščenko's lemma (see page 242 of [14] or Theorem 7.4 of [17]), there are only a countable number of minimal finite open overs of C by members of $\mathcal{U}$. List them as $\{\mathcal{V}(n) : n \geq 1\}$ and let $W(n) = \bigcup \mathcal{V}(n)$. Then $C \subseteq \bigcap\{W(n) : n \geq 1\}$. Suppose $p \in X - C$. For each $q \in C$, there is a member $U(q) \in \mathcal{U}$ with $q \in U(q) \subseteq X - \{p\}$. Then some sub-collection of $\{U(q) : q \in C\}$ is a finite minimal cover of C by members of $\mathcal{U}$ and we thereby obtain one of the collections $\mathcal{V}(n)$ with $C \subseteq W(n) = \bigcup \mathcal{V}(n) \subseteq X - \{p\}$, as required. □

Recall that the space X has a *quasi-G_δ-diagonal* provided there is a sequence $\langle \mathcal{G}(n) \rangle$ of collections of open sets with the property that, given distinct points $x, y \in X$, there is some n with $x \in \mathrm{St}(x, \mathcal{G}(n)) \subseteq X - \{y\}$. (If each $\mathcal{G}(n)$ is also a cover of X, then X has a G_δ-diagonal.)

Proposition 2.3 *Suppose that the Hausdorff space X has a quasi-G_δ-diagonal. Then any countably compact subset of X is a compact metrizable G_δ-subset of X.*

Proof: First, consider the case where M is a compact, metrizable subspace of X. Let $\langle \mathcal{G}(n) \rangle$ be a quasi-G_δ-diagonal sequence for X. We may assume that $\mathcal{G}(1) = \{X\}$. Being a compact, metrizable subset of X, M is hereditarily Lindelöf so that there is a countable sub-collection $\mathcal{H}(n) \subseteq \mathcal{G}(n)$ that covers $M \cap (\bigcup \mathcal{G}(n))$. Let $\mathcal{H} = \bigcup\{\mathcal{H}(n) : n \geq 1\}$. Because $\mathcal{H}$ is countable, we may index it as $\mathcal{H} = \{H_i : i \geq 1\}$. For each $x \in M$ and $n \geq 1$, let $W(x,n) = \bigcap\{H_i : i \leq n \text{ and } x \in H_i\}$. Let $V(n) = \bigcup\{W(x,n) : x \in M\}$. Then $M \subseteq \bigcap\{V(n) : n \geq 1\}$. For contradiction, suppose that there is some point $z \in \bigcap\{V(n) : n \geq 1\} - M$. Choose points $x_i \in M$ with $z \in W(x_i, i)$. Because $x_i \in M$, there is a cluster point p of $\langle x_i \rangle$ in M. Because $p \neq z$, we may find an $N \geq 1$ such that $p \in \mathrm{St}(p, \mathcal{G}(N)) \subseteq X - \{z\}$. Because $p \in M \cap (\bigcup \mathcal{G}(N)) \subseteq M \cap (\bigcup \mathcal{H}(N))$, some $H \in \mathcal{H}(N)$ has $p \in H$. Then H appears somewhere in the listing of $\mathcal{H}$ given above, say $H = H_k$. Because p is a cluster point of $\langle x_i \rangle$ and $p \in H = H_k$, there is some $j > k$ with $x_j \in H_k$. But then we have $z \in W(x_j, j) \subseteq H_k \subseteq \mathrm{St}(p, \mathcal{G}(N)) \subseteq X - \{z\}$ and that is impossible. Hence M is a G_δ-set in X.

Next, consider the case where C is a countably compact subset of X. Chaber (Corollary 3.A.1 of [12]) has proved that any countably compact space with a quasi-G_δ-diagonal is compact, and Hodel (Corollary 3.6 in [21]) proved that any paracompact wd-space with a quasi-G_δ-diagonal (and hence any compact Hausdorff space with a quasi-G_δ-diagonal) must be metrizable. Consequently, the set C is a compact metrizable subspace of X, and now the first paragraph of the proof applies to show that C is a G_δ-subset of X. □

We next show that any compact subset of a space with a base of countable order (BCO) must be a G_δ-set. We will need a characterization of BCO-spaces given by Worrell and Wicke [28]:

Proposition 2.4 *A regular space X has a BCO if there is a sequence $\langle \mathcal{B}(n) \rangle$ of bases for the topology of X such that if $p \in B_{n+1} \subseteq B_n \in \mathcal{B}(n)$ for each $n \geq 1$, then $\{B_n : n \geq 1\}$ is a local base at p.* □

Proposition 2.5 *If X is regular and has a BCO, then any compact subset of X is a G_δ-subset of X.*

Proof: Let $\mathcal{B}(n)$ be a the sequence of bases for X given by Proposition 2.4, and suppose $K \subseteq X$ is compact. Then K inherits a BCO (Theorem 6.4 in [17]), and any compact Hausdorff space with a BCO is metrizable. We recursively define open covers $\mathcal{D}(n)$ of K as follows. Let $\mathcal{D}(1)$ be any finite minimal (= irreducible) cover of K by members of $\mathcal{B}(1)$. If $\mathcal{D}(n)$ is defined, let

$$\mathcal{C}(n+1) = \{B \in \mathcal{B}(n+1) : \mathrm{cl}(B) \subseteq C \text{ for some } C \in \mathcal{D}(n)\}.$$

Let $\mathcal{D}(n+1)$ be any finite minimal cover of K by members of $\mathcal{C}(n+1)$. Let $W(n) = \bigcup \mathcal{D}(n)$. Then $K \subseteq \bigcap\{W(n) : n \geq 1\}$. For contradiction, suppose there is a point $p \in \bigcap\{W(n) : n \geq 1\} - K$. Let $\mathcal{E}(n) = \{B \in \mathcal{D}(n) : p \in B\}$. Then $\mathcal{E}(n) \neq \emptyset$ for each n, and if $C \in \mathcal{E}(n+1)$ then we may choose some $\pi_{n+1}(C) \in \mathcal{D}(n)$ with $\mathrm{cl}(C) \subseteq \pi_{n+1}(C)$ for some $\pi_{n+1}(C) \in \mathcal{D}(n)$. Note that $p \in C \subseteq \pi_{n+1}(C)$ means that $\pi_{n+1}(C) \in \mathcal{E}(n)$. Consequently, if we use the bonding maps $\pi_{n+1} : \mathcal{E}(n+1) \to \mathcal{E}(n)$, we have an inverse system of non-empty compact Hausdorff spaces (namely the sets $\mathcal{E}(n)$ with the discrete topology) so that Theorem 3.2.13 of [14] allows us to choose a sequence $C(n) \in \mathcal{E}(n)$ with the property that $\pi_{n+1}(C(n+1)) = C(n)$, i.e., $\mathrm{cl}(C(n+1)) \subseteq C(n)$. Because each $\mathcal{D}(n)$ is a minimal cover of K, $C(n) \cap K \neq \emptyset$ for each n. Choose $x_n \in C(n) \cap K$. Because K is compact, the sequence $\langle x_n \rangle$ has some cluster point $q \in K$. Because $\mathrm{cl}(C_{n+1}) \subseteq C_n$ for each n we see that the cluster point q belongs to each C_n. But then the collection $\{C_n : n \geq 1\}$ is forced to be a base at both $p \notin K$ and at $q \in K$, and that is impossible. □

Unlike the situation in Propositions 2.1, 2.2 and 2.3, Proposition 2.5 cannot be proved for countably compact sets. For example, $X = [0, \omega_1)$ with its usual topology has a BCO, and its subspace $C = \{\lambda \in X : \lambda \text{ is a limit ordinal}\}$ is a countably compact subset that is not metrizable and is not a G_δ-subset of X.

Our next two results provide easy ways to recognize that a given space $(X, \mathcal{T})$ has compact sets G_δ. The first is a "local implies global" proposition.

Proposition 2.6 *Suppose $\mathcal{U}$ is an open cover of the Hausdorff space X such that for every $U \in \mathcal{U}$, each compact subset of U is a G_δ-subset of U. Then each compact subset of X is a G_δ-subset of X.*

Proof: Suppose C is a compact subset of X. Choose finitely many sets $U(i) \in \mathcal{U}$ such that $C \subseteq \bigcup\{U(i) : i \leq n\}$. Then $\{U(i) \cap C : i \leq n\}$ is a finite relatively open cover of the compact Hausdorff space C so that there are closed subsets $D(i) \subseteq C$ such that $D(i) \subseteq U(i)$ and $C = \bigcup\{D(i) : i \leq n\}$. Each set $D(i)$ is compact and $D(i) \subseteq U(i)$ so that $D(i)$ is known to be a G_δ-subset of $U(i)$ and therefore a G_δ-subset of X. Hence $C = \bigcup\{D(i) : i \leq n\}$ is also a G_δ-subset of X. □

In applications of the next proposition, it often happens that the weaker topology $\mathcal{S}$ mentioned in the proposition is metrizable.

Proposition 2.7 *Suppose $\mathcal{S} \subseteq \mathcal{T}$ are topologies on X with the property that each compact subset of $(X,\mathcal{S})$ is a G_δ in $(X,\mathcal{S})$. Then each compact subset of $(X,\mathcal{T})$ is a G_δ-set in $(X,\mathcal{T})$.*

Proof: Any compact subset of $(X,\mathcal{T})$ is also compact in $(X,\mathcal{S})$. □

3 The CSS-property in general spaces

The next lemma shows that large classes of topological spaces are CSS. It was proved by H. W. Martin in his dissertation [24] and announced in [25].

Lemma 3.1 *The Hausdorff space $(X,\mathcal{T})$ is CSS provided any one of the following holds:*

a) X is a $\sigma^\#$-space, i.e., X has a σ-closure-preserving collection $\mathcal{C}$ of closed sets with the property that if $x \neq y$ are points of X, then some $C \in \mathcal{C}$ has $x \in C$ and $y \notin C$;

b) X has a G^*_δ-diagonal, i.e., there is a sequence $\langle \mathcal{G}(n) \rangle$ of open covers of X such that for each $x \in X$, $\bigcap\{\mathrm{cl}\,(St(x,\mathcal{G}(n))) : n \geq 1\} = \{x\}$;

c) X has a topology $\mathcal{S} \subseteq \mathcal{T}$ such that $(X,\mathcal{S})$ is CSS. □

To what extent do properties (a) or (b) in Lemma 3.1 characterize CSS-spaces? As the next example shows, (b) does not.

Example 3.2 *The space M^* of Example 4.5 is CSS but does not have a G_δ-diagonal.*

Question 3.3 *Is there a regular CSS space that is not a $\sigma^\#$-space?*

Question 3.3 was posed by H. Martin in his thesis [24]. It will follow from Proposition 4.6 that there cannot be a LOTS counterexample, but there might be a GO-space example of the desired type.

The definition of CSS-spaces refers to arbitrary compact subsets of a space. As our next lemma shows, there is a characterization of CSS-spaces that refers only to convergent sequences.

Lemma 3.4 *A topological space is CSS if and only if for each $x \in X$ there is a sequence $\langle g(n,x) \rangle$ of open sets such that*

a) $g(n+1,x) \subseteq g(n,x)$ for each $n \geq 1$;

b) $\bigcap\{g(n,x) : n \geq 1\} = \{x\}$*;*

c) if a sequence $\langle x_n \rangle$ *of distinct points of* X *converges to some* $y \in X$*, then* $\bigcap\{g(n,x_n) : n \geq 1\} \subseteq \{y\}$.

Proof: Suppose X is CSS with CSS function $G(n,K)$, defined for each compact subset K of X as in the Introduction. For any $x \in X$, let $g(n,x) = G(n,\{x\})$. Then both a) and b) of this lemma are satisfied. We verify assertion c). Suppose $\langle x_n \rangle$ is a sequence of distinct points of X that converges to a point $y \in X$. Let $q \in \bigcap\{g(n,x_n) : n \geq 1\}$. With $K = \{x_n : n \geq 1\} \cup \{y\}$ we have $q \in \bigcap\{g(n,x_n) : n \geq 1\} \subseteq \bigcap\{G(n,K) : n \geq 1\} = K$ so that either $q = y$ as required, or else $q = x_M$ for some M. If $q = x_M$ define $z_i = x_{M+i}$. Then $\langle z_i \rangle$ converges to y. Writing $L = \{z_i : i \geq 1\} \cup \{y\}$, we know that $q \in \bigcap\{g(n,x_n) : n \geq 1\} \subseteq \bigcap\{g(n,x_n) : n > M\} = \bigcap\{g(i,z_i) : i \geq 1\} \subseteq \bigcap\{G(i,L) : i \geq 1\} = L$ so that $q \neq x_M$, a contradiction. Therefore $q = y$, as required.

Conversely, suppose X has a function $g(n,x)$ with properties (a), (b), and (c). For any compact set K, let $G(n,K) = \bigcup\{g(n,x) : x \in K\}$. Clearly parts (ii) and (iii) of the definition of a CSS space (see the Introduction) are satisfied. We verify (i). Suppose $q \in \bigcap\{G(n,K) : n \geq 1\}$. If $q \notin K$, then for each n, choose $x_n \in K$ with $q \in g(x_n,n)$. Because $x_n \neq q$ for each n, by passing to a subsequence if necessary, we may assume that the points x_n are pairwise distinct. Note that because K is compact and each point of K is a G_δ-set, the subspace K is first countable. Hence there is a subsequence $\langle x_{n(k)} \rangle$ that converges to some point $z \in K$. Consequently, $q \in \bigcap\{g(n(k),x_{n(k)}) : k \geq 1\} \subseteq \{z\} \subseteq K$, so $q \in K$. That contradiction establishes part (i) of the definition of a CSS structure and completes the proof. □

As noted in the previous section, “local implies global” for the property “every compact set is a G_δ-set.” Example 4.2 shows that the CSS property does not satisfy a “local implies global” theorem. However, in the presence of a suitable covering condition, locally CSS *does* imply CSS, as our next result shows. (We thank the referee for showing us how a result of Gruenhage and Yajima can be used to remove the assumption of normality from an earlier version of the theorem.)

Proposition 3.5 *Suppose X is submetacompact (=* θ*-refinable). If X is locally CSS. then X is CSS.*

Proof: Let $\mathcal{W} = \{W(\alpha) : \alpha \in A\}$ be a cover of X by open subspaces, each of which is CSS in its relative topology, and let $\{g_\alpha(n,x) : n \geq 1,\ x \in W(\alpha)\}$ be a CSS function for the subspace $W(\alpha)$ as described in Lemma 3.4.

Because X is submetacompact, Theorem 2.1 of [19] shows that there is a filter $\mathcal{F}$ of subsets of ω and a sequence $\langle \mathcal{U}(n) \rangle$ of open covers of X, each refining $\mathcal{W}$ with the property that for every $x \in X$, $\{n < \omega : \operatorname{ord}(x, \mathcal{U}(n)) < \omega\} \in \mathcal{F}$. (In fact, the same filter $\mathcal{F}$ works for all open covers of all submetacompact spaces.) For each fixed n choose a function $f_n : \mathcal{U}(n) \to A$ such that $U \subseteq W(f_n(U))$ whenever $U \in \mathcal{U}(n)$. For each $\alpha \in A$ define $V(n,\alpha) = \bigcup\{U \in \mathcal{U}(n) : f_n(U) = \alpha\}$. Then the collection $\mathcal{V}(n) = \{V(n,\alpha) : \alpha \in A\}$ is an open cover of X that has $V(n,\alpha) \subseteq W(\alpha)$. Furthermore, for any $x \in X$, if $\operatorname{ord}(x, \mathcal{U}(n)) < \omega$, then $\operatorname{ord}(x, \mathcal{V}(n)) < \omega$ so that $\{m < \omega : \operatorname{ord}(x, \mathcal{U}(m)) < \omega\} \subseteq \{m < \omega : \operatorname{ord}(x, \mathcal{V}(m)) < \omega\}$. Because the first of those two sets belongs to the filter $\mathcal{F}$, so does the second. In summary, we now have open covers $\mathcal{V}(n) = \{V(n,\alpha) : \alpha \in A\}$ such that:

a) $V(n,\alpha) \subseteq W(\alpha)$ for each $\alpha \in A$;

b) for each $x \in X$ there is some $n \geq 1$ such that the set $\{\alpha \in A : x \in V(n,\alpha)\}$ is finite;

c) for each $x \in X$, the set $\{n < \omega : \operatorname{ord}(x, \mathcal{V}(n)) < \omega\} \in \mathcal{F}$; and therefore

d) whenever $x, y \in X$, the set $\{n < \omega : \operatorname{ord}(x, \mathcal{V}(n)) < \omega\} \cap \{n < \omega : \operatorname{ord}(y, \mathcal{V}(n)) < \omega\}$ is a member of $\mathcal{F}$ and is therefore nonempty, so that for some $n < \omega$, both $\operatorname{ord}(x, \mathcal{V}(n))$ and $\operatorname{ord}(y, \mathcal{V}(n))$ are finite.

For each $x \in X$ and $n \geq 1$, let $A(x,n)$ be defined as follows. If $\mathrm{ord}(x, \mathcal{V}(n))$ is finite, let $A(x,n) = \{\alpha \in A : x \in V(n,\alpha)\}$, and if $\mathrm{ord}(x, \mathcal{V}(n))$ is infinite, choose any $\alpha \in A$ with $x \in V(n,\alpha)$ and let $A(x,n) = \{\alpha\}$. Now define

$$g'(n,x) = \bigcap\{g_\alpha(n,x) \cap V(n,\alpha) : \alpha \in \bigcup\{A(x,i) : i \leq n\}\}.$$

Each $g'(n,x)$ is open because $\bigcup\{A(x,i) : i \leq n\}$ is finite. Let $H(n,m) = \{y \in X : \ \mathrm{ord}(y, \mathcal{V}(n)) < m\}$. Each $H(n,m)$ is closed in X so that the set $h(n,x) = X - \bigcup\{H(i,m) : i,m \leq n \text{ and } x \notin H(i,m)\}$ is open and contains x. Now let $g(n,x) = g'(n,x) \cap h(n,x)$.

We will show that the function $g(x,n)$ has the properties described in Lemma 3.4. Property (a) is clear. To verify (b), fix $x \in X$ and $\alpha(x) \in A(x,1)$. Then $g(n,x) \subseteq g'(n,x) \subseteq g_{\alpha(x)}(n,x)$ for each $n \geq 1$ so that $x \in \bigcap\{g(n.x) : n \geq 1\} \subseteq \bigcap\{g_{\alpha(x)}(n,x) : n \geq 1\} = \{x\}$

To verify (c), suppose $\langle x_n \rangle$ is a sequence of distinct points of X that converges to a point $y \in X$. For contradiction, suppose there is some $z \in X - \{y\}$ with $z \in g(n,x_n)$ for each $n \geq 1$. According to (d) above, there is some integer k such that the collection $\mathcal{V}(k)$ has finite order at both y and z.

Let $m_0 = \mathrm{ord}(z, \mathcal{V}(k))$. Suppose $n \geq \max(m_0, k)$. We claim that the cover $\mathcal{V}(k)$ has finite order at x_n. If not, then $x_n \notin H(k,m_0)$ while $z \in H(k,m_0)$ so that

$$z \in g(n,x_n) \subseteq h(n,x_n) \subseteq X - \bigcup\{H(i,m) : i,m \leq n, x_n \notin H(i,m)\} \subseteq X - H(k,m_0) \subseteq X - \{z\}$$

and that is impossible.

Fix any $V(k,\alpha) \in \mathcal{V}(k)$ with $y \in V(k,\alpha)$. Because $\langle x_n \rangle$ converges to y, there is some N such that $x_n \in V(k,\alpha)$ for each $n \geq N$. From the previous paragraph, for all $n \geq \max(k,m_0)$, $\mathrm{ord}(x_n, \mathcal{V}(k))$ is finite. Then for each $n \geq \max(k,m_0,N)$ we have $\alpha \in A(x_n,k)$ so that $z \in g(n,x_n) \subseteq g'(n,x_n) \subseteq g_\alpha(n,x_n)$.

Recall that the function g_α satisfies the conditions described in Proposition 3.4 for the subspace $W(\alpha)$. Because the sequence $\langle x_n : n \geq \max(k,m_0,N) \rangle$ consists of points of $V(k,\alpha) \subseteq W(\alpha)$ and converges to $y \in V(k,\alpha) \subseteq W(\alpha)$ we know that $\bigcap\{g_\alpha(n,x_n) : n \geq \max(k,m_0,N)\} \subseteq \{y\}$. But we have $z \neq y$ and $z \in \bigcap\{g(n,x_n) : n \geq \max(k,m_0,N)\} \subseteq \bigcap\{g_\alpha(n,x_n) : n \geq \max(k,m_0,N)\} = \{y\}$ and that contradiction completes the proof. □

In the previous section, we showed that any compact subset of X is a G_δ-subset of X provided X has a $\delta\theta$-base, or has a point-countable, T_1-point-separating open cover, or has a BCO. None of these properties is enough to guarantee that X is CSS, as shown by Example 4.7 (a paracompact, monotonically normal space with a point-countable base that is not CSS) and by Example 4.2 (a monotonically normal space with a BCO that is not CSS).

In Proposition 2.3 we showed that every compact subset of X is a G_δ-subset of X provided X has a quasi-G_δ-diagonal. Given certain additional covering conditions, we can prove that such a space must be CSS:

Proposition 3.6 *Suppose X is a Hausdorff space and has quasi-G_δ-diagonal $\langle \mathcal{G}(n) \rangle$ such that $\bigcap \mathcal{C}$ is open whenever $\mathcal{C} \subseteq \mathcal{G}(n)$. Then X is CSS.*

Proof: We may assume that $\mathcal{G}(1) = \{X\}$. For each $n \geq 1$ and each $x \in X$, let

$$C(n,x) = \bigcap\{G \in \mathcal{G}(i) : i \leq n \text{ and } x \in G\}.$$

Each $C(n,x)$ is open and $C(n+1,x) \subseteq C(n,x)$. For any compact set $K \subseteq X$ let $V(n,K) = \bigcup\{C(n,x) : x \in K\}$. Clearly $V(n,K)$ is monotonic in both n and K, and $K \subseteq \bigcap_{n=1}^\infty V(n,K)$. Let $z \in \bigcap_{n=1}^\infty V(n,K)$

and suppose $z \notin K$. There are points $x_n \in K$ with $z \in C(n, x_n)$. Because K is compact, the sequence $\langle x_n \rangle$ has a cluster point $p \in K$. Then $p \neq z$ so there is some n with $p \in \mathrm{St}(p, \mathcal{G}(n)) \subseteq X - \{z\}$. There is some $m > n$ with $x_m \in St(p, \mathcal{G}(n))$. Choose $G_0 \in \mathcal{G}(n)$ with $p, x_m \in G_0$. But then we have

$$z \in C(m, x_m) \subseteq C(n, x_m) \subseteq G_0 \subseteq \mathrm{St}(p, \mathcal{G}(n)) \subseteq X - \{z\}$$

and that is impossible. Hence X is CSS. □

What kinds of spaces satisfy the hypothesis of Proposition 3.6? Proposition 3.6 applies to any space that has a quasi-G_δ-diagonal and is hereditarily metacompact or (more generally) has the property that any open collection $\mathcal{U}$ in X has a σ-Q refinement $\mathcal{V}$ that covers the set $\bigcup \mathcal{U}$. [1] For example, any space with a σ-point-finite base has this property.

Any quasi-developable space has a δθ base, so that Proposition 2.1 shows that any compact subset of a quasi-developable space is a G_δ-set. The proof requires so many arbitrary choices that we cannot see how to prove that any quasi-developable space is CSS. It was announced in [26] that that any T_3, quasi-developable space is CSS, but the details of some steps in the proof are not completely clear. Therefore we ask:

Question 3.7 *Is it true that each quasi-developable T_3-space is CSS?*

As noted in [26], an affirmative answer to Question 3.7 would yield an affirmative answer to an old problem posed by Fletcher and Lindgren in [16], namely *Is every quasi-developable β-space developable?*.

We close this section with a result on the role of the CSS property in metrization theory. It is clear that a compact Hausdorff space that is CSS must be semi-stratifiable and hence must be metrizable. As with many other metrization theorems for compact Hausdorff spaces, that result holds for countably compact spaces and extends to the much larger class of paracompact p-spaces, as the next result shows. (The result is essentially due to Martin [25], although he did not state or prove it in exactly the following way.)

Proposition 3.8 *Let X be a completely regular space. Then:*

a) *X is developable if and only if X is submetacompact (= θ-refinable), a p-space in the sense of Arhangel'skii, and CSS;*
b) *X is metrizable if and only if X is a paracompact p-space and is CSS.*
c) *if X is a countably compact CSS-space, then X is compact and metrizable,*

Proof: Clearly (a) implies (b). To prove the harder half of (a), we recall that any submetacompact p-space is a β-space (Theorem 7.8 in [17]) and that any CSS β-space is semi-stratifiable (Theorem 3 of [25]). Hence any CSS submetacompact p-space is a semi-stratifiable p-space and is, therefore, a Moore space (Corollary 5.12 in [17]). To prove (c), recall that any countably compact space is a β-space and combine Martin's theorem [25] that a CSS β-space is semistratifiable with Creede's theorem that a semistratifiable countably compact space is compact and metrizable. □

[1] The collection $\mathcal{V}$ is a σ-Q-collection if $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}(n)$ where $\bigcap \mathcal{C}$ is open for each $\mathcal{C} \subseteq \mathcal{V}(n)$. For example, every σ-point-finite open collection is σ-Q, and any open collection in a GO-space has a σ-Q-refinement.

4 The CSS property in ordered spaces

Proposition 4.1 *Let S be a stationary subset of a regular uncountable cardinal κ. Then, in its relative topology, S is not CSS. Hence any monotonically normal CSS space is hereditarily paracompact. In particular, any CSS GO-space is hereditarily paracompact.*

Proof: Theorem 4.1 of [15] shows that no stationary set in a regular uncountable cardinal can be CSS and deduces hereditary paracompactness for GO-spaces that are CSS. The more general result about monotonically normal spaces follows from that stationary set argument in the light of a theorem of Balogh and Rudin [3]. $\Box$

Example 4.2 *There is a LOTS with a BCO that is locally CSS but not CSS.*

Proof: Proposition 4.1 shows that the usual space $X = [0, \omega_1)$ of all countable ordinals is not CSS. However every point of X has a compact, metrizable neighborhood, so that X is locally CSS. (Contrast this behavior with Proposition 2.6.) Also note that because "local implies global" for the BCO property (see [28]), X has a BCO. $\Box$

It was announced in [15] that a result on $\mathcal{G}$-Souslin diagonals could be used to prove that any GO-space with a quasi-G_δ-diagonal must be CSS. Our next proposition provides a direct proof of that result. We begin with a lemma that may be of use in its own right.

Lemma 4.3 *Suppose $(X, \tau, <)$ is a GO space. Then X has a quasi-G_δ-diagonal if and only if there is a GO-topology σ on $(X, <)$ that is quasi-developable and has $\sigma \subseteq \tau$.*

Proof: Any quasi-developable space has a quasi-G_δ-diagonal, so that if there is a quasi-developable topology $\sigma \subseteq \tau$ on X, then (X, τ) has a quasi-G_δ-diagonal.

To prove the harder implication, suppose that the GO-space $(X, \tau, <)$ has a quasi-G_δ-diagonal structure $\langle \mathcal{G}(n) \rangle$ as defined above. A stationary set argument (see [15]) shows that X is hereditarily paracompact so for each n there is a σ-disjoint collection that refines $\mathcal{G}(n)$ and covers $\bigcup \mathcal{G}(n)$. Therefore, we may assume that each $\mathcal{G}(n)$ is a pairwise disjoint collection of convex sets. The collection $\mathcal{B}$ of all finite intersections of sets from $\mathcal{G} = \bigcup \{ \mathcal{G}(n) : n \geq 1 \}$ is σ-disjoint and is the base for some topology σ on X having $\sigma \subseteq \tau$. If we can show that (X, σ) is Hausdorff, then, members of $\mathcal{B}$ being convex in $(X, <)$, we will have the required GO-topology.

To show σ is a Hausdorff topology, suppose p and q are distinct points of X. We may suppose $p < q$. If the open interval (p, q) is the empty set, choose members $G_p, G_q \in \mathcal{G}$ with $p \in G_p \subseteq X - \{q\}$ and $q \in G_q \subseteq X - \{p\}$. Then convexity forces $G_p \subseteq (\leftarrow, q)$ and $G_q \subseteq (p, \rightarrow)$ so that $G_p \cap G_q \subseteq (\leftarrow, q) \cap (p, \rightarrow) = \emptyset$. If $(p, q) \neq \emptyset$, choose any $z \in (p, q)$ and find $G_p, G_q \in \mathcal{G}$ with $p \in G_p \subseteq X - \{z\}$ and $q \in G_q \subseteq X - \{z\}$. Convexity forces $G_p \subseteq (\leftarrow, z)$ and $G_q \subseteq (z, \rightarrow)$ so that $G_p \cap G_q = \emptyset$, as required. $\Box$

Proposition 4.4 *Let X be a GO space. If X has a quasi-G_δ-diagonal, then X is CSS.*

Proof: In the light of Lemma 4.3, it is enough to prove that a quasi-developable GO-space is CSS. If X is a quasi-developable GO-space, then X has a σ-disjoint base (see [5] or [23]) so that Proposition 3.6 completes the proof. $\Box$

The space S^* in the next example shows that in the category of GO-spaces, the CSS property is not characterized by the existence of a quasi-G_δ-diagonal.

Example 4.5 *Each of the following linearly ordered topological spaces is CSS and is a $\sigma^{\#}$-space.*

LOTS	Additional Properties
M^*	CSS, quasi-developable, no G_δ-diagonal, not perfect, and contains the Michael line as a closed subspace
S^*	CSS, not quasi-developable, not perfect, no quasi-G_δ-diagonal, and contains the Sorgenfrey line as a closed subspace
the Big Bush	CSS, point-countable base, not perfect, no quasi-G_δ-diagonal, not quasi-developable

In the above table, $M^* = (\mathbb{R}\times\{0\})\cup(\mathbb{P}\times\mathbb{Z})$ and $S^* = \mathbb{R}\times\{n\in\mathbb{Z} : n\le 0\}$, both with the lexicographic order and the associated open-interval topology. It is easy to check (using Lemma 3.4) that each of these spaces is CSS. See [23] for the additional properties of M^* and S^*. The Big Bush is the lexicographically ordered set $B = \bigcup\{B_\lambda : \lambda < \omega_1 \text{ is a limit}\}$ where B_λ is the set of all functions $f : [0,\lambda]\to\mathbb{R}$ with the property that $f(\alpha)\in\mathbb{P}$ for each $\alpha<\lambda$ while $f(\lambda)\in\mathbb{Q}$. The additional properties of B are verified in [6]. To see that B is CSS, let $f\in B_\lambda$ and let

$$g(n,f) = \{h\in B : [0,\lambda]\subseteq \mathrm{dom}(h) \text{ and } h(\alpha)=f(\alpha) \text{ for all } \alpha<\lambda \text{ and } |f(\lambda)-h(\lambda)| < \frac{1}{n}\}.$$

Suppose that f_n is a sequence in B that converges to f and that $k\in g(n,f_n)$ for each $n\ge 1$. Write $\mathrm{dom}(f)=[0,\lambda]$ and $\mathrm{dom}(f_n)=[0,\lambda_n]$. We may assume that $f_n\in g(n,f)$ for all n. Hence $\lambda\le\lambda_n$, $f_n(\alpha)=f(\alpha)$ for all $\alpha<\lambda$, and the real-number sequence $f_n(\lambda)$ converges to $f(\lambda)$. If $\lambda<\lambda_n$ occurs for infinitely many values of n, then for all such n we have $k(\alpha)=f_n(\alpha)$ so that the real sequence $\langle f_n(\lambda)\rangle$ has a subsequence with a constant irrational value, showing that $f_n(\lambda)$ cannot converge to the rational number $f(\lambda)$. Therefore, we may assume that $\lambda_n=\lambda$ for all n. If $k(\lambda)\ne f(\lambda)$ find a positive integer m with $\frac{2}{m} < |f(\lambda)-k(\lambda)|$. Because $f_n(\lambda)$ converges to $f(\lambda)$ there is an integer $n>m$ with $|f_n(\lambda)-f(\lambda)|<\frac{1}{m}$. Because $k\in g(n,f_n)$ we have

$$|f(\lambda)-k(\lambda)| \le |f(\lambda)-f_n(\lambda)| + |f_n(\lambda)-k(\lambda)| < \frac{2}{m} < |k(\lambda)-f(\lambda)|$$

and that is impossible. Hence $k(\lambda)=f(\lambda)\in\mathbb{Q}$ so that $k=f$, as required to prove that B is CSS.

To see that neither the Big Bush nor S^* has a quasi-G_δ-diagonal, note that in any LOTS, the existence of a quasi-G_δ-diagonal is equivalent to quasi-developability. But the Big Bush is not quasi-developable (see [4], [6]) and S^* is not quasi-developable because it contains a copy of the Sorgenfrey line (a perfect, non-metrizable GO-space). To see that each of the three spaces is a $\sigma^{\#}$-space, we may apply Proposition 4.6, because each space is a LOTS (and not merely a GO-space). □

Proposition 4.6 *For any LOTS X, the following are equivalent:*

a) X is CSS;

b) X is a γ-space;

c) the topology of X can be generated by a non-Archimedean quasi-metric;

d) the topology of X can be generated by a quasi-metric;

e) X is a $\sigma^{\#}$-space.

Proof: The equivalence of (b), (c), and (d) in any GO-space was proved by Kofner in [22]. The equivalence of (e) and (c) is easy to prove in any LOTS, and was announced in [15], as was the equivalence of (a) and (b) in any LOTS. □

Example 4.7 *There is a LOTS with a point-countable base that is not CSS.*

Proof: In [18], Gruenhage constructed a LOTS with a point-countable base that is not quasi-metrizable. In the light of Proposition 4.6, that space cannot be CSS. □

Question 4.8 *Is there a GO-space that is CSS but not a $\sigma^{\#}$-space?*

Note that Question 4.8 is a special case of Question 3.3. Also note that such a space could not be a LOTS, in the light of Proposition 4.6. Finally, as can be seen from Example 4.12, the standard X^* construction for a GO-space X may fail to preserve the CSS property.

None of the three spaces in Example 4.5 are perfect and none have a G_δ-diagonal. This is no accident because for a very large class of perfect GO-spaces, being CSS is equivalent to having a G_δ-diagonal, as our next theorem shows. Recall that any GO-space having a σ-closed-discrete dense subset is perfect [7] and that there is no known ZFC example of a perfect GO-space that does not have a σ-closed-discrete dense set. (See [27] and [10] for related material. More recent work [9] has pointed out that there cannot be any ZFC example of a perfect GO-space that has local density ω_1 and does not have a σ-closed-discrete dense subset.) We begin with a lemma.

Lemma 4.9 *Suppose X is a GO-space.*

a) If X has a σ-closed-discrete dense subset, then there is a sequence $\langle \mathcal{V}(n)\rangle$ of convex open covers of X with the property that for each $p \in X$, $\bigcap\{St(p,\mathcal{V}(n)) : n \geq 1\}$ is a convex set with at most two points.

b) If $\langle \mathcal{V}(n)\rangle$ is any sequence of open covers of X as described in (a) and if the set K of all points such that $|\bigcap\{St(p,\mathcal{V}(n)) : n \geq 1\}| = 2$ is σ-closed-discrete in X, then X has a G_δ-diagonal.

c) If X is a perfect GO-space and if the set K in (b) is σ-relatively-discrete, then X has a G_δ-diagonal.

Proof: Assertion (a) is part of Theorem 2.1 of [7]. Assertion (c) follows directly from assertion (b) in any perfect GO-space. To prove (b), write $K = \bigcup\{K_n : n \geq 1\}$ where each K_n is closed and discrete in X. We may assume $K_n \subseteq K_{n+1}$ for all n. Use the fact that X is collectionwise normal to find a discrete collection $\mathcal{U}(n) = \{U(n,x) : x \in K_n\}$ of open sets with $x \in U(n,x)$. Define $\mathcal{W}(n) = \{V - K_n : V \in \mathcal{V}(n)\} \cup \{U(n,x) : x \in K_n\}$. Then $\mathcal{W}(n)$ is a G_δ-diagonal sequence of open covers of X, as required. □

Theorem 4.10 *Suppose $(X,\mathcal{S},<)$ is a GO-space with a σ-closed-discrete dense subset. Then X is CSS if and only is X has a G_δ-diagonal.*

Proof: Half of the proof follows from Proposition 4.4. For the converse, suppose X is CSS. Having a σ-closed-discrete dense subset, X is perfect and paracompact. We will begin by reducing the problem to a special case. Let $G = \bigcup\{U \in \mathcal{S} : U$ has a G_δ-diagonal in its relative topology $\}$. Because X is hereditarily paracompact, the subspace G has a G_δ-diagonal for its relative topology. Because X is perfect, G is a union of countably many closed G_δ-subsets, each with a G_δ-diagonal in its relative topology. Let $Y = X - G$. If we can show that Y has a G_δ-diagonal for its relative topology, then X is seen to be a countable union of closed, G_δ-subspaces, each with a G_δ-diagonal in its subspace topology, and that would be enough to guarantee that X has a G_δ-diagonal.

Note that the subspace $Y = X - G$ is CSS, has a σ-closed-discrete dense subspace (see [8]) and (most important) has no isolated points. To see that Y has no isolated points, suppose there is a point $p \in Y$ and an open set $V \subseteq X$ such that $V \cap Y = \{p\}$. Then $V - \{p\} \subseteq G$ so that $V - \{p\}$ has a G_δ-diagonal for its relative topology. Because $\{p\}$ is a G_δ-subset of X, it follows that the entire set V has a G_δ-diagonal for its relative topology, whence $V \subseteq G$ and $V \cap Y = \emptyset$. Therefore Y has no isolated points. Henceforth, we consider only the GO-space Y with its topology and ordering inherited from X. (Alternatively, the reader could assume that $Y = X$ so that X itself has no isolated points.)

Let $J = \{p \in Y : \exists q \in Y - \{p\}$ with $\mathrm{conv}\{p,q\} = \{p,q\}\}$, where $\mathrm{conv}\{p,q\}$ denotes the order-convex hull of the set $\{p,q\}$ in Y. Thus J is the set of jump-points in Y. Apply Lemma 4.9 to find convex open covers $\mathcal{V}(n)$ of Y with the property that whenever $p \in Y$ has $|\bigcap\{St(p,\mathcal{V}(n)) : n \geq 1\}| > 1$ then $p \in J$. In the light of Lemma 4.9 it will be enough to show that the set J is σ-relatively discrete in Y. For contradiction, suppose that J is *not* σ-relatively-discrete in Y.

Let $J_0 = \{p \in J : \exists q > p$ with $[p,q] = \{p,q\}\}$ and let $J_1 = J - J_0$. Each $p \in J_0$ has an immediate successor that we will call p^+ in $J - J_0 = J_1$ because Y has no isolated points. Because J is not σ-relatively-discrete, at least one of J_0 and J_1 must fail to be σ-relatively discrete. We will need more, namely that J_0 is not σ-relatively-discrete, and that follows from the stronger assertion (to be needed later) in the next claim.

Claim 1: Let $C \subseteq J_0$ and let $D = \{p^+ : p \in C\}$. Then both of the sets C and D are σ-relatively-discrete if and only if one of C and D is σ-relatively-discrete. Half of Claim 1 is trivial. To prove the non-trivial half, suppose C is σ-relatively-discrete. Because Y is perfect, it follows that C can be written as $C = \bigcup\{C_n : n \geq 1\}$ where each C_n is a closed and discrete subset of Y. Let $D_n = \{p^+ : p \in C_n\}$. We claim that D_n is relatively discrete. If not, then there is a sequence $q_k \in D_n$ that converges to a point $q \in D_n$. We may assume that the points q_k are distinct. For each k, find $p_k \in C_n$ with $q_k = p_k^+$. Note that $[q,\rightarrow)$ is an open set in Y so we may assume that $q < q_k$ for all k. But then we must have $q < p_k < q_k$ so that the sequence p_k must also converge to q. But that is impossible because the points p_k were chosen from the closed, discrete subset C_n. Thus D_n is relatively discrete. Hence $D = \bigcup\{D_n : n \geq 1\}$ is σ-relatively-discrete, as claimed. An analogous argument shows that if D is σ-relatively-discrete, then so is C.

Next we collapse the jumps of Y. For $a,b \in Y$, define $a \sim b$ to mean that either $a = b$ of else $\mathrm{conv}\{a,b\} = \{a,b\}$ (i.e., a and b are the endpoints of a jump of Y). Because Y has no isolated points, $\sim$ is an equivalence relation on Y. Let $Z = Y/\sim$ and let $\mathcal{T}$ be the quotient topology and $\prec$ the induced ordering of Z. Then by Proposition 1.2.3 of [29] $(Z,\mathcal{T},\prec)$ is a GO-space and the natural projection map $\pi : Y \rightarrow Z$ has the property that $y_1 \leq y_2$ in Y implies $\pi(y_1) \preceq \pi(y_2)$ in Z. Consequently, the GO-space Z has a σ-closed-discrete dense set and it is easy to verify that there are no jumps in the set $(Z,\prec)$. Applying Lemma 4.9, we see that the GO-space $(Z,\mathcal{T})$ has a G_δ-diagonal. It follows from a theorem of Przymusinski (quoted in [1]) that there is a metrizable topology $\mathcal{M}$ on Z such that $\mathcal{M} \subseteq \mathcal{T}$ and such that $(Z,\mathcal{M},\prec)$ is a GO-space. Let d be a metric on Z that is compatible with $\mathcal{M}$.

The subspace $(Y,\mathcal{S}_Y)$ is CSS because the CSS property is hereditary. Therefore we can find a CSS function $g(n,y)$ for Y and we may assume that each $g(n,y)$ is convex, that if $[y,\rightarrow) \in \mathcal{S}_Y$ (respectively if $(\leftarrow,y] \in \mathcal{S}_Y$) then $g(n,y) \subseteq [y,\rightarrow)$ (respectively $g(n,y) \subseteq (\leftarrow,y]$), and that $\{g(n,y) : n \geq 1\}$ is a neighborhood base at y.

Claim 2: It cannot happen that for some $y \in J_0$ there are points $x_n \in g(n,y)$ with the property that $g(n,x_n) \not\subseteq (\leftarrow,y]$. For suppose that the points y and x_n exist. Then $\langle x_n \rangle$ converges to y. Also the convexity of $g(n,x_n)$ combines with $g(n,x_n) \not\subseteq (\leftarrow,y]$ to show that y^+, the immediate successor of y in Y, belongs to each $g(n,x_n)$. That contradicts $\bigcap\{g(n,x_n) : n \geq 1\} \subseteq \{y\}$ (see Lemma 3.4). Therefore Claim 2 is established and we conclude

$$(**) \quad \text{for each } y \in J_0,\ \exists m = m(y) \text{ such that if } x \in g(m,y) \text{ then } g(m,x) \subseteq (\leftarrow,y].$$

For each positive integer r, define $C(r) = \{y \in J_0 : m(y) = r\}$. Because J_0 is not σ-relatively-discrete there is some r_0 such that $C(r_0)$ is not σ-relatively-discrete. For future reference, let us record that $y \in C(r_0)$ if and only if

$$(***) \quad x \in g(r_0,y) \Rightarrow g(r_0,x) \subseteq (\leftarrow,y].$$

Suppose $y \in J_0$ and $u \in Y$ has $u < y$. Then in the quotient space Z we have $\pi(u) \prec \pi(y)$ so that $(\pi(u),\rightarrow)$ is an open set in the metric GO-topology $\mathcal{M}$. Hence there is an $\varepsilon > 0$ such that, if $B_d(\pi(y),\varepsilon)$ denotes the ε-ball with respect to the metric d that was chosen to be compatible with $\mathcal{M}$, then $B_d(\pi(y),\varepsilon) \cap (\leftarrow,\pi(y)]$ has the property that if $v \in Y$ has $v < y$ and $\pi(v) \in B_d(\pi(y),\varepsilon)$ then $v \in (u,y]$. In particular, if $y \in C(r_0)$ there is some positive integer $n = n(y)$ such that if $v < y$ has $\pi(v) \in B_d(\pi(y),\frac{1}{n})$ then $v \in g(r_0,y)$.

For each integer $s \geq 1$ let $C(r_0,s) = \{y \in C(r_0) : n(y) = s\}$. Because $C(r_0)$ is the union of all the sets $C(r_0,s)$ and because $C(r_0)$ is not σ-relatively-discrete, there must exist an integer s_0 such that the set $C(r_0,s_0)$ is not σ-relatively-discrete. For future reference we record the key property of the set $C(r_0,s_0)$, namely

$$(****) \quad \text{if } y \in C(r_0,s_0) \text{ and } v < y \text{ has } \pi(v) \in B_d(\pi(y),\frac{1}{s_0}) \text{ then } v \in g(r_0,y).$$

Let $D(r_0,s_0) = \{p^+ : p \in C(r_0,s_0)\}$. In the light of Claim 1, $D(r_0,s_0)$ cannot be relatively discrete, so there must be a sequence of distinct points $q_i \in D(r_0,s_0)$ that converges to the point $q \in D(r_0,s_0)$. Because the set $[q,\rightarrow)$ is open in Y, we may assume that $q < q_i$ for each $i \geq 1$. Because no point of Y is isolated in Y, each set $[q,q_i)$ must be infinite. Consequently, the fact that $q_i = p_i^+$ for some $p_i \in C(r_0,s_0)$ yields $q < p_i < q_i$ for each i and therefore the sequence $\langle p_i \rangle$ converges to q in Y.

Projecting into the quotient space $(Z,\mathcal{T})$ we see that $\langle \pi(p_i) \rangle$ converges to $\pi(q)$. Because $\mathcal{M} \subseteq \mathcal{T}$ we know that $\langle \pi(p_i) \rangle$ converges to $\pi(q)$ in the metric space $(Z,\mathcal{M})$. Therefore we may assume that $d(\pi(q),\pi(p_i)) < \frac{1}{s_0}$ for each $i \geq 1$. Because $p_i \in C(r_0,s_0) \subseteq C(r_0)$ and $q < p_i$ in Y, it now follows from (****) that $q \in g(r_0,p_i)$ for each $i \geq 1$. But then $p_i \in C(r_0)$ forces $g(r_0,q) \subseteq (\leftarrow,p_i]$ for each $i \geq 1$ so that $g(r_0,q) \subseteq (\leftarrow,q]$ because $q = \inf\{p_i : i \geq 1\}$. But that is impossible because $g(r_0,q)$ is a neighborhood of q while $(\leftarrow,q]$ is not (because Y has no isolated points). That contradiction completes the proof that the subset J of jump points of Y must be σ-relatively-discrete and, in the light of Lemma 4.9, that is enough to show that Y has a G_δ-diagonal. □

Example 4.11 *If there is a Souslin line, then there is a perfect GO space that is CSS but does not have a quasi-G_δ-diagonal, showing that the existence of a σ-closed-discrete dense subset is a necessary part of the proof of Theorem 4.10.*

Proof: In [20], R. W. Heath showed that if there is a Souslin line (something that is undecidable in ZFC) then there is a quasi-metrizable Souslin line. In the light of Proposition 4.6, Heath's space is CSS. Because Heath's space is a Souslin line, it cannot have a G_δ-diagonal, or even a quasi-G_δ-diagonal. □

It is known that any GO-space X embeds as a closed subspace of a LOTS X^* in a canonical way, and that for many topological properties P, if X has P then so does X^* [23]. Our next example shows that being a CSS space, and being a $\sigma^\#$-space, are not properties of that type.

Example 4.12 *There is a GO space X that is both CSS and a $\sigma^{\#}$-space, and yet the LOTS X^* is neither.*

Proof: Let X be the GO-space constructed on $\mathbb{R}$ by making $[x,\rightarrow)$ open for each $x \in \mathbb{P}$ and $(\leftarrow,q]$ open for each $q \in \mathbb{Q}$. Then X is a separable GO space with a weaker metrizable topology, so that X is both CSS and a $\sigma^{\#}$-space. The LOTS extension of X is

$$X^* = (\mathbb{P} \times \{n \in \mathbb{Z} : n \leq 0\}) \cup (\mathbb{Q} \times \{n \in \mathbb{Z} : n \geq 0\})$$

with the lexicographic ordering. For contradiction, suppose X^* is CSS and that $g(n,(x,i))$ is a CSS function for X^* as in Lemma 3.4. We may assume that each $g(n,(x,i))$ is convex. If there is some $x \in \mathbb{P}$ such that for all $n \geq 1$ some $q_n \in [x, x+\frac{1}{n})$ has $g(n,(q_n,0)) \cap ((x,0),\rightarrow) \neq \emptyset$, then $(x,1) \in X^*$ because $x \in \mathbb{P}$ so that convexity of $g(n,(q_n,0))$ gives

$$(x,1) \in \bigcap\{g(n,(q_n,0)) : n \geq 1\} \subseteq \{(x,0)\}$$

and that contradicts Lemma 3.4 because $(q_n,0)$ converges to $(x,0)$.

Therefore, for each $x \in \mathbb{P}$ there is some $n = n(x)$ such that if $q \in \mathbb{Q} \cap [x, x+\frac{1}{n})$ then $g(n,(q,0)) \subseteq [(x,0),\rightarrow)$. Let $P(k) = \{x \in \mathbb{P} : n(x) = k\}$. Baire Category theory yields an open interval $(a,b) \subseteq \mathbb{R}$ and an integer k_0 such that $P(k_0)$ is dense in (a,b). Choose any $q \in \mathbb{Q} \cap (a,b)$ and, for $i \geq k_0$, choose a point $x_i \in (q-\frac{1}{i}, q) \cap P(k_0)$. Then we have $g(k_0,(q,0)) \subseteq [(x_i,0),\rightarrow)$ for each $i \geq k_0$ so that $g(k_0,(q,0)) \subseteq [(q,0),\rightarrow)$. But that is impossible because the latter set is not a neighborhood of $(q,0)$ in X^*. Hence X^* is not CSS. Because any $\sigma^{\#}$-space is CSS (see 3.1), it follows that X^* is not a $\sigma^{\#}$-space.

We remark that there is a Lindelöf example of this type: let B and C be complementary Bernstein sets in $\mathbb{R}$, and make a GO-space Y by requiring that $(\leftarrow,x]$ open for each $x \in B$, and $[x,\rightarrow)$ open for each $x \in C$. The resulting GO-space Y is Lindelöf, CSS, and a $\sigma^{\#}$-space, and the LOTS extension Y^* is Lindelöf but neither CSS nor a $\sigma^{\#}$-space. □

References

[1] Alster, K., Subparacompactness in Cartesian products of generalized ordered spaces, *Fundamenta Math.* 87(1975), 7-28.

[2] Aull, C., A generalization of a theorem of Aquaro, *Bull. Australian Math. Soc.* 9(1973), 105-108.

[3] Balogh, Z., and Rudin, M., Monotone normality, *Topology and its Applications* 47(1992), 115-127.

[4] Bennett, H., Quasi-developable spaces, Ph.D. Thesis, Arizona State University, Tempe, AZ, 1968.

[5] Bennett, H., On quasi-developable spaces. *General Topology and Appl.* 1(1971), 253–262.

[6] Bennett, H., Point-countability in ordered spaces, *Proc. Amer. Math. Soc.* 28(1971), 598-606.

[7] Bennett, H., Heath, R., and Lutzer, D., GO-spaces with σ-closed-discrete dense subspaces, *Proc. Amer. Math. Soc.* 129(2000), 931-939.

[8] Bennett, H. Lutzer, D., and Purisch, S., On dense subspaces of generalized ordered spaces, *Topology and its Applications* 93(1999), 191-205.

[9] Bennett, H., and Lutzer, D., On a question of Maarten Maurice, *Topology Appl.*, to appear.

[10] Bennett, H., and Lutzer, D., Recent developments in the topology of ordered spaces, in *Recent Progress in General Topology* ed. by. M. Hušek and J. van Mill, Elsevier, Amsterdam, 2002.

[11] Burke, D., and Lutzer, D., Recent advances in the theory of generalized metric spaces, pp. 1-70 in *Topology (Proc. Ninth Annual Spring Topology Conference, Memphis State Univ., Memphis, Tenn., 1975*, ed. by S. Franklin and B. Thomas, *Lecture Notes in Pure and Appl. Math.*, Vol. 24, Dekker, New York, 1976.

[12] Chaber, J., Conditions which imply compactness in countably compact spaces, *Bull. Acad. Polon. Sci., Ser. Sci. Math., Astronom., Phy.* 24(1976), 993-998.

[13] Creede, G., Concerning semi-stratifiable spaces, *Pacific J. Math.* 32(1970) 47-54.

[14] Engelking, R., *General Topology*, Heldermann-Verlag, Berlin, 1989.

[15] Engelking, R., and Lutzer, D., Paracompactness in ordered spaces, *Fundamenta Math.* 94(1976), 49-58.

[16] Fletcher, P., and Lindgren, W., On $w\Delta$-spaces, $w\sigma$-spaces, and $\Sigma^{\#}$-spaces, *Pacific J. Math.* 71(1977), 419-428.

[17] Gruenhage, G., Generalized metric spaces, in *Handbook of Set Theoretic Topology* ed. by K. Kunen and J. Vaughan, North Holland, Amsterdam, 1984.

[18] Gruenhage, G. A note on quasi-metrizability, *Canadian J. Math.* 29(1977), 360-366.

[19] Gruenhage, G., and Yajima, Y., A filter property of submetacompactness and its applications to products, *Topology Appl.* 36(1990), 43-55.

[20] Heath, R.W., A construction of a quasi-metric Souslin space with a point-countable base, pp. 219-224 in *Set-Theoretic Topology*, ed by G.M Reed, Academic Press, New York, 1977.

[21] Hodel, R., Metrizability of topological spaces, *Pacific J. Math* 55(1974), 441-458.

[22] Kofner, J., Transitivity and the γ-space conjecture in ordered spaces, *Proc. Amer. Math. Soc.* 81(1981), 629-635.

[23] Lutzer, D., On generalized ordered spaces, *Dissertationes Math.* 89(1971).

[24] Martin, H., Metrization and submetrization of topological spaces, Ph.D. Thesis, University of Pittsburgh, Pittsburgh, PA, 1973.

[25] Martin, H., Metrizability of M-spaces, *Canad. J. Math.* 4(1973), 840-841.

[26] Mohamad, A., A note on developability and metrizability, *New Zealand J. Math.* 32(2003), 67-71.

[27] Qiao,Y-Q., and Tall, F., Perfectly normal non-metrizable non-archimedean spaces are generalized Souslin lines, *Proc. Amer. Math. Soc.* 131(2003), 3929-3636.

[28] Worrell, J., and Wicke, H., Characterizations of developable topological spaces, Canad. J. Math. 17(1965), 820-830.

[29] van Wouwe, J., *GO-spaces and Generalizations of Metrizability*, Mathematical Centre Tracts No. 104, Mathematical Center, Amsterdam, 1979.