Draft of Jan 4, 2013

Choban Operators in Generalized Ordered Spaces

by

Harold Bennett[1], Dennis Burke[2], and David Lutzer[3]

Abstract: In this paper we investigate generalized ordered (GO) spaces that have a flexible diagonal in the sense of Arhangel'skii [2]. Spaces with a flexible diagonal are generalizations of topological groups and include spaces that are continuously homogeneous, Choban spaces, and rotoid spaces. We prove some some paracompactness and metrization theorems for such spaces and construct examples of generalized ordered spaces that clarify how the types of spaces with a flexible diagonal are interrelated. We show, for example, that any GO Choban space is hereditarily paracompact, that any continuously homogeneous, first-countable GO-space is metrizable, that the space of real numbers is the only non-degenerate connected LOTS that is a Choban space, and that the Sorgenfrey line and the Michael line are Choban spaces. We extend some results of Arhangel'skii and pose a family of questions.

MathSciNet classification numbers Primary = 54H11; secondary = 54F05, 54B05

1 Introduction

As outlined in Arhangel'skii's paper [2] on flexible diagonals, recent research has shown that it is useful to generalize certain properties of topological groups to the class of topological spaces. In this paper we study several of these generalizations in the class of generalized ordered spaces, focusing primarily on Choban spaces. We extend some results from [2] and pose a family of questions.

A space X is a *Choban space* ([9], [2]) provided there is a continuous function $h : X^2 \to X$ (called a *Choban operator*) that satisfies:

a-1) there is a point $e \in X$ with $h(x, x) = e$ for all $x \in X$ (we will say that h maps the diagonal Δ of X^2 to e); and

a-2) for each $x \in X$ the function h maps the subspace $Vert(x) = \{x\} \times X$ of X^2 onto X in a one-to-one way.

As Arhangel'skii points out in [2], Choban operators are a topological generalization of the function $h(a, b) = a * b^{-1}$ in a topological group $(G, *)$. Each $Vert(x)$ in the above definition is a copy of X inside of X^2, and in many of our results, it will be convenient to think of $Vert(x)$ as a "vertical line" because X will be a kind of line called a generalized ordered space.

Recall that a *linearly ordered topological space* (LOTS) is a triple $(X, <, \mathcal{I})$ where $<$ is a linear ordering of X and $\mathcal{I}$ is the open-interval topology of that ordering. A *generalized ordered space* (GO-space) is a triple $(X, <, \mathcal{T})$ where $<$ is a linear ordering of X and $\mathcal{T}$ is a Hausdorff topology on X that has a base of order-convex sets[4]. E. Cech proved that X is a GO-space if and only if X is homeomorphic to a subspace

[1]Mathematics Department, Texas Tech University, Lubbock, TX 79409, harold.bennett@ttu.edu
[2]Mathematics Department, Miami University, Oxford, OH 45156, burkedk@muohio.edu
[3]Mathematics Department, College of William and Mary, Williamsburg, VA 23187, lutzer@math.wm.edu

[4]A subset $C \subset X$ is *order-convex* if $x \le y \le z$ and $\{x, z\} \subseteq C$ implies $y \in C$.

of some LOTS. The most famous GO-spaces are the Sorgenfrey line and the Michael line. These spaces, and their subspaces, have proved to be fundamental tools in the study of product spaces.

Our paper is organized as follows. Section 2 shows how some of the recent generalizations of topological groups (the rectifiable, continuously homogeneous, Choban, and rotoid spaces described in [2]) are related and uses GO examples to clarify the relations between such spaces. Section 3 presents some results about Choban spaces in general. For example, we note that a first-countable Choban space has a G_δ-diagonal and we use GO examples to show that there is a Choban space $X \times Y$ such that X is not a Choban space. Section 4 describes the role of some recent generalizations of topological groups in the context of GO-spaces. We prove that any GO-space with a Choban operator is hereditarily paracompact and that a first-countable LOTS with a Choban operator is metrizable. That metrization theorem fails if we consider GO-spaces rather than LOTS. However, when we consider the stronger property of continuous homogeneity, we prove that a continuously homogeneous first-countable GO space is metrizable. We also show that a perfect GO-space with a Choban operator must have a σ-closed-discrete dense set, something that is undecidable in ZFC without the Choban operator [8], and that a non-degenerate connected LOTS with a Choban operator must be homeomorphic to $\mathbb{R}$. Sections 5 through 8 are devoted to constructing examples. In Section 5 we discuss which subspaces of the real line $\mathbb{R}$ are Choban spaces and we show that there are Bernstein sets B_1, B_2 of $\mathbb{R}$ such that B_1 has a Choban operator and, under a mild set-theoretic hypothesis, B_2 does not. In Sections 7 and 8 we show that the Sorgenfrey line and the Michael line are both Choban spaces.

We want to thank the referee for suggestions that improved our paper.

Special notation The sets of real, rational, and irrational numbers will be denoted by $\mathbb{R}, \mathbb{Q}$, and $\mathbb{P}$, and $\mathbb{N}$ denotes the set of positive integers. The cardinal number 2^ω is denoted by $\mathfrak{c}$. For any x in any linearly ordered set X, the *cofinality of x in X*, denoted $cf(x)$, is the least cardinal κ such that the set $(\leftarrow, x)$ has a cofinal subset of size κ. Note that if x has an immediate predecessor in X, then $cf(x) = 1$. The *coinitiality of x in X*, denoted $ci(x)$, is analogously defined.

2 Relations among recent generalizations of topological groups

The Choban spaces defined above are just one of several recent generalizations of topological groups. In this section, we remind readers of the definitions of rectifiable, continuously homogeneous, and (strong) rotoid spaces, and use results from this paper and from [6] to show how the various generalizations are related.

Rectifiable spaces were introduced by Shapirovskii [22] and later studied by M. Choban in [10], [11], [12] and Uspenskii [23] and A. Gulko [18]. The following characterization of rectifiable spaces is given by Arhangel'skii in Proposition 8.12 of [2]: a space X is *rectifiable* if there is a special point $e \in X$ and a homeomorphism H from X^2 onto itself satisfying:

- b-1) $H(x, e) = (x, x)$ for each $x \in X$;
- b-2) $H(e, x) = (e, x)$ for each $x \in X$;
- b-3) for each $x, y \in X$ there is some $z \in X$ with $H(x, y) = (x, z)$.

Assertion b-3) says that for all $x, H[Vert(x)] \subseteq Vert(x)$, where $Vert(x) = \{x\} \times X$. Because H is surjective, that is equivalent to

- b-4) For each $x \in X, H[Vert(x)] = Vert(x)$.

In [2], a space is said to be *continuously homogeneous* if there is a special point $e \in X$ and for each $y \in X$ there is a homeomorphism ψ_y from X onto itself with the property that the function $f(x,y) := \psi_y(x)$ is continuous from X^2 to X and has $f(x,x) = e$ for each $x \in X$.

A space X is a *rotoid* if there is a special point $e \in X$ and a homeomorphism H from X^2 onto itself such that

c-1) for each $x \in X$, $H(x,e) = (x,x)$

c-2) for each $x \in X$, $H(e,x) = (e,x)$

If it happens that any point of X can be used as the special point e in the definition of rotoid, then X is called a *strong rotoid*. Reference [6] studies GO-spaces that are rotoids.

Every topological group is rectifiable, and the classes of rectifiable, continuously homogeneous, Choban and rotoid spaces are related as follows:

$$rectifiable \Rightarrow^1 continuously\ homogeneous \Rightarrow^2 homogeneous\ Choban\ space \Rightarrow^3 Choban\ space; \text{ and}$$

$$rectifiable \Rightarrow^4 strong\ rotoid \Rightarrow^5 rotoid$$

NOTES:

1) That every rectifiable space is continuously homogeneous was noted on page 103 of [2].

2) That every continuously homogeneous space is a Choban space is Proposition 3.5 in [2]. The Sorgenfrey line shows that this arrow cannot be reversed (see Proposition 7.3).

3) In the light of Proposition 8.1, the Michael line M is a Choban space, but its special point $e \in M$ cannot be irrational (because the diagonal of M is not an open set in M^2), so that M is not homogeneous.

4) That every rectifiable space is a strong rotoid is Proposition 8.12 in [2] . The Sorgenfrey line is a strong rotoid (see [6]) but, being first-countable and non-metrizable, a theorem of Gulko shows that the Sorgenfrey line cannot be rectifiable [18].

5) The usual interval $[-1,1]$ is a rotoid (see [6]) but not a strong rotoid because the points ± 1 cannot be used as the special point $e \in [-1,1]$. See [6].

Rotoids can fail to be Choban spaces (the usual closed interval $[-1,1]$ is a rotoid [6] but is not a Choban space by Proposition 5.2) and Choban spaces can fail to be rotoids (the Michael line is a Choban space by Proposition 8.1 but is not a rotoid [6]).

Question 2.1 *Is there a GO-space that is continuously homogeneous but not rectifiable?*

3 Generalities about Choban spaces

Our paper deals primarily with Choban operators in GO-spaces, but it will be useful to list some general lemmas first. The proofs are mostly straightforward. Some of the results (e.g., Examples 3.3, 3.7, and 3.8) show how GO-spaces can be used to provide counterexamples in the general theory of Choban operators.

Lemma 3.1 *If a space X is homeomorphic to a space Y that has a Choban operator, then X also has a Choban operator.*$\square$

Lemma 3.2 *Suppose that $\{X_i : i \in I\}$ is a family of spaces each having a Choban operator. Then $\Pi\{X_i : i \in I\}$ also has a Choban operator.* $\square$

Example 3.3 *The converse of Lemma 3.2 is false: there exist spaces X and Y such that $X \times Y$ has a Choban operator but X does not.*

Proof: Let $X := \{0\} \cup (\frac{1}{4}, \frac{3}{4})$. By Example 5.5, the space X does not have a Choban operator. Consider the space $X \times \mathbb{Z}$, which is a disjoint union of countably many copies of X and that disjoint union is homeomorphic to the space of Example 8.4, so that $X \times \mathbb{Z}$ has a Choban operator. $\square$

Question 3.4 *Do there exist spaces X and Y such that $X \times Y$ has a Choban operator and yet neither X nor Y has a Choban operator?*

Lemma 3.5 *: For any cardinal $\kappa \geq 1$ there is a topological group G of cardinality κ.*

Proof: If κ is finite use the cyclic group $(\mathbb{Z}_\kappa, +)$ with the discrete topology. If κ is infinite, let X be any set with cardinality κ and let G be the free group on the set X. Then $|G| = \kappa$ and we let G have the discrete topology. $\square$

Lemma 3.6 *Suppose X is a topological space with a Choban operator and κ is any cardinal ≥ 1. Let Y be the topological sum of κ-many copies of X. Then Y has a Choban operator.*

Proof: By Lemma 3.5, there is a discrete topological group G of cardinality κ. Then G has a Choban operator so that, by Lemma 3.2, $X \times G$ also has a Choban operator. But $X \times G$ is the topological sum of κ-many copies of X, namely the pairwise disjoint open subsets $X \times \{g\} \subseteq X \times G$, where $g \in G$.$\square$

Example 3.7 *It is crucial in Lemma 3.6 that we consider only sums of copies of a fixed Choban space. For example, the subspace $X = (0, \rightarrow)$ of $\mathbb{R}$ is a topological group under multiplication, so X is a Choban space, and the single point space $Y = \{-1\}$ is also a Choban space, but as shown in Example 5.5, their topological sum is not.*

Example 3.8 *Open subspaces, and closed subspaces, of Choban spaces may fail to be Choban spaces.*

Proof: Let X be the space obtained when $\mathbb{R}$ is re-topologized by isolating each integer and letting other points have their usual neighborhoods. Then X has a Choban operator (see Example 8.4) but its open subspace $Y = \{-1\} \cup (0, 1)$ does not (see Example 3.7). The space $\mathbb{R}$ has a Choban operator, but its closed subspace $Y = \{0\} \cup \{\frac{1}{n} : n \geq 1\}$ does not. $\square$

Lemma 3.9 *Suppose X has a Choban operator and has countable pseudo-character at its special point e (i.e., $\psi(e, X) \leq \omega$). Then X has a G_δ-diagonal.*

Proof: Suppose e is the special point of X and $f : X^2 \rightarrow X$ is the Choban operator. Let $\{V_n : n \geq 1\}$ be a family of open sets with $\{e\} = \bigcap\{V(n) : n \geq 1\}$. Then $\Delta = \bigcap\{f^{-1}[V_n] : n \geq 1\}$ because f is 1-1 on each subspace $Vert(x)$. $\square$

4 GO-spaces with Choban operators

In this section, we show that the existence of a Choban operator in a given GO-space imposes some restrictions on the structure of the space. Results in this section show that none of the following spaces can have Choban operator: an uncountable ordinal space; Alexandroff's double arrow; the lexicographic square; the Big Bush space $Bush(S,T)$ where S and T are disjoint dense subsets of $\mathbb{R}$ [5]; a Souslin line (which exists in certain models of ZFC).

We begin with a lemma that is a consequence of the Pressing Down Lemma.

Lemma 4.1 *Suppose S is a stationary set in a regular uncountable cardinal κ and that $\{W_i : i \in I\}$ is a family of open sets in S^2 with $\Delta_S \subseteq \bigcap\{W_i : i \in I\}$. If $|I| < \kappa$ then there is some $\gamma < \kappa$ with $([\gamma,\kappa) \cap S)^2 \subseteq \bigcap\{W_i : i \in I\}$.* □

Proposition 4.2 *Suppose X is a GO-space with a Choban operator. Then X is hereditarily paracompact.*

Proof: Suppose that $f : X^2 \to X$ is a Choban operator and that $e \in X$ has $f(x,x) = e$ for all $x \in X$. For contradiction, suppose X is not hereditarily paracompact. Then by [15] there is a regular uncountable cardinal κ and a stationary set $S \subseteq [0,\kappa)$ such that S embeds in X under an order-preserving or order reversing homeomorphism. Without loss of generality suppose the embedding is order preserving, and write $S \subseteq X$. We will need to distinguish between Δ_X, the diagonal of X, and its subset Δ_S, the diagonal of the subspace S. Clearly $\Delta_S = \Delta_X \cap S^2$.

Note that the set S^d of all points of S that are limit points of S must also be stationary in κ and that if $s \in S^d$ then in the subspace $Vert(s) = \{s\} \times X \subseteq X^2$, the point (s,s) is the limit of the net $\langle (s,\alpha) : \alpha < s\rangle$. Consequently $e = f(s,s) = \lim\langle f(s,\alpha) : \alpha < s\rangle$.

If there exist $s_1, s_2 \in S^d$ such that cofinally many points of $\langle f(s_1,\alpha) : \alpha < s_1\rangle$ lie below e while cofinally many points of $\langle f(s_2,\alpha) : \alpha < s_2\rangle$ lie above e (or vice versa), then the point e has a neighborhood base $\{V_i : i \in I\}$ where $|I| \le \max(\mathrm{cf}(s_1), \mathrm{cf}(s_2)) < \kappa$. Because f is 1-1 on each vertical line $Vert(x)$, we conclude that $\Delta_X = \bigcap\{f^{-1}[V_i] : i \in I\}$. Consequently the diagonal Δ_S of the subspace S has

$$\Delta_S = \Delta_X \cap (S^2) = \bigcap\{f^{-1}[V_i] \cap S^2 : i \in I\}.$$

But that contradicts Lemma 4.1. Therefore, one of the following two statements is true:

a) for each $s \in S^d$ there is some $\sigma(s) < s$ with $f(s,\alpha) < e$ for each $\alpha \in [\sigma(s), s)$, or

b) for each $s \in S^d$ there is some $\sigma(s) < s$ with $e < f(s,\alpha)$ for all $\alpha \in [\sigma(s), s)$.

Without loss of generality, assume that a) holds. Fix $s_0 \in S^d$. The net $\langle (s_0,\alpha) : \alpha < s_0\rangle$ converges to (s_0,s_0) so that $e = f(s_0,s_0)$ is the limit of the net $\langle f(s_0,\alpha) : \alpha < s_0\rangle$. Because f is 1-1 on $Vert(s_0)$ we know that $f(s_0,\alpha) \ne f(s_0,s_0) = e$ and therefore we conclude that $\mathrm{cf}_X(e) \le \mathrm{cf}_S(x_0) < \kappa$. Hence there are convex open sets V_j in X with $[e,\to) = \bigcap\{V_j : j \in J\}$ where $|J| < \kappa$.

For each $j \in J$ we have $\Delta \subseteq W_j := S^2 \cap f^{-1}[V_j]$. Because $|J| < \kappa$, Lemma 4.1 gives is some $\gamma < \kappa$ such that $(\gamma,\kappa)^2 \subseteq \bigcap\{f^{-1}[V_j] : j \in J\}$. Choose $s \in S^d$ with $\gamma < s$. With $\sigma(s) < s$ as in a), above, we may find $\alpha < s$ with $\max(\sigma(s),\gamma) < \alpha$. Then $f(s,\alpha) < e$ by a), and $(s,\alpha) \in (\gamma,\kappa)^2 \subseteq \bigcap\{f^{-1}[V_j] : j \in J\}$, showing that $f(s,\alpha) \in \bigcap\{V_j : j \in J\} = [e,\to)$. That contradiction completes the proof. □

Proposition 4.2 suggests a question about monotonically normal spaces with Choban operators. For a topological space (X,τ), let $\mathcal{P} := \{(A,U) : A \subseteq U, A \text{ closed, and } U \in \tau\}$. The space (X,τ) is *monotonically normal* if for each pair $(A,U) \in \mathcal{P}$ there is an open set $G(A,U)$ such that

i) $A \subseteq G(A, U) \subseteq \mathrm{cl}_X(G(A, U)) \subseteq U$, and

ii) if $(B, V) \in \mathcal{P}$ and $A \subseteq B$ and $U \subseteq V$, then $G(A, U) \subseteq G(B, V)$.

GO-spaces are a special subclass of monotonically normal spaces [19], and Balogh and Rudin [4] proved that the characterization of paracompactness in a monotonically normal space is the same as the characterization in a GO-space: a monotonically normal space X is paracompact if and only if no closed subspace of X is homeomorphic to a stationary subset of a regular initial ordinal. Consequently, we have

Question 4.3 *Suppose X is a monotonically normal space with a Choban operator. Must X be paracompact? hereditarily paracompact?*

We have a partial answer to that question, provided we consider only first-countable spaces.

Proposition 4.4 *Suppose X is a first-countable monotonically normal space with a Choban operator. Then X is hereditarily paracompact.*

Proof: Being a first-countable space with a Choban operator, X has a G_δ-diagonal (see Proposition 3.9), and therefore so does every subspace Y of X. Because monotone normality is a hereditary property, if the subspace Y is not paracompact, then the Balogh-Rudin theorem mentioned above gives a subspace of Y that is a stationary set in a regular uncountable cardinal. But no such subspace can have a G_δ-diagonal by Lemma 4.1. □

If we restrict attention to first-countable LOTS with a Choban operator, our next result gives a metrization theorem.

Proposition 4.5 *Any first-countable LOTS with a Choban operator is metrizable.*

Proof: Lemma 3.9 shows that X has a G_δ-diagonal. The proposition now follows from the metrization theorem in [21]. □

Proposition 4.5 does not generalize to GO-spaces because as we show later in our paper, the Sorgenfrey and Michael lines are first-countable GO-spaces with Choban operators that are not metrizable. Furthermore, the hypothesis of first-countability is crucial in Proposition 4.5 because there is a non-metrizable LOTS that is a topological group (and therefore has a Choban operator), as our next example shows.

Example 4.6 *There is a non-metrizable LOTS of cardinality $\mathfrak{c}$ that is a topological group and therefore has a Choban operator.*

Proof: The following construction is standard so we omit details. Let $\mathcal{U}$ be a free ultrafilter on ω and let C be the ring of functions from ω to $\mathbb{R}$ with pointwise operations. Let $M := \{f \in C : \{n \in \omega : f(n) = 0\} \in \mathcal{U}\}$. Then M is a maximal ideal of C so that the quotient ring C/M is a field. With $<$ being the usual order on $\mathbb{R}$, we define an order on C/M as follows: $f + M \prec g + M$ if and only if $\{n : f(n) < g(n)\} \in \mathcal{U}$. Then $\prec$ is a linear ordering on C/M and the operations of addition and subtraction in C/M are continuous when C/M carries the open interval topology of the ordering $\prec$, making C/M a LOTS that is a topological group. Because $\mathcal{U}$ is a free ultrafilter on ω, $(C/M, \prec)$ is an η_1-set in the sense of [17], so C/M with the order topology is a LOTS and a topological group (and therefore has a Choban operator) that is not first-countable, as required. □

As noted in Section 2, the property "X is continuously homogeneous" is stronger than the property "X has a Choban operator." For this stronger property we can improve Proposition 4.5 and we have the following metrization theorem for GO-spaces.

Proposition 4.7 *Suppose that (X, τ) is a first-countable GO-space that is continuously homogeneous. Then X is metrizable.*

Proof: If a continuously homogeneous space has a single isolated point, then every point is isolated and there is nothing to prove. So suppose (X, τ) has no isolated points.

As Arhangel'skii pointed out in [2], any continuously homogeneous space has a Choban operator. Therefore Lemma 3.9 gives a sequence of open covers $\mathcal{G}(n)$ such that for each $x \in X$,

$$\bigcap\{St(x, \mathcal{G}(n)) : n \geq 1\} = \{x\}.$$

Then X is first-countable and Proposition 4.2 shows that X is paracompact so that we may find locally finite open covers $\mathcal{H}(n)$ of X that refine $\mathcal{G}(n)$. But then the collection $\bigcup\{\mathcal{H}(n) : n \geq 1\}$ contains a σ-locally finite base at each Euclidean point x of (X, τ) where, a "Euclidean point" of (X, τ) is a point $x \in X$ such that each neighborhood of x contains an interval (a, b) with $a < x < b$. Because X has no isolated points, the proof will be complete if we can show that the sets $R := \{y \in X : [y, \rightarrow) \in \tau\}$ and $L := \{y \in X : (\leftarrow, y] \in \tau\}$ are both σ-closed discrete in (X, τ). Once we have that R and L are σ-closed-discrete, we use collectionwise normality and first countability of X to obtain a σ-discrete collection of open sets in X that is a base at each point of $R \cup L$. (This is essentially the proof of Faber's metrization theorem for GO-spaces. See Theorems 3.1 and 3.2 in [16] and also Theorem 1.3.4 of [24].)

The proofs that R and L are σ-closed-discrete are analogous, so we will consider only the set R. Fix any point $x \in R$. Let $\{V_n : n \geq 1\}$ be a local base at the point $e \in X$. We claim that for some $n \geq 1$, the set $f^{-1}[V_n]$ contains no point (x, y) with $y < x$. If that is not the case, then for each $n \geq 1$ choose $y_n < x$ in such a way that $(x, y_n) \in f^{-1}[V_n]$. Then $f(x, y_n) \in V_n$ so that the sequence $\langle f(x, y_n)\rangle$ converges to $e = f(x, x)$. But $x \in R$ so that the sequence $\langle (x, y_n)\rangle$ cannot converge to (x, x) and that is impossible because the function $f|_{Vert(x)}$ is a homeomorphism.

Let $R_n := \{x \in R : f^{-1}[V_n] \cap Vert(x) \subseteq \{x\} \times [x, \rightarrow)\}$. The previous paragraph shows that $R = \bigcup\{R_n : n \geq 1\}$. It remains to show that each fixed R_n is closed and discrete. For contradiction, suppose that $z \in X$ is a limit point of R_n. Then there is a monotonic sequence of points $x_k \in R_n$ that converges to z. Without loss of generality, we may assume $x_1 < x_2 < \cdots$. Consequently every neighborhood of z contains a nonempty interval $(a, z]$ with $a < z$. Therefore, for some $b < z$ we have $((b, z])^2 \subseteq f^{-1}[V_n]$ so that for some k we have $b < x_k < x_{k+1} < z$. But then $(x_{k+1}, x_k) \in ((b, z])^2 \subseteq f^{-1}[V_n] \cap Vert(x_{k+1})$, contrary to $x_{k+1} \in R_n$. Therefore, each R_n is closed and discrete in (X, τ). □

Topological groups are homogeneous, and so are the rectifiable spaces studied by Gulko (see [18] or [2]). In contrast, we already know that spaces with a Choban operator may fail to be homogeneous (see Example 5.5) because they can have isolated points without being discrete. However, the existence of a Choban operator for a GO-space X does impose a degree of uniformity on X, as the next result shows. We need slight modifications of the cardinal invariants $cf(x)$ and $ci(x)$ that are defined in the Introduction. Let $(X, <, \tau)$ be a GO-space and let $x \in X$. The *topological cofinality* of x, denoted $tcf(x)$, is defined as follows: If $[x, \rightarrow) \in \tau$ then $tcf(x) = 1$ and if $[x, \rightarrow) \notin \tau$ then $tcf(x) = cf(x)$. The *topological co-initiality* of x, denoted $tci(x)$, is an analogous modification of $ci(x)$. If τ is the usual open interval topology of $<$, then $tcf = cf$ and $tci = ci$.

Proposition 4.8 *Let X be a GO-space with a Choban operator $f : X^2 \rightarrow X$. Then the set $\{tcf(x), tci(x) : x \in X\}$ has at most two distinct infinite members.*

Proof: Suppose $\kappa < \lambda$ are the first two infinite members of the set $\{tcf(x), tci(x) : x \in X\}$. Note that both κ and λ are regular cardinals. Then there are points $x, y \in X$ and well-ordered nets $\langle x_\alpha : \alpha < \kappa\rangle$

and $\langle y_\beta : \beta < \lambda \rangle$ in $X - \{x, y\}$ that converge to x and y respectively. Therefore $\langle f(x, x_\alpha) : \alpha < \kappa \rangle$ and $\langle f(y, y_\beta) : \beta < \lambda \rangle$ converge to $f(x, x) = e = f(y, y)$.

Without loss of generality, we may assume that a cofinal sub-net of $\langle f(x, x_\alpha) : \alpha < \kappa \rangle$ lies in $(\leftarrow, e)$ so that $tcf(e) \leq \kappa$. Let $\{z_\gamma : \gamma < cf(e)\}$ be a strictly increasing cofinal net in $(\leftarrow, e)$ with supremum e. We claim that the set $\{\beta < \lambda : f(y, y_\beta) \in (e, \rightarrow)\}$ must be cofinal in λ. Otherwise there is some $\beta_0 < \lambda$ such that $\{f(y, y_\beta) : \beta_0 < \beta < \lambda\} \subseteq (\leftarrow, e)$ (we know that $f(y, y_\beta) \neq e$ for all β, because f is 1-1 on the vertical line $Vert(y)$). For each z_γ there is some $\beta(\gamma)$ with $\beta_0 < \beta(\gamma) < \lambda$ such that $f(y, y_\beta) \in (z_\gamma, e]$ whenever $\beta(\gamma) < \beta < \lambda$. Because $\kappa < \lambda$ are regular cardinals, there is a $\delta < \lambda$ with $\sup\{\beta(\gamma) : \gamma < \kappa\} < \delta < \lambda$. Consider any $\beta \in (\delta, \lambda)$. Then $f(y, y_\beta) \in (\leftarrow, e)$ and yet $z_\gamma < f(y, y_\beta)$ for each $\gamma < cf(e)$, which is impossible because the set of all z_γ is cofinal in $(\leftarrow, e)$. Therefore some cofinal sub-net of $\langle f(y, y_\beta) : \beta < \lambda \rangle$ must lie in $(e, \rightarrow)$, showing that $tci(e) \leq \lambda$.

Because $\kappa < \lambda$ we see that the point e has a neighborhood base $\{U_i : i \in I\}$ where $|I| = \lambda$. But then $\Delta = \bigcap\{f^{-1}[U_i] : i \in I\}$ so that every point of X has topological coinitiality and cofinality at most λ, as required. □

Question 4.9 *Is there a LOTS X with a Choban operator that has two different infinite cofinalities and coinitialities.*

Question 4.10 *Is there an η_1-set that, in its order topology, does not have a Choban operator? (Under CH, the answer is "No" for η_1-sets of cardinality $\mathfrak{c}$ because under CH any pair of η_1-sets are order isomorphic if they both have cardinality $\mathfrak{c}$ (see [17]), and the space of Example 4.6 is one η_1 set of cardinality $\mathfrak{c}$ that has a Choban operator.)*

Proposition 4.11 *Suppose $(X, \tau, <)$ is a perfect GO-space that has a Choban operator. Then X has a σ-closed-discrete dense set. Therefore, any GO-space that has a Choban operator and satisfies the countable chain condition must be separable.*

Proof: Being perfect, X is first-countable and therefore has a G_δ-diagonal by Lemma 3.9. A theorem of Przymusinski (see [1]) guarantees that there is a metrizable GO-topology μ on X such that $\mu \subseteq \tau$. Then the metric space (X, μ) has a dense subset D that is σ-closed-discrete in (X, μ) and therefore also σ-closed discrete in (X, τ). Let I be the set of all isolated points of the space (X, τ). Because (X, τ) is perfect, the set I is σ-closed-discrete in (X, τ). Hence so is $I \cup D$. To complete the proof, suppose U is any non-void convex τ-open set. If $|U| = 1$ then $U \subseteq I$. If $|U| \geq 3$ then U has non-void μ-interior (because μ is a GO topology on $(X, <)$) and therefore meets D. If $|U| = 2$, write $U = \{a, b\}$ with $a < b$ and $(a, b) = \emptyset$. Then $\{a\} = U \cap (\leftarrow, b)$ showing that $a \in I$. Therefore, in all cases, $U \cap (I \cup D)$ is non-empty.

The second statement of 4.11 holds because any GO-space that satisfies the countable chain condition must be hereditarily Lindelöf [7] and therefore perfect, and any σ-closed discrete subset of such an X must be countable. □

Our next result extends Theorem 7.9 in [2] which asserted that a non-degenerate connected LOTS that is rectifiable (a property defined in Section 2 that is stronger than having a Choban operator) must be homeomorphic to the real line.

Proposition 4.12 *Suppose X is a connected LOTS with a Choban operator. Then either $|X| = 1$ or else X is homeomorphic to $\mathbb{R}$.*

Proof: Suppose $|X| \geq 2$ and suppose $g : X^2 \to X$ is a Choban operator where X is a connected LOTS, with $g[\Delta] = \{e\}$. Recall that if Y is any connected LOTS and $f : Y \to Z$ is a 1-1 continuous function from Y to a LOTS Z, then either f is strictly increasing (i.e., for all $y_1 < y_2$ in Y, $f(y_1) < f(y_2)$ in Z) or else f is strictly decreasing (i.e., for all $y_1 < y_2$ in Y, $f(y_1) > f(y_2)$ in Z). For each $x \in X$, denote the function $y \to g(x, y)$ by $g(x, *)$. Then $g(x, *)$ is either strictly increasing or strictly decreasing. Let $I := \{x \in X : g(x, *) \text{ is strictly increasing}\}$. We claim that I is closed in X. Fix any $c < d$ in X and suppose $\langle x_\alpha : \alpha \in A\rangle$ is a net of points of I that converges to a point $x \in X$. Because g is continuous, we have $g(x, c) = \lim\langle g(x_\alpha, c) : \alpha \in A\rangle$ and $g(x, d) = \lim\langle g(x_\alpha, d) : \alpha \in A\rangle$. But $x_\alpha \in I$ gives $g(x_\alpha, c) < g(x_\alpha, d)$ for each $\alpha \in A$ so that

$$g(x, c) = \lim\langle g(x_\alpha, c) : \alpha \in A\rangle \leq \lim\langle g(x_\alpha, d) : \alpha \in A\rangle = g(x, d).$$

Because $c \neq d$ we know that $g(x, c) \neq g(x, d)$ so that $g(x, c) < g(x, d)$. Because this holds for each choice of $c < d$ in X, we see that $g(x, *)$ is strictly increasing, so that $x \in I$. Hence I is closed. Similarly, the set $J := \{x \in X : g(x, *) \text{ is strictly decreasing}\}$ is closed in X. Because X is connected and $X = I \cup J$ with $I \cap J = \emptyset$ we know that one of the closed sets I and J is empty. Consider the case where $J = \emptyset$ so that $X = I$. The other case is analogous.

Because X is connected and has more than one point, X is infinite and we may choose a bounded sequence $y_1 < y_2 < \cdots$. Connectedness of X gives us a point $x = \sup\{y_n : n \geq 1\}$. Similarly there is a strictly decreasing sequence $\hat{y}_1 > \hat{y}_2 > \cdots$ whose infimum is a point $\hat{x} \in X$. Because the function $g(x, *)$ is strictly increasing we know that $g(x, y_n) < g(x, x)$ and that $\langle g(x, y_n) : n \geq 1\rangle$ has limit $g(x, x) = e$. Similarly $g(\hat{x}, \hat{y}_n)$ converges to $g(\hat{x}, \hat{x}) = e$ from above. Therefore the point e has a countable neighborhood base in X so that as in Lemma 3.9, the space X has a G_δ-diagonal and is, therefore, metrizable. But any connected metrizable LOTS is separable so that X is homeomorphic to a connected subspace of the real line. Up to homeomorphism, there are only four possibilities, namely, $[0, 1], [0, 1), (0, 1]$, and $(0, 1)$. In Example 5.2, we will show that none of the first three can have a Choban operator, so we conclude that X is homeomorphic to $(0, 1)$ which is homeomorphic to $\mathbb{R}$. □

Proposition 4.13 *Suppose X is a locally connected GO-space with a Choban operator. Then X is the topological sum of a set I of isolated points (possibly $I = \emptyset$) and a collection of connected subspaces of the usual space of real numbers.*

Proof: First observe that each component of X is open, so that X is a topological sum of its components. Next, suppose $f : X^2 \to X$ is a Choban operator with $f[\Delta] = e$. Consider the component C of X that contains e. If $|C| = 1$ then the space X is discrete (because then $f^{-1}[C] = \Delta$ is open), so suppose $|C| > 1$.

As in Proposition 4.12, the function $f|_{C\times C}$ has the property that either $f|_{\{x\}\times C} : \{x\} \times C \to X$ is strictly increasing for each $x \in C$, or it is strictly decreasing for each $x \in C$. Suppose $f|_{\{x\}\times C} : \{x\}\times C \to X$ is strictly increasing for each $x \in C$.

If e is not an endpoint of C, then there is a strictly increasing sequence $x_n \in C \cap (\leftarrow, e)$ with supremum $x \in C$, and a decreasing sequence $y_n \in C \cap (e, \rightarrow)$ with infimum $y \in C$. Then $\langle f(x, x_n) : n \geq 1\rangle$ is a strictly increasing sequence with limit $f(x, x) = e$, and similarly $\langle f(y, y_n) : n \geq 1\rangle$ converges to $f(y, y) = e$ from above, showing that X is first countable at e. If e is an endpoint of C (say the right endpoint), then there is a strictly increasing sequence $\langle x_n : n \geq 1\rangle$ in C that converges to some point $x \in C$, so that $\langle f(x, x_n) : n \geq 1\rangle$ is a sequence of distinct points of $f[C]$ that converges to $f(x, x) = e$. Because e is the right endpoint of C, the set $(\leftarrow, e]$ is open, so that X must be first-countable at e. Therefore, in any case, X is first-countable at e and therefore X has a G_δ-diagonal.

We now know that each component of X is a connected LOTS with a G_δ-diagonal, and therefore is either a singleton or is homeomorphic to a connected interval of real numbers, as claimed. □

Proposition 4.13 does not give a characterization of locally connected GO-spaces that have Choban operators, as can be seen from Example 5.5 in the next section.

5 Examples in $\mathbb{R}$

As noted above, for any topological group $(G, *)$, the function $g(a, b) = a * b^{-1}$ is a Choban operator on G. Therefore certain subspaces of $\mathbb{R}$ automatically have Choban operators because they are (homeomorphic to) topological groups.

Example 5.1 *The following subspaces of* $\mathbb{R}$ *have Choban operators:* $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{P}$, $\mathbb{Z}$, *the Cantor Set,* $(0, \infty)$, *and* $\mathbb{R} - \{0\}$.

Proof: $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ are topological groups under addition. The space $\mathbb{P}$ is homeomorphic to the topological group $\mathbb{Z}^\omega$ and the Cantor Set is homeomorphic to the topological group $\{-1, 1\}^\omega$. The subspaces $(0, \infty)$ and $\mathbb{R} - \{0\}$ are both topological groups under multiplication. $\Box$

Example 5.2 *None of the intervals* $[0, 1], [0, 1), (0, 1] \subseteq \mathbb{R}$ *can have a Choban operator.*

Proof: We consider the space $X = [0, 1)$, the others being analogous. Suppose $g : X^2 \to X$ is a Choban operator with $g[\Delta] = \{e\} \subseteq [0, 1)$. Then, as in the proof of Proposition 4.12 above, either the function $g(x, *)$ is strictly increasing for all $x \in [0, 1)$, or the function $g(x, *)$ is strictly decreasing for all $x \in [0, 1)$. Consider the function $g(0, *) : Vert(0) \to [0, 1)$ – it cannot be strictly decreasing, so it is strictly increasing and must have $e = g(0, 0) = 0$. Hence every function $g(x, *) : Vert(x) \to [0, 1)$ is strictly increasing. Consider $g(\frac{1}{2}, *) : Vert(\frac{1}{2}) \to [0, 1)$. We know that $g(\frac{1}{2}, \frac{1}{2}) = g(0, 0) = 0$ so that for any $y < \frac{1}{2}$ we must have $g(\frac{1}{2}, y) < g(\frac{1}{2}, \frac{1}{2}) = 0$ and that is impossible. $\Box$

Example 5.3 *Let* $A = \bigcup\{(n, n + \frac{1}{2}) : n \in \mathbb{Z}\}$, $B = \bigcup\{[n, n + \frac{1}{2}) : n \in \mathbb{Z}\}$ *and* $C = \bigcup\{[n, n + \frac{1}{2}] : n \in \mathbb{Z}\}$ *be topologized as subspaces of* $\mathbb{R}$. *Then* A *and* B *have Choban operators, while* C *does not.*

Proof: Lemma 5.4 shows that A has a Choban operator. Notice that the space B is homeomorphic to the GO-space $\hat{B}$ obtained from the usual space of real numbers by letting each $n \in \mathbb{Z}$ have a neighborhood base of the form $\{[n, n + \epsilon) : \epsilon > 0\}$. To simplify notation, we will replace B by it homeomorphic copy $\hat{B}$ for the rest of this proof.

Consider the basic vertical strip $Y_0 = [0, 1) \times B$. We will define a function $f_0 : Y_0 \to B$ and then use f_0 to define a Choban operator $f : B^2 \to B$.

For $(a, b) \in Y_0$ we define $f_0(a, b)$ as follows:

1) If $(a, b) \in Y_0$ with $b \geq 1$, let $f_0(a, b) = b$.

2) If $(a, b) \in Y_0$ with $b < -1$, let $f_0(a, b) = b + 1$.

3) If $(a, b) \in Y_0$ with $a \leq b < 1$ let $f_0(a, b) = \frac{1}{2} + \frac{b-a}{2-2a}$ and note that f_0 maps the vertical interval from (a, a) to $(a, 1)$ (excluding the top point) onto the segment $[\frac{1}{2}, 1)$ of B in a continuous 1-1 way.

4) If $(a, b) \in Y_0$ with $0 \leq b \leq a$, let $f_0(a, b) = \frac{1}{2} + \frac{b-a}{2}$ so that f_0 maps the vertical interval from $(a, 0)$ to (a, a) onto the segment $[\frac{1}{2} - \frac{a}{2}, \frac{1}{2}]$ in a continuous, 1-1 way.

5) If $(a, b) \in Y_0$ with $-1 \leq b < 0$ define $f_0(a, b) = \frac{(b+1)(1-a)}{2}$, so that f_0 maps the vertical interval from $(a, -1)$ to $(a, 0)$ onto the segment $[0, \frac{1}{2} - \frac{a}{2}]$ in a continuous, 1-1 way.

Because the five parts of the definition of f_0 agree where they overlap, the function $f_0 : Y_0 \to B$ is continuous. Furthermore, for each fixed $a \in [0,1)$, the function f_0 maps $Vert(a)$ onto B in a continuous, 1-1 way.

For each $n \in \mathbb{Z}$, let $Y_n = [n, n+1) \times B$. The sets Y_n are pairwise disjoint open subsets of B^2 and we define f as follows. For $(a,b) \in B^2$ choose the unique integer n with $(a,b) \in Y_n$. Now let $f(a,b) = f_0(a-n, b-n)$. This function $f : B^2 \to B$ is the required Choban operator.

Finally, consider the space C. For contradiction, suppose that there is a Choban operator $f : C^2 \to C$. For notational simplicity, write $[a_n, b_n] = [n, n+\frac{1}{2}]$ for each $n \in \mathbb{Z}$. Find $n \in \mathbb{Z}$ with $e = g[\Delta] \in [a_n, b_n]$. Fix any $x \in [a_n, b_n]$ and note that $Vert(x) = \bigcup\{\{x\} \times [a_i, b_i] : i \in \mathbb{Z}\}$. Define $g_x(y) = f(x,y)$. Then g_x is continuous and $g_x(x) = f(x,x) = e \in [a_n, b_n]$ so that $g_x([a_n, b_n]) \subseteq [a_n, b_n]$. Because each set $g_x([a_i, b_i])$ is connected, we see that either $g_x([a_i, b_i]) \subseteq [a_n, b_n]$ or else $g_x([a_i, b_i])$ is disjoint from $[a_n, b_n]$ so that $[a_n, b_n]$ is the union of the intervals $g_x([a_j, b_j])$ that are subsets of $[a_n, b_n]$. Because the closed interval $[a_n, b_n]$ cannot be written as the union of any pairwise disjoint collection of closed subintervals that are proper subsets of $[a_n, b_n]$, we see that $g_x([a_n, b_n]) = [a_n, b_n]$ for each $x \in [a_n, b_n]$.

As in Proposition 4.12, we know that either the function g_x is strictly increasing for every $x \in [a_n, b_n]$, or else g_x is strictly decreasing for all such x. The cases are analogous, so we consider the first. We know that $g_{a_n}(a_n) = e \in [a_n, b_n]$ so that e must be the left endpoint of $[a_n, b_n]$. But we also know that $e = g_{b_n}(b_n)$ so that e must also be the right endpoint of $[a_n, b_n]$ and that is impossible □

In general, open subspaces of Choban spaces may fail to have a Choban operator (see Example 3.8). However, in the space $\mathbb{R}$ we have:

Lemma 5.4 *Any nonempty open subspace of* $\mathbb{R}$ *has a Choban operator.*

Proof: Any open subset U of $\mathbb{R}$ can be written as the countable disjoint union of its components, each of which is an open interval (a_i, b_i) (or an open half-line) and therefore each is homeomorphic to $\mathbb{R}$. Since each of these open intervals has a Choban operator, so does their disjoint union, by Lemma 3.6. □

Example 5.5 *: The subspace* $X = (0,1) \cup \{2\} \subseteq \mathbb{R}$ *does not have a Choban operator and the subspace* $Y = \mathbb{Z} \cup \bigcup\{(n+\frac{1}{4}, n+\frac{3}{4}) : n \in \mathbb{Z}\} \subseteq \mathbb{R}$ *does have a Choban operator.*

Proof: If $g : X^2 \to X$ is a Choban operator, then $g[\Delta]$ cannot be the isolated point 2 of X (otherwise the diagonal would be an open set in X^2, making X discrete). Therefore for some $e \neq 2$ we have so $g[\Delta] = e$. Because $g(2,2) = e \neq 2$ there must be some $x \in X$ with $g(2,x) = 2$ since $g(2,*)$ maps $Vert(2)$ onto X. But 2 is isolated in X and $(2,x)$ is not isolated in $Vert(2)$, so the function $g(2,*)$, which is known to be 1-1 on $Vert(2)$, cannot be continuous on $Vert(2)$. Hence X cannot have a Choban operator. In the Proposition 8.4, we construct a Choban operator on a space that is homeomorphic to Y, so the subspace Y of $\mathbb{R}$ has a Choban operator. □

In the rest of this section, we describe two subspaces of $\mathbb{R}$. The first is a Bernstein set[5] that has a Choban operator (because it is a topological group) and the second is a Bernstein set designed in such a way that (under the Continuum Hypothesis and also under certain other axioms) it cannot have a Choban operator. A set similar to the Bernstein set described in the first example appears in Example 8.5 of [3]. The construction of the second example uses a technique of van Mill for finding rigid subspaces of $\mathbb{R}$.

[5] A subset $B \subseteq \mathbb{R}$ is a Bernstein set provided both B and $\mathbb{R} - B$ contain points of every uncountable compact subset of $\mathbb{R}$.

Proposition 5.6 *: There is a subspace $B \subseteq \mathbb{R}$ that is a Bernstein set and a topological group, and therefore has a Choban operator.*

Proof: For any subset $S \subseteq \mathbb{R}$, let $Span(S)$ be the $\mathbb{Q}$-linear span of S. Note that if S is infinite, then $Span(S)$ has the same cardinality as S.

Let $\mathcal{K} := \{K_\alpha : \alpha < \mathfrak{c}\}$ be a listing of all uncountable compact subsets of $\mathbb{R}$. Note that each member of $\mathcal{K}$ has cardinality $\mathfrak{c}$. Choose $x_0 \in K_0$ with $x_0 \neq 0$ and let $Span(x_0) = \{qx_0 : q \in \mathbb{Q}\}$ be the $\mathbb{Q}$-linear span of x_0. Then $K_0 - Span(x_0) \neq \emptyset$ so we may choose $y_0 \in K_0 - Span(x_0)$.

Next we may choose $x_1 \in K_1 - Span(\{x_0, y_0\})$ and $y_1 \in K_1 - Span(\{x_0, y_0, x_1\})$. Observe that $y_0 \notin Span(\{x_0, x_1\})$ because otherwise there would be rational numbers a_0, a_1 with $y_0 = a_0x_0 + a_1x_1$. We know that $y_0 \notin Span(\{x_0\})$ so that $a_1 \neq 0$. Therefore we would have $x_1 = \frac{1}{a_1}y_0 - \frac{a_0}{a_1}x_0$ and that is impossible because $x_1 \notin Span(\{x_0, y_0\})$

Suppose $\alpha < \mathfrak{c}$ and for each $\beta < \alpha$ we have distinct points x_β, y_β with

a) $x_\beta, y_\beta \in K_\beta$

b) $y_\beta \notin Span(x_\gamma : \gamma < \alpha)$

The set $Span(\{x_\beta, y_\beta : \beta < \alpha\})$ has cardinality less than $\mathfrak{c}$ (because $\alpha < \mathfrak{c}$ and $\mathbb{Q}$ is countable) so that we can choose $x_\alpha \in K_\alpha - Span(\{x_\beta, y_\beta : \beta < \alpha\})$ and $y_\alpha \in K_\alpha - (Span(\{x_\beta, y_\gamma : \beta \leq \alpha, \gamma < \alpha\})$. We claim that if $\beta < \alpha$ then $y_\beta \notin Span(\{x_\gamma : \gamma \leq \alpha\})$. Otherwise there exist a positive integer n, ordinals $\gamma_i \leq \alpha$ for $i \leq n$, and rational numbers $a_i \neq 0$ with

$$(*) \quad y_\beta = a_1x_{\gamma_1} + a_2x_{\gamma_2} + \cdots + a_nx_{\gamma_n}.$$

We may assume that $\gamma_1 < \gamma_2 < \cdots < \gamma_n \leq \alpha$. Then $\gamma_i < \alpha$ for $i < n$. But we know that $y_\beta \notin Span(\{x_\delta : \delta < \alpha\}$ so we conclude that $\gamma_n = \alpha$ and $a_n \neq 0$. This allows us to solve for $x_{\gamma_n} = x_\alpha$, expressing it as a $\mathbb{Q}$-linear combination of y_β and $x_{\gamma_1}, \cdots, x_{\gamma_{n-1}}$ contrary to our choice of x_α. Therefore the recursion continues.

Our recursion produces a set $B := Span(\{x_\alpha : \alpha < \mathfrak{c}\})$ that is a topological vector space under the usual addition and scalar multiplication, and therefore is a topological group under addition. By construction, B intersects each uncountable compact set K_α. Because $B \cap \{y_\beta : \beta < \mathfrak{c}\} = \emptyset$ and $\{y_\beta : \beta < \mathfrak{c}\}$ also intersects every K_α, we see that B is the required Bernstein set. $\square$

Our next example requires some set theory beyond ZFC. The key property that we need is

> (*) If X is a complete separable metric space that is dense-in-itself, then X cannot be written as the union of fewer than $\mathfrak{c} = 2^\omega$ many closed nowhere-dense subsets.

The Baire Category theorem guarantees that (*) holds under the Continuum Hypothesis, and (*) also holds under MA + notCH [20]. However, there are other models of ZFC in which (*) fails [13].

It follows from (*) that if Y is a non-empty open subspace of the dense-in-itself, complete, separable metric space X, then Y cannot be a subset of a union of fewer than $\mathfrak{c}$-many closed nowhere dense subsets of X.

Proposition 5.7 *In any model of ZFC in which (*) holds, there is a Bernstein set $T \subseteq \mathbb{R}$ such that T does not have a Choban operator.*

Proof: There are $\mathfrak{c}$ many dense G_δ-subsets of $\mathbb{R}$, and for each dense G_δ-subset $D \subseteq \mathbb{R}$ there are $\mathfrak{c}$ many continuous functions from D into $\mathbb{R}$. Consequently, there are $\mathfrak{c}$ many functions f whose domain $dom(f)$ is a dense G_δ-subset of $\mathbb{R}$ and where $f : dom(f) \to \mathbb{R}$ is continuous. Let $\mathcal{F}$ be the set of all functions satisfying:

i) $dom(f)$ is a dense G_δ subset of $\mathbb{R}$;

ii) $f : dom(f) \to \mathbb{R}$ is continuous;

iii) there is a dense subset of $dom(f)$ on which f is 1-1;

iv) there is some $x \in dom(f)$ with $f(x) \neq x$.

Then $|\mathcal{F}| = \mathfrak{c}$ so we may index $\mathcal{F}$ as $\{f_\alpha : \alpha < \mathfrak{c}\}$. Write $D_\alpha = dom(f_\alpha)$ and let $E_\alpha \subseteq D_\alpha$ be the dense subset of D_α on which f_α is 1-1. For each $\alpha < \mathfrak{c}$ let $move(f_\alpha) = \{x \in D_\alpha : f_\alpha(x) \neq x\}$. Notice that each set $move(f_\alpha)$ is a non-empty open subset of the subspace D_α. Because D_α is completely metrizable (being a dense G_δ-subset of $\mathbb{R}$) and dense-in-itself, the set $move(f_\alpha)$ has cardinality $\mathfrak{c}$ and, by (*), cannot be written as the union of fewer than $\mathfrak{c}$ closed nowhere-dense subsets of D_α. Let $\{K_\alpha : \alpha < \mathfrak{c}\}$ be a listing of all uncountable compact subsets of $\mathbb{R}$.

We claim that that if $\alpha < \mathfrak{c}$ and $r \in \mathbb{R}$ are fixed, then the set $f_\alpha^{-1}(r)$ is a closed nowhere-dense subset of $D_\alpha = dom(f_\alpha)$. In case $f_\alpha^{-1}(r) = \emptyset$, there is nothing to prove. In case $f_\alpha^{-1}(r) \neq \emptyset$, suppose U is a relatively open nonempty subset of D_α with $U \subseteq f_\alpha^{-1}(r)$. Then $U \cap E_\alpha \neq \emptyset$ where E_α is the dense subset of D_α defined above. Because D_α is dense in $\mathbb{R}$, it has no relatively isolated points, so that we have distinct $e_1, e_2 \in E_\alpha \cap U$. But then $f_\alpha(e_1) = r = f_\alpha(e_2)$ contradicting the fact that f_α is 1-1 on the set E_α. Hence $f_\alpha^{-1}(r)$ is closed and nowhere dense in the subspace D_α. We will apply this observation throughout the following construction.

We are now ready to initialize our recursion. For each $q \in \mathbb{Q}$, the set $f_0^{-1}(q)$ is a closed nowhere-dense (possibly empty) subset of D_0 and the set $move(f_0)$ is a non-empty relatively open set in D_0. Because D_0 is completely metrizable we have $move(f_0) \not\subseteq \bigcup\{f_0^{-1}(q) : q \in \mathbb{Q}\}$ and this allows us to choose $x_0 \in move(f_0) - \bigcup\{f_0^{-1}(q) : q \in \mathbb{Q}\}$. Compute $y_0 = f_0(x_0)$. Then $y_0 \neq x_0$ and $y_0 \notin \mathbb{Q}$. Choose distinct $a_0, b_0 \in K_0 - (\mathbb{Q} \cup \{x_0, y_0\})$.

For induction hypothesis, suppose that $\alpha < \mathfrak{c}$ and that for each $\beta < \alpha$ we have chosen points $x_\beta \in move(f_\beta) \subseteq D_\beta$, $y_\beta = f_\beta(x_\beta)$, and $a_\beta, b_\beta \in K_\beta$ in such a way that $(\{x_\beta, a_\beta : \beta < \alpha\} \cup \mathbb{Q}) \cap \{y_\beta, b_\beta : \beta < \alpha\} = \emptyset$.

The set $move(f_\alpha)$ is a non-empty open subset of the separable completely metrizable space D_α and for each $z \in \mathbb{R}$, the set $f_\alpha^{-1}(z)$ is closed and nowhere dense in D_α. Therefore $\alpha < \mathfrak{c}$ combines with (*) to guarantee that

$$move(f_\alpha) - (f_\alpha^{-1}[\mathbb{Q} \cup \{x_\beta, y_\beta, a_\beta, b_\beta : \beta < \alpha\}] \cup \mathbb{Q} \cup \{x_\beta, y_\beta, a_\beta, b_\beta : \beta < \alpha\}) \neq \emptyset.$$

Choose x_α in that non-empty set and note that neither x_α nor $y_\alpha := f_\alpha(x_\alpha)$ is in the set $\mathbb{Q} \cup \{x_\beta, y_\beta, a_\beta, b_\beta : \beta < \alpha\}$. Because $|K_\alpha| = \mathfrak{c}$ and $\alpha < \mathfrak{c}$ we may choose distinct points $a_\alpha, b_\alpha \in K_\alpha - (\{x_\gamma, y_\gamma : \gamma \leq \alpha\} \cup \{a_\beta, b_\beta : \beta < \alpha\})$. Then $(\mathbb{Q} \cup \{x_\gamma, a_\gamma : \gamma < \alpha + 1\}) \cap \{y_\gamma, b_\gamma : \gamma < \alpha + 1\} = \emptyset$. Therefore the recursion continues.

This recursion gives points $x_\alpha, y_\alpha, a_\alpha, b_\alpha$ for each $\alpha < \mathfrak{c}$ and we let $T = \mathbb{Q} \cup \{x_\alpha, a_\alpha : \alpha < \mathfrak{c}\}$. Observe that if $\alpha < \mathfrak{c}$ then $y_\alpha = f_\alpha(x_\alpha) \notin T$. Also note that $a_\alpha \in T \cap K_\alpha$ and $b_\alpha \in (\mathbb{R} - T) \cap K_\alpha$ for each α, showing that T is a Bernstein set.

For contradiction, suppose that the subspace T of $\mathbb{R}$ has a Choban operator $g : T \times T \to T$. Then there is a $d \in T$ such that $g(x, x) = d$ for all $x \in T$. Fix $z \in T - \{d\}$ and consider the function

$f(y) = g(z, y)$ defined for $y \in T$. Note that $f(z) = g(z, z) = d \neq z$ and that f is 1-1 on the set T. Let $O(n) = \{x \in \mathbb{R} : \inf\{diam(f[(x-\epsilon, x+\epsilon) \cap T]) : \epsilon > 0\} < \frac{1}{n}\}$. Then $O(n)$ is an open set in $\mathbb{R}$. We see that $O(n)$ is dense in $\mathbb{R}$ because $T \subseteq O(n)$ for each n (since f is continuous on the dense subspace T). Let $M := \bigcap\{O(n) : n \geq 1\}$. Then M is a dense G_δ-subset of $\mathbb{R}$, $T \subseteq M$, and the function f extends to a continuous function $F : M \to \mathbb{R}$ [14].

We claim that $F \in \mathcal{F}$. We must show that for some $x \in dom(F)$ we have $F(x) \neq x$ and that there is a dense subset of $dom(F)$ on which F is 1-1. To verify the first assertion, recall that $z \in T$ and that $F(z) = f(z) = g(z, z) = d \neq z$. To verify the second, note that $T \subseteq M$ and $F(x) = f(x)$ for each $x \in T$, showing that T is a dense subset of M on which F is 1-1. Consequently, $F \in \mathcal{F}$.

Therefore there is some $\alpha < \mathfrak{c}$ with $F = f_\alpha$. Then $x_\alpha \in T$ and we have $g(z, x_\alpha) = f(x_\alpha) = F(x_\alpha) = f_\alpha(x_\alpha) = y_\alpha \notin T$, contrary to $g : T \times T \to T$. We conclude that T cannot have a Choban operator. $\square$

In the proof of Proposition 5.7, it is not surprising that one may use *any* $z \in T - \{d\}$ to obtain a contradiction, because the set T has no continuous 1-1 onto self-mapping except the identity map.

Question 5.8 *Characterize those subspaces $X \subseteq \mathbb{R}$ that have Choban operators in their relative topology.*

6 Step-function constructions and a technical lemma

Lemma 6.1 *Suppose $a < b$ and $c < d$ are real numbers. Then there is a continuous, onto, order-preserving 1-1 function $f : [a, b] \to [c, d]$ satisfying:*

i) $f(a) = c, f(b) = d$
ii) $f[(a, b) \cap \mathbb{Q}] = (c, d) \cap \mathbb{Q}$
iii) $f[(a, b) \cap \mathbb{P}] = (c, d) \cap \mathbb{P}$

Proof: A standard recursion produces an order-preserving function $g : (a, b) \cap \mathbb{Q} \to (c, d) \cap \mathbb{Q}$ that is 1-1 and onto. For $x \in (a, b) \cap \mathbb{P}$ define $f(x) = \sup\{f(q) : q \in \mathbb{Q} \cap (a, x)\}$. Then $f(x) \in \mathbb{P} \cap (c, d)$. Finally, define $f(a) = c$ and $f(b) = d$. $\square$

We now construct a special family of step functions and a system of notation for naming them that will be used in the next two sections. In $\mathbb{R}^2$ draw the diagonal $y = x$, the lines $y = x + \frac{1}{m}$, and the lines $y = x + n$ for each $m, n \in \mathbb{Z}$ with $m \neq 0$. We will name these parallel lines by their y -intercepts, so that the line $y = x - 5$ is called L_{-5} while the line $y = x + \frac{1}{3}$ is called $L_{\frac{1}{3}}$. Under this system the diagonal should be called L_0 but we will call it Δ instead.

Notation: Let $T = \{n \in \mathbb{Z} : n \neq 0\} \cup \{\frac{1}{m} : m \in \mathbb{Z}, m \neq 0\}$ be the set of indices used to name the parallel lines constructed in the previous paragraph. For any $t \in T$, there is a $t^+ \in T$ that is the first element of T greater than t. For example, if $t = 3$ then $t^+ = 4$; if $t = \frac{-1}{5}$ then $t^+ = \frac{-1}{6}$; and if $t = \frac{1}{2}$ then $t^+ = 1$.

Between each pair of adjacent parallel lines L_t and L_{t^+} we construct a special step function $S : \mathbb{R} \to \mathbb{R}$ satisfying:

a) The values of the step function S are all irrational, i.e., $\{S(x) : x \in \mathbb{R}\} \subseteq \mathbb{P}$.

b) If v is in the range of S, then $S^{-1}(v)$ is an interval of the form $[a, b)$ where $a, b \in \mathbb{P}$; thus each horizontal portion of the graph of S is a half-open interval containing a left endpoint but no right endpoint, and both endpoints are irrational.

c) For each step function S, the set of left endpoints of the intervals mentioned in b) is a closed discrete set in the usual space of real numbers.

d) The graph of S lies strictly between the lines L_t and L_{t^+} for some $t \in T$.

We name the step functions in the following way, using names from the set T defined above: the step function constructed between L_{-3} and L_{-2} will be called S_{-3} and the step function constructed between $L_{\frac{1}{3}}$ and $L_{\frac{1}{2}}$ will be called $S_{\frac{1}{3}}$. In other words, the step function S_t is named after the line L_t that lies immediately below it.

For any $x \in \mathbb{R}$ we let $Vert(x) = \{x\} \times \mathbb{R}$. Notice that $Vert(x)$ meets the graph of each of the step functions S_t defined above. Suppose $t \in T$ is one of the subscripts used to identify one of our step functions. Then let $Vert(x, t)$ be the vertical segment on $Vert(x)$ that lies between the graph of the step function S_t and the graph of the step function S_{t^+} that lies immediately above S_t, with bottom endpoint included, and top endpoint excluded. Then $Vert(x, t)$ is a vertical subinterval on $Vert(x)$ and the second coordinates of its endpoints are irrational. We will write $Vert(x, t) = \{x\} \times [u(x, t), v(x, t))$ where $u(x, t) := S_t(x)$ and $v(x, t) := S_{t^+}(x)$ are both in $\mathbb{P}$.

7 Choban operators in Sorgenfrey lines

The Sorgenfrey line S is the set of real numbers re-topologized so that $\{[a, b) : a < b\}$ is a local base at each real number a. In this section we will show that S and some of its subspaces have Choban operators. Most of this section will be devoted to proving

Proposition 7.1 *The Sorgenfrey line S has a Choban operator.*

Notation

The subintervals $I(t) \subseteq S$: In the Sorgenfrey line S consider the following intervals:

a) For each non-zero integer $n \in T$ let $I(n) = [n, n+1)$ (e.g., $I(2) = [2, 3)$ and $I(-1) = [-1, 0)$).

b) For each $\frac{1}{m} \in T$ with $m \geq 2$ let $I(\frac{1}{m}) = [\frac{1}{m}, \frac{1}{m-1})$ and divide that interval into two halves using the interval's midpoint M, obtaining $BotI(\frac{1}{m}) = [\frac{1}{m}, M)$ and $TopI(\frac{1}{m}) = [M, \frac{1}{m-1})$. For example, $I(\frac{1}{3}) = [\frac{1}{3}, \frac{1}{2})$ and $BotI(\frac{1}{3}) = [\frac{1}{3}, \frac{5}{12})$.

c) Note that we do not have any interval corresponding to negative fractional values in T (e.g., there is no interval $I(\frac{-1}{3})$).

We already noted that for each x, the vertical line $Vert(x) = \{x\} \times S$ is broken into vertical subintervals $Vert(x, t)$ by the step functions S_t and its immediate successor S_{t^+} and we wrote $Vert(x, t) = \{x\} \times [u(x, t), v(x, t))$ where $u(x, t), v(x, t)$ are the irrational endpoints of the vertical interval $Vert(x, t)$.

The functions $h_{a,b,t}$

Let $[a, b)$ be any interval in the real line with $a < b$. For each integer $t \in T$ let $h_{a,b,t} : [a, b) \to I(t)$ be an order-preserving 1-1 onto function taking the left endpoint a to the left endpoint of $I(t)$ and the right endpoint b to the right endpoint of $I(t)$, as in Lemma 6.1. For example, $h_{2,3,-5}$ would map the interval $[2, 3)$ onto $I(-5) = [-5, -4)$. For each positive fraction $t \in T$ let $h_{a,b,t} : [a, b) \to TopI(t)$ be an order-preserving 1-1 continuous onto mapping that preserves left and right endpoints as in Lemma 6.1, and for a negative fraction $t \in T$, let $h_{a,b,t} : [a, b) \to BotI(|t|)$ be an order-preserving 1-1 continuous onto mapping that preserves left and right endpoints. For example, $h_{a,b,\frac{1}{2}}$ maps $[a,b)$ onto $TopI(\frac{1}{2}) = [\frac{3}{4}, 1)$ and $h_{a,b,\frac{-1}{2}}$ maps $[a,b)$ onto $BotI(\frac{1}{2}) = [\frac{1}{2}, \frac{3}{4})$, preserving left and right endpoints.

Outline of the proof:

(a) We will use $e = 0$ as the special point of the Sorgenfrey line. (Because S is homogeneous, we could have used any point of S as the point e.)

(b) Our strategy will be to use these homeomorphisms $h_{u,v,t}$ to map the vertical segments $Vert(x,t) = \{x\} \times [u,v)$ onto pieces $I(t)$ of the Sorgenfrey line by a homeomorphism that depends only on t and the two irrational endpoints $u = u(x,t)$ and $v = v(x,t)$ of $Vert(x,t)$ Therefore, if it should happen that $Vert(x,t)$ and $Vert(x',t)$ have the same endpoints, then the same homeomorphism will be used to map both $Vert(x,t)$ and $Vert(x',t)$. This is a crucial issue in establishing continuity of our Choban operator.

(c) The segments $Vert(x,t)$, where $t \in T$ is a negative fraction, will be mapped to the bottom half of the interval $I(|t|)$ and segments $Vert(x,t)$ where $t \in T$ is a positive fraction will be mapped onto $TopI(t)$, both of which lie above zero in the Sorgenfrey line. As x is allowed to vary, this will insure that a band containing a Euclidean neighborhood of the diagonal will be mapped into each interval $[0,\epsilon)$ and that is crucial for continuity of our function $g : S^2 \to S$ at points of the diagonal.

Definition of the Choban operator $g : S \times S \to S$.

For each fixed vertical line $Vert(x) \subseteq S^2$ define $g(x,y)$ as follows:

(i) If $(x,y) \in Vert(x,t)$ with t an integer, then $Vert(x,t) = \{x\} \times [u(x,t), v(x,t))$ and we let $g(x,y) = h_{u(x,t),v(x,t),t}(y)$ where the function $h_{u(x,t),v(x,t),t} : [u(x,t), v(x,t)) \to I(t)$ is the 1-1 continuous, onto, order-preserving mapping found above.

(ii) If $(x,y) \in Vert(x,t) = \{x\} \times [u(x,t), v(x,t))$ where t is a fraction, we let $g(x,y) = h_{u(x,t),v(x,t),t}(y)$, which gives us a point in the top half or bottom half of the interval $I(t)$, depending upon the sign of t.

(iii) If $x = y$, then let $g(x,y) = 0$.

Continuity of g

It is clear that the function g is well-defined and continuous when restricted to the subspace $Vert(x)$ of S^2 (with the Sorgenfrey topology), and that g maps $Vert(x)$ onto S in a one-to-one way.

To verify continuity at a point (z,z) of the diagonal, note that any sequence $\langle (x_k, y_k) \rangle$ that converges to (z,z) must eventually get between the step-function S_{-t} and S_t for arbitrarily small fractions $t > 0$. Consequently, the values $g(x_k, y_k)$ lie in arbitrarily small intervals $I(t) \subseteq [0,\epsilon)$ so that g is continuous at each (z,z).

To verify continuity of g at a point $(x,y) \in S^2$ with $x \neq y$, note that there is a step-function S_t such that (x,y) lies on or above the graph of S_t and strictly below the graph of the step function S_{t^+} (where t^+ is the immediate successor of t in T.) Because no horizontal segment of the graph of S_t or S_{t+} contains a right endpoint of itself, there must be a positive δ such that for each $x' \in [x, x+\delta)$, the point (x',y) lies between the graphs of S_t and S_{t+} and the second coordinates of the endpoints of the vertical segments $Vert(x,t)$ and $Vert(x',t)$ are the same irrational numbers. Because (x,y) lies strictly below the graph of S_{t^+} there is a positive ϵ such that the entire segment $\{x\} \times [y, y+\epsilon)$ lies below the graph of S_{t^+}. Now consider any point $(x',y') \in [x, x+\delta) \times [y, y+\epsilon)$. Because $Vert(x,t)$ and $Vert(x',t)$ have exactly the same endpoints $u < v$, the same function $h_{u,v,t}$ was used to define $g(x,y')$ and $g(x',y')$ and we have $g(x,y') = h_{u,v,t}(y') = g(x',y')$.

Therefore, on the open neighborhood $[x, x+\delta) \times [y, y+\epsilon)$ of (x, y) we have $g(x', y') = h_{u,v,t}(y')$, so that g is continuous at (x, y). □

Notice that at each point $(x, y) \in S^2$ with $x \neq y$, the Choban operator g constructed above is a local homeomorphism from part of $Vert(x)$ onto part of S, and is continuous but is not a local homeomorphism at points (x, x) of the diagonal. Proposition 4.7 shows that this kind of behavior is inevitable and allows us to say that the Sorgenfrey line is a homogeneous Choban space that is not continuously homogeneous (see Section 2).

Corollary 7.2 *The Sorgenfrey line is a homogeneous space with a Choban operator, but it is not continuously homogeneous.*

In fact we can say a bit more, using the ideas from the proof of Proposition 4.7.

Proposition 7.3 *: Let S be the Sorgenfrey line. For any Choban operator $h : S^2 \to S$, there is a dense G_δ-subset D of the diagonal such that if $(x, x) \in D$ then the restriction h to $Vert(x)$ is not a homeomorphism.* □

The key to Proposition 7.1 was our ability to find suitable mappings $h_{a,b,c,d} : [a, b] \to [c, d]$. These mappings needed to be 1-1, onto, continuous and order-preserving and have $h(a) = c$ and $h(b) = d$. In fact, it was enough to have such mappings when a and b are the values of consecutive step functions $S_t(x)$ and $S_{t^+}(x)$ for various x-values.

Note that if we replace the Sorgenfrey line S by the set $T = \mathbb{P}$ of irrational numbers or the set $T = \mathbb{Q}$ of all rational numbers (both with the Sorgenfrey topology) then one can still choose functions $h_{a,b,c,d} : [a, b) \cap T \to [c, d) \cap T$. Thus, the proof of Proposition 7.1 also gives:

Corollary 7.4 *: Let $T = \mathbb{Q}$ or $T = \mathbb{P}$. Then, with the Sorgenfrey topology, T has a Choban operator.* □

Remark 7.5 *:* Of course, we do not need Corollary 7.4 to handle the case of the rational numbers with the Sorgenfrey topology, because the space of rational numbers with the Sorgenfrey topology is homeomorphic to the space of rationals with the usual topology and the latter is a topological group. However, the space of irrationals with the Sorgenfrey topology is not homeomorphic to any topological group because it is first-countable and not metrizable.

Under suitable set theory, there are certain strange subspaces of the Sorgenfrey line that have Choban operators. For example, the Proper Forcing Axiom (PFA) guarantees that any $\aleph_1$-dense subset[6] D of $\mathbb{R}$ has the property that for $a < b$ and $c < d$ there is an order-preserving 1-1 mapping $h_{a,b,c,d}$ from $(a, b) \cap D$ onto $(c, d) \cap D$. There is no harm in assuming that $0 \in D$ – with trivial modifications, any point of D can replace 0. Modify the step functions S_t to have their values in the dense set $\mathbb{R} - D$, and change the intervals $I(t)$ in $\mathbb{R}$ to have both endpoints in $\mathbb{R} - D$. Then the proof of Corollary 7.1 will show that with the Sorgenfrey topology, any $\aleph_1$-dense set $D \subseteq \mathbb{R}$ has a Choban operator in any model of ZFC + PFA.

Recall from the previous section that assertion (*) is "If X is a complete, dense-in-itself, separable metric space, then X cannot be written as the union of fewer than $\mathfrak{c}$ many closed nowhere-dense sets."

Proposition 7.6 *: Assume that Assertion (*) holds and let $T = \{x_\alpha : \alpha < \omega_1\}$ be the Bernstein set constructed in Proposition 5.7. Let σ be the Sorgenfrey topology on T. Then (T, σ) cannot have a Choban operator.*

[6] A subset $D \subseteq \mathbb{R}$ is $\aleph_1$-dense provided $|D \cap (a, b)| = \aleph_1$ for each nonempty open interval (a, b).

Proof: Let $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{c}\}$ be the collection of functions defined in Example 5.7 (using the usual topology λ for $\mathbb{R}$, not the topology σ). We will need one crucial fact about the set T, namely that as a subspace of $\mathbb{R}$, the set T is a Baire space. We prove this assertion by showing that $T \cap G \neq \emptyset$ for each dense G_δ-subset G of the usual real line. Fix a dense G_δ-subset G of $\mathbb{R}$. Define $k : G \to \mathbb{R}$ by $k(x) = x + 1$. Then k, with domain G, is one of the functions in the collection $\mathcal{F}$, say $k = f_\alpha$ so that $x_\alpha \in T \cap G$. Hence $(T, \lambda|_T)$ is a Baire space.

For contradiction, suppose that $g : (T, \sigma)^2 \to (T, \sigma)$ is a Choban operator. There is a point $d \in T$ with $d = g(x, x)$ for all $x \in T$. Fix any $z \in T - \{d\}$ and define $h(x) = g(z, x)$ for each $x \in T$. Then $h : (T, \sigma) \to (T, \sigma)$ is 1-1 and continuous. Also, $h(d) = g(z, d) \neq g(z, z) = d$.

Define $O(n) = \{x \in \mathbb{R} :$ for some $a < x < b, diam(h[(a, b) \cap T]) < \frac{1}{n}\}$ as in Proposition 5.7. Clearly $O(n)$ is a dense open subset of $\mathbb{R}$. We cannot assert that $T \subseteq O(n)$ for all n but we can easily prove that $T \cap O(n)$ is a dense relatively open subset of $(T, \lambda|_T)$ where $\lambda|_T$ is the subspace topology that T inherits from $\mathbb{R}$. As noted above $(T, \lambda|_T)$ is a Baire space so that the set $E = \bigcap\{T \cap O(n) : n \geq 1\}$ is a dense subset of $(T, \lambda|_T)$. In addition, the function $h|_E$ can be extended to be a continuous function H on the set $M = \bigcap\{O(n) : n \geq 1\}$, and M is a dense G_δ-subset of $\mathbb{R}$. We cannot claim that $d \in E$. However, because E is dense in T and $h(d) \neq d$, there is some $e \in E$ with $h(e) \neq e$ so that $H(e) = h(e) \neq e$. Also note that E is a dense subset of the domain of H on which H is 1-1. Therefore $H \in \mathcal{F}$, so that $H = f_\alpha$ for some $\alpha < \omega_1$. Then T contains the point x_α which lies in $dom(f_\alpha) = M$, giving $x_\alpha \in M \cap T = E$. Therefore we have

$$g(z, x_\alpha) = h|_E(x_\alpha) = H(x_\alpha) = f_\alpha(x_\alpha) = y_\alpha \notin T$$

and that contradicts $g : T^2 \to T$. Therefore, (T, σ) cannot have a Choban operator. □

8 The Michael line has a Choban operator

Retopologize the set of real numbers by isolating every irrational and letting rationals have their usual open-interval neighborhoods. This space is called the *Michael line*. It is a GO-space but not a LOTS (no matter what ordering is used).

Proposition 8.1 *The Michael line M has a Choban operator.*

Proof: Let $a < b$ and $c < d$. As in Lemma 6.1 choose a continuous order-preserving function from $[a, b]$ onto $[c, d]$ that maps $(a, b) \cap \mathbb{Q}$ onto $(c, d) \cap \mathbb{Q}$ and $(a, b) \cap \mathbb{P}$ onto $(c, d) \cap \mathbb{P}$. This function is a homeomorphism from (a, b) onto (c, d) where both intervals carry the Michael line topology. If all of $a, b, c, d \in \mathbb{P}$ then the function is a homeomorphism from $[a, b]$ onto $[c, d]$ topologized as subsets of M. We will arrange that the intervals that we will use for domain and range of our functions will have all endpoints irrational.

Constructing $f : M \times M \to M$.

We will use the step-functions S_t for $t \in T$ constructed above. We will need to replace the intervals I(t) used in Proposition 7.1 with a new family of intervals that we will call $J(t) \subseteq \mathbb{R}$.

For any integer $t \in T$ with $t \geq 1$ or with $t \leq -2$ let $J(t) = [t\pi, (t + 1)\pi)$. For example, $J_{-2} = [-2\pi, -\pi)$ and $J_7 = [7\pi, 8\pi)$.

Let $J(-1) = [-\pi, \frac{-\pi}{2})$.

For any fraction $t \in T$, recall that t^+ is the immediate successor of t in T, so that if $t = \frac{1}{3}$ then $t^+ = \frac{1}{2}$ and if $t = \frac{-1}{8}$ then $t^+ = \frac{-1}{9}$. For each fractional $t \in T$ let $J(t) = [t\pi, t^+\pi)$. For example $J(\frac{1}{3}) = [\frac{\pi}{3}, \frac{\pi}{2})$ and $J(\frac{-1}{8}) = [\frac{-\pi}{8}, \frac{-\pi}{9})$.

We will use $e = 0$ as the special point of M and will define our function $f : M^2 \to M$ by defining it on each vertical line $Vert(x)$. Suppose x is fixed and $(x, y) \in Vert(x)$. If $x = y$ define $f(x, y) = 0$ (so that $f[\Delta] = 0$). If $x \neq y$ there there exist $t \in T$ so that (with t^+ being the immediate successor of t in T) we have $S_t(x) \leq y < S_{t^+}(x)$. Then $(x, y) \in Vert(x, t)$ where $Vert(x, t) = \{x\} \times [u(x, t), v(x, t))$ with $u(x, t) = S_t(x)$ and $v(x, t) = S_{t^+}(x)$. Writing $u = u(x, t), v = v(x, t)$ let $h_{u,v,t}$ be the order-preserving 1-1 onto mapping from $[u, v)$ onto the interval $J(t)$ found in Lemma 6.1. Note that both endpoints u, v and both endpoints of $J(t)$ are irrational and that $h_{u,v,t}$ takes rational numbers to rational numbers and irrational numbers to irrational numbers. Now define $f(x, y) = h_{u,v,t}(y)$.

It is clear that the function $f : M^2 \to M$ is well-defined. Also, recall that the endpoints of each $Vert(x, t)$ and of each $J(t)$ are irrational, and that $h_{u,v,t}$ maps rational numbers to rational numbers and irrational numbers to irrational numbers so that when $Vert(x, t)$ carries the subspace topology from M^2, the function f restricted to $Vert(x, t)$ is continuous at each point of $Vert(x, t) - \{(x, x)\}$. Also, the restriction of f to $Vert(x)$ is clearly 1-1 and onto the Michael line M.

It remains to prove that f is continuous at each point of M^2. There are two cases. Consider any point $(x, x) \in M^2$. If $x \in \mathbb{P}$ then (x, x) is isolated and there is nothing to prove. If $x \in \mathbb{Q}$ then basic neighborhoods of $(x, x) \in M^2$ have the form $(x - \epsilon, x + \epsilon)^2$ where $\epsilon > 0$. Consider any sequence (x_k, y_k) that converges to (x, x). For each fractional $t \in T$, this sequence is eventually between the step functions S_t and S_{-t} and therefore $f(x_k, y_k)$ will eventually lie arbitrarily close to $f(x, x) = 0$.

Finally consider the case where $(x, y) \in M^2$ with $x \neq y$. If $x \in \mathbb{P}$, then $Vert(x)$ is an open subset of M^2 and we already know that f is continuous when restricted to that set. Therefore, suppose $x \in \mathbb{Q}$. There are two sub-cases, depending upon whether $y \in \mathbb{Q}$ or $y \in \mathbb{P}$.

Consider the first sub-case, where both x and y are rational. Then (x, y) does not lie on the graph of any of our step functions (because the values produces by each step function are all irrational), so we may find $t \in T$ such that (x, y) lies strictly between the graphs of the consecutive step functions S_t and S_{t^+} and $(x, y) \in Vert(x, t) = \{x\} \times [u(x, t), v(x, t))$. Write $u = u(x, t), v = v(x, t)$. Because $y \in \mathbb{Q}$ while $u, v \in \mathbb{P}$ we have $(x, y) \in \{x\} \times (u, v)$ Next recall that the endpoints of the horizontal sections of the graphs of our step functions have irrational endpoints, so that the rational number x is not one of those endpoints. Therefore there is a $\delta > 0$ such that for each $x' \in (x - \delta, x + \delta)$ the point (x', y) is trapped between the graphs of S_t and S_{t^+}, showing that $u(x', t) = u(x, t)$ and $v(x', t) = v(x, t)$. Therefore the same function $h_{u,v,t}$ was used to define $f(x', y') = h_{u,v,t}(y')$ for each $(x', y') \in (x - \delta, x + \delta) \times (u, v)$ and that makes f continuous at each $(x, y) \in \mathbb{Q}^2$.

The second sub-case is where $x \in \mathbb{Q}$ and $y \in \mathbb{P}$. Then for some $t \in T$, $(x, y) \in Vert(x, t) = \{x\} \times [u, v)$ where $u = u(x, t), v = v(x, t)$. As above, x cannot be an endpoint of any horizontal segment of graph of any of our step functions, so there is a $\delta > 0$ such that for each $x' \in (x - \delta, x + \delta)$ we have $(x', y) \in Vert(x', t) = \{x'\} \times [u, v)$. Therefore the same function $h_{u,v,t}$ was used to define both $f(x, y)$ and $f(x', y)$ and we have $f(x, y) = h_{u,v,t}(y) = f(x', y)$ for all $x' \in (x - \delta, x + \delta)$. But because $y \in \mathbb{P}$, the set $(x - \delta, x + \delta) \times \{y\}$ is a neighborhood of (x, y) in M^2 and therefore f is continuous at (x, y), as required. □

The construction of the Michael line can be modified by choosing sets other than $\mathbb{P}$ to become the set of isolated points.

Question 8.2 *Let Y be a subset of $\mathbb{R}$ and let $M(Y)$ be the space obtained by isolating all points of Y and letting points of $\mathbb{R} - Y$ have their usual open-interval neighborhoods. Characterize those subsets Y such that $M(Y)$ has a Choban operator.*

With minor modifications, the proof of Proposition 8.1 shows the following:

Proposition 8.3 *Let X be the space obtained from $\mathbb{R}$ by isolating each rational number and letting irrationals have their usual neighborhoods. The X has a Choban operator.*

Proof: It is convenient to modify the step functions S_t in the proof of Proposition 8.1 so that they have rational rather than irrational values and so that all jumps occur at rational rather than irrational numbers. Also, the number 0 in that proof must be replaced by an irrational, say π and the intervals $J(t)$ must be shifted so that for fractional $t \in T$, the intervals $J(t)$ approach π. $\square$

Example 8.4 *Let X be the GO-space obtained by retopologizing $\mathbb{R}$ by isolating each integer and letting other points have their usual open-interval neighborhoods. Then X has a Choban operator.*

Proof: The space X^2 is the disjoint union of clopen subspaces $[n, n+1) \times X$. We will define a function h on the basic vertical strip $Y_0 = [0, 1) \times X$ and then define a Choban operator $H : X^2 \to X$ on strips $Y_n = [n, n+1) \times X$ after shifting them linearly onto Y_0 in such a way that the point (n, n) is moved to $(0, 0)$. We will use the special point $e = \frac{1}{2} \in X$.

Consider the basic strip Y_0. For $0 < x < 1$ let $B(x) = \{x\} \times (-\infty, x]$ be the points on or below the diagonal and let $A(x) = \{x\} \times [x, \infty)$ be the points on or above the diagonal.

a) If $x \le t \le 1$ let h map (x, t) linearly in the variable t onto $[\frac{1}{2}, 1]$ with $h(x, x) = \frac{1}{2}$ and $h(x, 1) = 1$.

b) If $1 \le t$ let $h(x, t) = t$. (Now h is defined on all of $A(x)$.)

c) If $0 \le t \le x$, let h map (x, t) linearly in the variable t onto $[0, \frac{1}{2}]$ in such a way that $h(x, 0) = 0$ and $h(x, x) = \frac{1}{2}$.

d) If $t \le 0$ let $h(x, t) = t$. (Now h is defined on all of $B(x)$.)

e) Define h on the clopen subspace $Vert(0) = \{0\} \times X$ of X^2 as follows:

$h(0, t) = t + 1$ if $t \le -1$;
h maps $\{0\} \times [-1, 0]$ linearly onto $[0, \frac{1}{2}]$ with $h(0, -1) = 0$ and $h(0, 0) = \frac{1}{2}$;
h maps $\{0\} \times [0, 1]$ linearly onto $[\frac{1}{2}, 1]$ with $h(0, 0) = \frac{1}{2}$ and $h(0, 1) = 1$
$h(0, t) = t$ if $1 \le t$;

The resulting function h is well-defined on the basic strip $Y_0 = [0, 1) \times X$, and because $Vert(0)$ is an open subset of Y_0, the function h is continuous on Y_0 as a subspace of X^2. Furthermore, $h(x, x) = \frac{1}{2}$ for each $x \in [0, 1)$ and for $0 \le x < 1$, h maps each $Vert(x)$ onto X is a 1-1 way.

We are now ready to define $H : X^2 \to X$ as follows. If $(x, t) \in [n, n+1) \times X$, define $H(x, t) = h(x - n, t - n)$. Then $H : X^2 \to X$ is continuous, $H(x, x) = \frac{1}{2}$ for all x, and H maps each vertical line $Vert(x)$ onto X in a 1-1 way. Consequently H is the required Choban operator. $\square$

Acknowledgment: The first author was supported by a faculty research grant from Texas Tech University during completion of this research.

References

[1] Alster, K., Subparacompactness in Cartesian products of generalized ordered topological spaces, *Fundamenta Mathematicae* 87(1975), 7-28.

[2] Arhangel'skii, A., Topological spaces with a flexible diagonal, *Questions and Answers in General Topology* 27(2009), 83-105.

[3] Arhangel'skii, A. and Choban, M., Completeness type properties of semitopological groups and the theorems of Montgomery and Ellis, *Topology Proceedings* 37(2011), 33-60.

[4] Balogh, Z. and Rudin M.E., Monotone normality, *Topology and its Applications* 47(1997), 115-127.

[5] Bennett, H., Burke, D., and Lutzer, D., The big bush machine, *Topology and its Applications* 159(2012), 1514-1528.

[6] Bennett, H., Burke, D., and Lutzer, D., Some questions of Arhangel'skii on rotoids, *Fundamenta Math.* 216(2012) 147-161.

[7] Bennett, H. and Lutzer, D., Separability, the countable chain condition, and the Lindelöf property in linearly orderable topological spaces, *Proceedings Amer. Math. Soc.* 23(1969), 664-667.

[8] Bennett, H. and Lutzer, D., On a question of Maarten Maurice, *Topology and its Applications*, 153(2006), 1631-1638.

[9] Choban, M., Some topics in topological algebra, *Topology and its Applications* 54(1993)183-202.

[10] Choban, M., The structure of locally compact algebras, *Baku Internat. Topology Conf.*, (1987) Abstract, p.334.

[11] Choban, M., On topological homogeneous algebras, *Interim Report of the Prague Topological Symposium* (1987), p.26.

[12] Choban, M., The structure of locally compact algebras, *Serdica* 18(1992), 129-137.

[13] van Douwen, E. K., The integers and topology, pp. 113-167 in *Handbook of General Topology* ed. by K. Kunen and J. Vaughan, North Holland, Amsterdam, 1984.

[14] Engelking, R., *General Topology*, Heldeman Verlag, Berlin, 1989.

[15] Engelking, R. and Lutzer D., Paracompactness in ordered spaces, *Fundamenta Mathematicae* 94(1976), 49-58.

[16] Faber, M.J., Metrizability in generalized ordered spaces, Mathematisch Centrum, Amsterdam, 1974.

[17] Gillman, L., and Jerrison, M., *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

[18] Gulko, A., Rectifiable Spaces, *Topology and its Applications* 68(1996), 107-112.

[19] Heath, R., Lutzer, D., and Zenor, P., Monotonically normal spaces, *Transactions of the American Mathematical Society*, 178(1973), 41-193.

[20] Kunen, K., Random and Cohen Reals, pp. 887-912 in *Handbook of General Topology* ed. by K. Kunen and J. Vaughan, North Holland, Amsterdam, 1984.

[21] Lutzer, D., A metrization theorem for linearly ordered topological spaces, *Proc. Amer. Math. Soc.* 22(1969), 557-8.

[22] Shapirovskii, B.E., Special types of embeddings in Tychonoff cubes, *Colloq. Math. Soc. Janos Bolyai* 23 (1978), 1055-1086.

[23] Uspenskii, V., Topological groups and Dugundji spaces *Math. Sb.* 180 (1989), 1092-1118 and *Math USSR Sbornik* 67(1990), 555-580.

[24] van Wouwe, J.M., GO-spaces and generalizations of metrizability, Mathematisch Centrum, Amsterdam, 1978.