Branch Space Representations of Lines

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Abstract: Todorčevic noted that any linearly ordered set $(X,<)$ is isomorphic to the branch space of some tree, provided the tree is allowed to be as complicated as X itself. In this paper we investigate representing linearly ordered sets as branch spaces of trees where the trees satisfy certain natural restrictions designed to make them less complicated than the set we are seeking to represent. We show that any uncountable order-complete linearly ordered set X can be represented as the branch space of a tree T that is more simple than X and that if X is representable as a branch space, then any G_δ subset of X is also representable as a branch space. However, we show that even though the usual set $\mathbb{R}$ of real numbers can be represented as the branch space of a tree with countable height and countable levels, most subsets of $\mathbb{R}$ cannot be represented in this way. We characterize those limit ordinals λ that can be represented by trees whose nodes do not contain copies of λ or λ^*. We also study topological properties of branch spaces. Our results show that if T is a tree of height ω_1 that does not contain any ω_1-branches, then the branch space of T must be hereditarily paracompact. We investigate branch spaces of Aronszajn trees, showing that any such branch space is hereditarily paracompact, first-countable, Lindelöf, non-separable, non-metrizable, and provided T does not contain any Souslin subtree, then its branch space is not perfect (i.e., has a closed subset that is not a G_δ-set). We also study the existence of σ-disjoint π bases and σ-disjoint bases in branch spaces of trees.

Key Words and Phrases linear order, tree, branch space, ordinals, Aronszajn tree, Aronszajn line, paracompact, Lindelöf, perfect space, σ-disjoint π-base, σ-disjoint base

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1 Introduction

By a *line* we mean a linearly ordered set $(X,<)$. Todorčevic pointed out that any line X is order isomorphic to the branch space of some tree, provided the tree is allowed to be as complicated as X itself. The problem studied in this paper is "Which lines $(X,<)$ can be represented, up to order isomorphism, as the branch space of a tree that is less complicated than X?"

Our terminology and notation for trees generally follow [9]. Recall that a *tree* is a partially ordered set $(T,\leq_T)$ such that for each $t\in T$ the set $T_t=\{s\in T : s\neq t, s\leq_T t\}$ is well-ordered by $\leq_T$. The order type of T_t is called the *level* of t, abbreviated $lv(t)$. For any ordinal α, the set $T_\alpha=\{t\in T : lv(t)=\alpha\}$ is the αth level of T. Because T is a set, there must be some ordinal α with $T_\alpha=\emptyset$; the first such ordinal is called the *height* of the tree and is denoted by $ht(T)$.

By a *path* in T we mean a subset $\rho\subseteq T$ that is linearly ordered by $\leq_T$ and has the property that if $s\leq_T t$ and $t\in\rho$, then $s\in\rho$. Each non-maximal path ρ determines a *node* of the tree by $Node(\rho)=\{t\in T : T_t=\rho\}$.

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Sometimes we will have a point $t \in T$ and we will want to look at the node of T to which t belongs. We will use the notation $N(t, T) = \{s \in T : T_t = T_s\}$.

Let $\mathcal{N}(T)$ be the collection of all nodes of the tree T. Observe that for any node $N \in \mathcal{N}(T)$, the points of N are incomparable with respect to $\leq_T$. For each node N we will choose a linear ordering $<_N$ of N. There is no necessary relation between the orderings of one node and another.

By a *branch* of T we mean a maximal path in T and we denote the set of all branches of T by $\mathcal{B}_T$. For any branch $b \in \mathcal{B}_T$ let $ht(b)$ be the order type of the well-ordered set b (ordered as a subset of $(T, \leq_T)$). For $\alpha < ht(b)$ let $b(\alpha)$ be the unique point of $b \cap T_\alpha$. For distinct $b, c \in \mathcal{B}_T$, there is a first ordinal $\delta = \Delta_T(b, c)$ such that $b(\delta) \neq c(\delta)$. Then $b(\delta), c(\delta)$ belong to the same node N of T and we define $b <_{\mathcal{B}} c$ if and only if in the ordering $<_N$ chosen for N, we have $b(\delta) <_N c(\delta)$. The *branch space* of the tree is the linearly ordered set $(\mathcal{B}_T, <_{\mathcal{B}})$. The term "branch space of T" is actually a misnomer, because the linear orderings chosen for the nodes have at least as much influence on the structure of the branch space as does the tree itself. If $t \in T$ and N is a node of T, then both $[t]_T = \{b \in \mathcal{B}_T : t \in b\}$ and $[N]_T = \bigcup\{[t]_T : t \in N\}$ are convex subsets of $\mathcal{B}_T$. One cannot (in general) assume that each $[t]_T$ has $|[t]_T| \geq 3$: see Proposition 2.3 in Section 2.

With that terminology and notation in hand, we can describe Todočevic's observation in [9] showing that any line is the branch space of a tree provided one is willing to allow the tree to be as complicated as the line one seeks to represent. For any linearly ordered set $(X, <)$ let $T = T_0 = X$ and use equality as the partial order on T. Then T_0 is the only node of T and we linearly order T_0 to make it a copy of $(X, <)$. The branch space of the resulting tree is a copy of $(X, <)$.

The above example shows that if one wants a reasonable branch space representation theory for linearly ordered sets, one needs to restrict the trees used to represent a given $(X, <)$ to make sure that they are less complicated than X itself. The literature contains several well-known ways to impose such restrictions on a tree T. One could impose cardinality constraints on the height of T, or on each level of T, or on each node of T, or on each anti-chain of T. Alternatively one could constrain the node orderings needed to define the linear ordering of the branches of T. For a given linearly ordered set $(L, <)$, we will say that the node-orderings of T are *L-non-degenerate* if for each node N of T, the linearly ordered set $(N, <_N)$ does not contain an order-isomorphic copy of $(L, <)$.

Throughout this paper we reserve the symbols $\mathbb{R}$, $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{Z}$ for the sets of real, irrational, and rational numbers, and for the set of all integers, respectively. If $f : X \to Y$ and $S \subseteq X$, then we abuse notation slightly by writing $f(S) = \{f(s) : s \in S\}$ in lieu of the more familiar $f[S]$ because notation involving square brackets already has too many meanings in our paper.

The authors want to thank the referee whose suggestions significantly improved our paper. In particular, the proofs given for Lemmas 2.1 and 6.1 are much clearer and shorter than the ones we originally proposed.

2 Simplifying Trees for Branch Space Constructions

In this section we present two technical lemmas that describe how we can simplify certain trees without changing their branch spaces. The first deals with trees in general, and the second deals with trees whose branch spaces are order isomorphic to sets of real numbers. The lemmas may be known, but we have not been able to find proofs in the literature. In the first reading of the next lemma, readers may want to ignore the references to L-non-degeneracy. This idea is not needed until the end of Section 4.

Lemma 2.1 *: Let $(L, <_L)$ be a linearly ordered set. Let $(T, \leq_T)$ be a tree and $\{<_N : N \in \mathcal{N}(T)\}$ a family of node orderings of T each of which is L-non-degenerate. Let $(\mathcal{B}_T, <_{\mathcal{B}_T})$ be the corresponding branch space. Then there is a subtree $(V, \leq_V)$ of T such that*

> *a)* $\mathcal{N}(V) = \{N \in \mathcal{N}(T) : |N| > 1\}$
>
> *b) if nodes of V are linearly ordered exactly as they are ordered in T, then the nodes of V are L-non-degenerate and the branch space $(\mathcal{B}_V, <_{\mathcal{B}_V})$ is order-isomorphic to $(\mathcal{B}_T, <_{\mathcal{B}_T})$.*
>
> *c) each node of V has at least two elements, so that each non-maximal element of V splits in V.*

Proof: Once assertions (a) and (b) are established, (c) is immediate.

Let $\mathcal{M}(T) = \{N \in \mathcal{N}(T) : |N| > 1\}$ and let $V = \bigcup \mathcal{M}(T)$. Let V carry the partial order induced by the partial order of T. We first show that $\mathcal{N}(V) = \mathcal{M}(T)$.

Claim 1: If N is a node of V, then for some $M \in \mathcal{M}(T),\ N \subseteq M$. Fix a node N of V and let $x \in N$. Then there is some $M \in \mathcal{M}(T)$ with $x \in M$. For contradiction, suppose that $N \not\subseteq M$. Choose $y \in N$ with $y \notin M$. Because $x, y \in N$ we have

$$(*) \quad \{z \in V : z < y\} = \{z \in V : z < x\}$$

and because $y \notin M$ we have

$$(**) \quad \{w \in T : w < y\} \neq \{w \in T : w < x\}.$$

For contradiction, suppose that $\{w \in T : w < y\} \subseteq \{w \in T : w < x\}$. Let r be the unique point of the set $\{w \in T : w < x\}$ whose level in T is the same as $lv_T(y)$. If $y = r$, then $y < x$ and that contradicts $(*)$. Hence $r \neq y$ and we have $\{w \in T : w < r\} = \{w \in T : w < y\}$ so that some node P of T contains both y and r. Then $|P| \geq 2$ so that $r \in P \subseteq V$ and that contradicts $(*)$.

An analogous argument shows that $\{w \in T : w < x\} \subseteq \{w \in T : w < y\}$ is also impossible.

At this stage, we know that neither $\{w \in T : w < x\}$ nor $\{w \in T : w < y\}$ is a subset of the other. Then $(**)$ allows us to choose the first ordinal α so that the sets $\{w \in T : w < x\}$ and $\{w \in T : w < y\}$ contain distinct points r, s, respectively, with $lv_T(r) = lv_T(s) = \alpha$. Then $\{w \in T : w < r\} = \{w \in T : w < s\}$ so that r and s are distinct members of the same node Q of T. But then $r, s \in Q \subseteq V$ and once again we have contradicted $(*)$. Therefore, $N \subseteq M$ and Claim 1 is established.

Claim 2: If $M \in \mathcal{M}(T)$ then some $N \in \mathcal{N}(V)$ has $M \subseteq N$. This follows from the fact that if $x, y \in M$ then $\{w \in T : w < x\} = \{w \in T : w < y\}$ and the facts that $\{z \in V : z < x\} = \{w \in T : w < x\} \cap V$ and $\{w \in T : w < y\} \cap V = \{z \in V : z < y\}$.

Together, Claim 1 and Claim 2 establish (a).

Now linearly order the nodes of V using the node orderings of T and let $(\mathcal{B}_V, <_{\mathcal{B}_V})$ denote the branch space of V. If $b \in \mathcal{B}_T$, it is easy to see that $b \cap V \in \mathcal{B}_V$. Therefore, the rule $f(b) = b \cap V$ defines a function from $\mathcal{B}_T$ to $\mathcal{B}_V$. To show that f is strictly increasing, suppose $a, b \in \mathcal{B}_T$ with $a <_{\mathcal{B}_T} b$. Let $\delta = \Delta(a, b)$ and choose distinct $x \in a,\ y \in b$ with $lv_T(x) = lv_T(y) = \delta$. But then $\{w \in T : w < x\} = \{w \in T : w < y\}$ so that x and y are distinct points of the same node M of T. Therefore $x, y \in M \subseteq V$ showing that $a \cap V$ precedes $b \cap V$ in the ordering of $\mathcal{B}_V$. Finally, suppose that $c \in \mathcal{B}_V$. Then c is a linearly ordered subset of T so there is some $b \in \mathcal{B}_T$ with $c \subseteq b$. But then $f(b) = c$ so that f is also seen to be onto. Therefore, $\mathcal{B}_T$ is isomorphic to $\mathcal{B}_V$, as required. $\square$

Lemma 2.2 *Suppose $(T, \le_T)$ is a tree with countable nodes and that $\{<_N : N \in \mathcal{N}(T)\}$ is a fixed family of node orderings for T. Suppose that the resulting branch space $(\mathcal{B}_T, <_{\mathcal{B}_T})$ is known to be order isomorphic to some set X of real numbers. Then there is a subtree $(U, \le_U)$ of T having countable levels and height $\le \omega$ and a set of node orderings $\{<_M : M \in \mathcal{N}(U)\}$ such that the corresponding branch space is order isomorphic to X.*

Proof: Recall that a subset $C \subseteq X \subseteq \mathbb{R}$ is *relatively convex* in X if given three points $x < y < z$ of X with $x, z \in C$, we have $y \in C$.

We may assume that T satisfies Lemma 2.1. Let $f : \mathcal{B}_T \to X$ be an order isomorphism from the branch space of T onto a set $X \subseteq \mathbb{R}$.

Claim 1: Each branch of T has height $< \omega_1$ and hence T has height $\le \omega_1$. For suppose some branch $b \in \mathcal{B}_T$ has height $\ge \omega_1$. Then either $\{\inf_{\mathbb{R}}(f([t]_T) : t \in b\}$ or $\{\sup_{\mathbb{R}}(f([t]_T)) : t \in b\}$ contains an uncountable strictly increasing (respectively, decreasing) ω_1 sequence in $\mathbb{R}$ and that is impossible. Thus, each branch las height $< \omega_1$ and therefore the height of T is $\le \omega_1$.

Claim 2: Each level of T is countable. In the light of Lemma 2.1, each node of T has at least two points. Fix an ordinal α and consider the collection $\mathcal{N}_\alpha$ consisting of all nodes of T at level α and for $N \in \mathcal{N}_\alpha$ let $[N]$ be the set of all branches b of T with $b \cap T_\alpha \subseteq N$. Then $\{conv(f([N])) : N \in \mathcal{N}_\alpha\}$ is a pairwise disjoint collection of convex subsets of $\mathbb{R}$, where $conv(S)$ denotes the convex hull in $\mathbb{R}$ of a set S. Such a collection must be countable, so that $\mathcal{N}_\alpha$ must be countable. Because each individual node of T is known to be countable, each level of T must be countable. $\square$

Claim 3: The height of T is less than ω_1. For contradiction, suppose that $ht(T) = \omega_1$. Because $X \subseteq \mathbb{R}$, there is a countable set $D \subseteq X$ that is *order dense* in X, i.e., if $x < y$ belong to X, then $[x, y] \cap D \ne \emptyset$. (Note: this is not the same as being topologically dense in X.) Let $\alpha_0 = \sup(\{ht(b) : b \in \mathcal{B}_T, f(b) \in D\})$. Because D is countable and f is 1-1, Claim 1 shows that $\alpha_0 < \omega_1$. Because $\alpha_0 < \omega_1$, $T_{\alpha_0+1} \ne \emptyset$. Choose $t \in T_{\alpha_0+1}$. The node N of T to which t belongs has at least two members, so that $f([N])$ is a non-degenerate, relatively convex subset of X. However, $f([N]) \cap D = \emptyset$ and that is impossible because D is an order-dense subset of X. Therefore, $ht(T) < \omega_1$ as claimed.

For each $t \in T$, let I_t be the convex hull in $\mathbb{R}$ of the set $f([t]_T) = \{f(b) : t \in b\}$. If s and t are distinct and belong to the same level of T, then I_s and I_t are disjoint subsets of $\mathbb{R}$. For each $b \in \mathcal{B}_T$, let $K_b = \bigcap\{I_t : t \in b\}$.

Claim 4: For each $b \in \mathcal{B}_T$ the set K_b is a closed and bounded subset of $\mathbb{R}$. Compute $f(b) \in \mathbb{R}$. For each $t \in b$ choose branches $b_0^t,\ b_1^t \in [t]$ with $b_0^t \le_{\mathcal{B}_T} b \le_{\mathcal{B}_T} b_1^t$ with strict inequalities whenever possible. Then $K_b = \bigcap\{I_t : t \in b\} = \bigcap\{[f(b_0^t), f(b_1^t)] : t \in b\}$, where the second equality follows from the fact that whenever possible, we used strict inequalities in choosing the branches b_i^t. Therefore the set K_b is closed and bounded in $\mathbb{R}$.

For each $b \in \mathcal{B}_T,\ f(b) \in K_b$. If b_1 and b_2 are distinct branches of T and if $\delta = \Delta_T(b_1, b_2)$, then $b_1(\delta) \ne b_2(\delta)$. Write $t_i = b_i(\delta)$. Then $K_{b_i} \subseteq I_{t_i}$ forces $K_{b_1} \cap K_{b_2} = \emptyset$.

Define $\pi : \mathbb{R} \to \mathbb{R}$ by the rule that if $x \in K_b$ for some $b \in \mathcal{B}_T$, then $\pi(x) = f(b)$ and for all other $x \in \mathbb{R}$ define $\pi(x) = x$. Then $Y = \pi[\mathbb{R}] \subseteq \mathbb{R}$.

Claim 5: The set $Y = \pi[\mathbb{R}]$ with the order inherited from $\mathbb{R}$ is order isomorphic to $\mathbb{R}$. Because $|Y| > 1$, to prove this assertion, it will be enough to show that Y has a countable order-dense subset, has no endpoints, is densely ordered, and has the least upper bound property (because that list of properties characterizes the ordered set $\mathbb{R}$). Because each set K_b is bounded, Y has no end points. That Y has a countable order-dense subset follows from $Y \subseteq \mathbb{R}$. To see that Y is densely ordered, suppose $y_1 < y_2$ in Y. Because π is weakly increasing (i.e., if $x_1 < x_2$

in $\mathbb{R}$ then $\pi(x_1) \leq \pi(x_2)$ in Y) we know that if $x_i \in \pi^{-1}(y_i)$ then $x_1 < x_2$. Furthermore, the sets $\pi^{-1}(y_i)$ are each compact and therefore the numbers $x_1 = \sup(\pi^{-1}(y_1))$ and $x_2 = \inf(\pi^{-1}(y_2))$ both exist in $\mathbb{R}$ and $x_i \in \pi^{-1}(y_i)$. Hence $x_1 < x_2$. Choose any $z \in (x_1, x_2)$. Then $\pi(z) \in Y \cap (\pi(x_1), \pi(x_2)) = Y \cap (y_1, y_2)$. Thus Y is densely ordered.

Finally, Y has the least upper bound property because $\mathbb{R}$ has that property and π is weakly increasing and $\pi^{-1}(y)$ is a compact, convex subset of $\mathbb{R}$ for each $y \in Y$. Hence, $Y = \pi(\mathbb{R})$ is order isomorphic to $\mathbb{R}$. Let $g : \pi(\mathbb{R}) \to \mathbb{R}$ be an order isomorphism.

Claim 6: For any branch $b \in \mathcal{B}_T$, $\bigcap\{\pi(I_t) : t \in b\} = \pi(K_b) = \{f(b)\}$. Clearly $\pi(K_b) \subseteq \bigcap\{\pi(I_t) : t \in b\}$. For a given $t \in b$ note that $\pi(I_t) \subseteq I_t$ and therefore $\bigcap\{\pi(I_t) : t \in b\} \subseteq \bigcap\{I_t : t \in b\} = K_b$. But then $\bigcap\{\pi(I_t) : t \in b\} \subseteq K_b \cap \pi(\mathbb{R}) = \pi(K_b)$, as required.

Define a function $d : Y \times Y \to [0, \infty)$ by the rule that if $y_1, y_2 \in \pi[\mathbb{R}]$, then $d(y_1, y_2) = |g(y_1) - g(y_2)|$. Then d is a metric on the set $\pi(\mathbb{R})$. We use d to define the diameter of a set in the usual way.

Claim 7: For any branch b of T, $\inf\{diam_d(\pi(I_t)) : t \in b\} = 0$. Fix $b \in \mathcal{B}_T$ and suppose that $u, v \in \pi(\mathbb{R})$ have $u < \pi(f(b)) < v$ in $\pi(\mathbb{R})$. Choose $x_u = \max \pi^{-1}(u)$ and $x_v = \min \pi^{-1}(v)$. Because the set $\pi^{-1}(y)$ is compact for every $y \in \pi(\mathbb{R})$, both x_u and x_v exist and belong to $\pi^{-1}(u)$ and $\pi^{-1}(v)$ respectively. Then $K_b \subseteq (x_u, x_v)$. Because the sets I_t are convex and have $\bigcap\{I_t : t \in b\} = K_b$, some $t \in b$ has $I_t \subseteq (x_u, x_v)$, showing that the d-diameter of $\pi(I_t)$ is less than the d-diameter of any interval (u, v) with $u, v \in \pi(\mathbb{R})$ and $\pi(f(b)) \in (u, v)$. Therefore, $\inf\{diam_d(\pi(I_t)) : t \in b\} = 0$ as claimed.

For each $n \geq 1$ let $V_n = \{t \in T : diam_d(\pi(I_t)) < \frac{1}{n}$ and $s <_T t \Rightarrow diam_d(\pi(I_s)) \geq \frac{1}{n}\}$.

Claim 8: Each V_n is a maximal anti-chain of T.

Clearly V_n is an anti-chain. To prove maximality, consider any $t_0 \in T$. Choose any branch b_0 of T that contains t_0. According to Claim 8, some $t \in b_0$ has $diam_d(\pi(I_t)) < \frac{1}{n}$. Let t_1 be the first member of b_0 with that property. Then $t_1 \in V_n$ and the points t_0 and t_1 are comparable in T because both belong to the branch b_0. Hence V_n is maximal.

Now define $U = \bigcup\{V_n : n \geq 1\}$ and partially order U by restricting the partial order of T. From earlier claims we know that T has countable height and countable levels, so that $|U| \leq |T| \leq \omega$. (Notice that we do *not* claim that the sets V_n are the levels of U.)

Claim 9: If $v_i \in V_i$ and $v_j \in V_j$ have $v_i <_T v_j$ then $i < j$. If $j \leq i$ then $\frac{1}{i} \leq \frac{1}{j}$ so that $diam_d(\pi(I_{v_i})) < \frac{1}{i} \leq \frac{1}{j}$. But $v_i <_T v_j$ and that contradicts the minimality condition built into the definition of V_j.

Claim 10 The height of U is $\leq \omega$.

Otherwise there would be some element $u^* \in U$ with height ω in U. List the predecessors of u^* in U as $\{u_k : k \geq 1\}$ and choose integers n_k with $u_k \in V_{n_k}$. Because each V_{n_k} is an anti-chain, no more than one of the points u_k can belong to any one set V_{n_k}, showing that the set $N^* = \{n_k : k \geq 1\}$ must be infinite. Because $u^* \in U$ there is some integer $n^* \geq 1$ with $u^* \in U_{n^*}$. But then Claim 9 shows that $n_k \leq n^*$ for each k, so the set N^* must be finite. That contradiction completes the proof of Claim 10.

Claim 11: Let c be any branch of the tree $(U, \leq_U)$ and let $\psi(c) = \{t \in T : \exists u \in c \text{ having } t \leq_T u\}$. Then $\psi(c)$ is a branch of T. If that is not true, there is a branch b of T having $\psi(c)$ properly contained in b. Choose t^* from the lowest possible level of T having $t^* \in b$ and $u <_T t^*$ for every $u \in c$. Because V_n is a maximal anti-chain in T, there is some $u_n \in V_n$ that is comparable to t^*. Consider any $n \geq 1$. If it happens that $t^* \leq_T u_n$ then we have found an element $u_n \in U$ that lies strictly above every member of the branch c of U, and that is impossible. Hence $u_n <_T t^*$ so that $u_n \in b$. But then $u_n \in U \cap b = c$ for every $n \geq 1$. Therefore,

$diam_d(\pi(I_{t^*})) \leq diam_d(\pi(I_{u_n})) < \frac{1}{n}$ for each $n \geq 1$ showing that $diam_d(\pi(I_{t^*}) = 0$ and hence that exactly one branch of T contains t^*. Therefore t^* is a maximal element of T and the set $b(t^*) = \{t \in T : t \leq t^*\}$ is a branch of T.

Because T satisfies Lemma 2.1, the node of T to which t^* belongs has at least two elements. Choose s^* distinct from t^* in that node. Then with notation as in the previous paragraph, $u_n < s^*$ for each n so that s^* is also maximal in T and the set $b(s^*) = \{t \in T : t \leq s^*\}$ is a branch of T that is distinct from $b(t^*)$. Then $f(b(s^*)) \neq f(b(t^*))$ and $\pi(f(b(s^*))) \neq \pi(f(b(t^*)))$ are elements of the set $\pi(I_{u_n})$ for each n, showing that $diam_d(\pi(I_{u_n}))$ cannot be made arbitrarily small, contrary to $u_n \in V_n$. Therefore, Claim 11 is established.

Fix a node M of U. We linearly order M as follows. Let $s, t \in M$ be distinct. Then s and t are incomparable in T so that in the branch space $\mathcal{B}_T$ of T, the set $[s]_T = \{b \in \mathcal{B}_T : s \in b\}$ and the analogously defined set $[t]_T$ are disjoint convex sets. Define $s <_M t$ if and only if every point of $[s]_T$ precedes every point of $[t]_T$ in $(\mathcal{B}_T, <_{\mathcal{B}_T})$.

The next two claims complete the proof by showing that $\mathcal{B}_U$ is order isomorphic to $\mathcal{B}_T$.

Claim 12: The function $\psi : \mathcal{B}_U \to \mathcal{B}_T$ is strictly increasing. Let $c_1 <_{\mathcal{B}_U} c_2$ be branches of U. Let $\delta = \Delta_U(c_1, c_2)$ be the first level of U where the branches c_1 and c_2 differ. Then $c_1(\delta), c_2(\delta)$ belong to the same node M of U and we know that $c_1(\delta) <_M c_2(\delta)$. But then in the branch space of T we know that each branch belonging to the set $[c_1(\delta)]_T$ precedes each branch belonging to $[c_2(\delta)]_T$ and we know that $\psi(c_i) \in [c_i(\delta)]_T$. Hence ψ is strictly increasing.

Claim 13: The function $\psi : \mathcal{B}_U \to \mathcal{B}_T$ is onto. Let $b \in \mathcal{B}_T$. Because $\inf\{diam_d(\pi(I_t)) : t \in b\} = 0$, we know that $b \cap V_n \neq \emptyset$ for every $n \geq 1$. Let $c = b \cap U$. Then c is a branch of U and $\psi(c) = b$. □

The first two results in this section have shown that one can assume that the trees involved have been "cleaned up" without changing their branch spaces, and another result of this type will appear in the section on Aronszajn lines, below. One might wonder whether it is possible to make other assumptions about the trees being used, to make branch space constructions smoother. For example, could we always assume that each element of the tree splits? The next result from [1] provides a negative answer. Its hypothesis is even weaker than "everything splits": it holds, for example, if every $t \in T$ has at least three branches that contain t.

Proposition 2.3 *: Let $(T, \leq_T)$ be a tree with a family of node orderings and suppose that, when the branch space $\mathcal{B}_T$ of T is endowed with its open interval topology, each set $[t]_T$ has non-empty interior in $\mathcal{B}_T$. Then $\mathcal{B}_T$ is a Baire space, i.e., the intersection of countably many dense open sets is dense.*

In fact, one can show that the branch space of a tree as in Proposition 2.3 is actually α-favorable [1]. See Section 4 for definitions.

3 Branch Space Representation of Compact and Čech-complete Lines

Recall that a linearly ordered set $(X, <)$ is *order-complete* if every subset of X, including $\emptyset$ and X, has a least upper bound in X. It is well-known that $(X, <)$ is order-complete if and only if X is compact when endowed with its usual open interval topology. A completely regular topological space is said to be *Čech-complete* if it is a G_δ-subset of some (equivalently, of each) of its compactifications. For a linearly ordered set $(X, <)$, that is equivalent to saying that X is a G_δ-subset of its Dedekind completion X^+ when X^+ carries its open interval topology.

Our first theorem will make use of a standard line to tree construction called a *partition tree*. This idea is in widespread use, but the literature contains many different descriptions of it. We will construct our partition trees as follows. For an order-convex subset C of a linearly ordered set X and for an ordinal α, we will have a collection $P_\alpha(C)$ of pairwise disjoint convex subsets of C. Usually $P_\alpha(C)$ is expected to cover C, but sometimes it does not. Some of the members of $P_\alpha(C)$ might be degenerate, i.e., might be singleton convex sets. In some cases the subscript α is irrelevant, and then we suppress it, writing $P(C)$. We construct the partition tree T recursively. Let $T_0 = \{X\}$. If T_α is defined for some ordinal α, let $T_{\alpha+1} = \bigcup\{P_{\alpha+1}(C) : C \in T_\alpha, |C| \geq 2\}$. If λ is a limit ordinal and if T_α is defined for each $\alpha < \lambda$, then let

$$T_\lambda = \{D = \bigcap\{C_\alpha : \alpha < \lambda\} : C_\alpha \in T_\alpha, |D| \geq 2\}.$$

Partially order T by reverse inclusion. Then T is a tree. Because X is a set, there is some α with $T_\alpha = \emptyset$. The height of the tree is the least such α and $T = \bigcup\{T_\alpha : T_\alpha \neq \emptyset\}$.

Next we will go from the partition tree T to its branch space $(\mathcal{B}_T, <_\mathcal{B})$, something that requires us to linearly order each node. Observe that each node of the partition tree T is a family of pairwise disjoint convex subsets of $(X, <)$ and therefore inherits a natural ordering that we call the *precedence ordering* from X. That is, if C, D are distinct members of a node N, we say that $C <_N D$ if and only if every pair of points $x \in C,\ y \in D$ has $x < y$ in the ordering of X. Let $(\mathcal{B}_T, <_{\mathcal{B}_T})$ be the resulting branch space. There is a natural strictly increasing function $i : (X, <) \to (\mathcal{B}_T, <_{\mathcal{B}_T})$ given by $i(x) = \{t \in T : x \in t\}$. Unfortunately, this function is not necessarily onto, and is not necessarily continuous when both X and $\mathcal{B}_T$ carry their usual open interval topologies. See Section 2 of [1] for details.

We use a certain cardinal invariant of the ordered set X to impose constraints on T. It is a relative of the familiar cardinal invariant of a topological space X called cellularity [3]. Recall that a topological space X has *cellularity* $c(X)$ if $c(X)$ is the least cardinal such that if $\mathcal{U}$ is a family of pairwise disjoint, non-empty open subsets of X, then $|\mathcal{U}| \leq c(X)$. For a linearly ordered set $(X, <)$ the *order cellularity* of $(X, <)$ is the least cardinal $orc(X)$ such that every family $\mathcal{C}$ of pairwise disjoint non-degenerate (= has more than one point) convex subsets of X has cardinality $\leq orc(X)$. These two cardinal invariants can be different. For example, in the lexicographically ordered set $X = \mathbb{R} \times \{0, 1\}$ with its usual open interval topology, we have $c(X) = \omega < 2^\omega = orc(X)$. In general, $orc(X)$ is the maximum of the topological cellularity of X (equipped with its open interval topology) and the number of jumps in X (i.e. pairs of consecutive points of X). Another familiar cardinal invariant of the linearly ordered set $(X, <)$ is $supcf(X)$ (for "supremum of cofinalities"), defined to be the least cardinal number κ such that for each $x \in X,\ cf(\{y \in X : y < x\}) \leq \kappa$. The analog for co-initialities is $supci(X)$ and is analogously defined.

Proposition 3.1 *: Suppose that $(X, <)$ is an infinite order-complete linearly ordered set. Then there is a tree T such that:*

a) each node N of T has $|N| \leq supcf(X)$;

b) each level T_α of T has $|T_\alpha| \leq orc(X)$;

c) each branch b of T has $|b| \leq \max\{supcf(X), supci(X)\} \leq orc(X)$; and

d) there is a node ordering for T so that the associated branch space of T is order isomorphic to X.

Proof: For any interval $[a, b) \subseteq X$ with at least two points, the cardinal number $cf(b)$ is finite if and only if b has an immediate predecessor b^- in X and in that case we define $P[a, b) = \{[a, b^-), \{b^-\}\}$. If the cardinal $cf(b)$ is infinite, then there is a strictly increasing net $\{x_\alpha : 0 \leq \alpha < cf(b)\}$ that is cofinal in $[a, b)$, has $x_0 = a$, and has the property that each set $[x_\alpha, x_{\alpha+1})$ has at least two points. Because X is order-complete, we may also assume that for each limit ordinal $\lambda < cf(b)$ we have $x_\lambda = \sup_X\{x_\alpha : \alpha < \lambda\}$. Then define $P[a, b) = \{[x_\alpha, x_{\alpha+1}) : 0 \leq \alpha < cf(b)\}$. For an interval $[a, b]$, define $P[a, b] = \{[a, b), \{b\}\}$.

We now define sets T_α recursively. Because X is order-complete, there are points a_0, b_0 with $X = [a_0, b_0]$. Let $T_0 = \{[a_0, b_0]\}$ and $T_1 = \{[a_0, b_0), \{b_0\}\}$. Next, suppose $\gamma > 1$ and suppose that T_α is defined for all $\alpha < \gamma$ in such a way that:

1) for any non-limit ordinal $\beta < \gamma$, each non-degenerate member of T_β can be written as $[a, b)$;

2) for any limit ordinal $\beta < \gamma$, each member of T_β can be written in the form $[a, b]$ and has at least two points; and

3) if $\beta_1 < \beta_2 < \gamma$ and if $B_i \in T_{\beta_i}$, then either $B_1 \cap B_2 = \emptyset$ or else $\mathrm{cl}(B_1) \subseteq B_2$. (We will refer to this last property as "strong nesting.")

In case γ is a limit ordinal, define $T_\gamma = \{D = \bigcap\{C_\beta : \beta < \gamma\} : C_\beta \in T_\beta, |D| \geq 2\}$. The strong nesting property in the induction hypothesis guarantees that each $D \in T_\gamma$ is compact and convex, and hence can be written as $D = [a, b]$ for suitably chosen $a, b \in X$. In case $\gamma = \beta + 1$ let $T_\gamma = \bigcup\{P(C) : C \in T_\beta, |C| \geq 2\}$.

Let $T = \bigcup\{T_\alpha : T_\alpha \neq \emptyset\}$ and partially order T by reverse inclusion. Order the nodes of T by the precedence order from X. Define $i : X \to \mathcal{B}_T$ by $i(x) = \{t \in T : x \in t\}$. Then i is 1-1 and increasing. Because X is compact and each branch of T is strongly nested, we see that i is also onto, as required to prove (d).

Observe that every node of T is either finite or has cardinality equal to the cofinality of some point of X. This proves (a).

We prove (b) by induction. Clearly (b) holds for T_0 and T_1 defined above. Suppose that (b) holds for each $\beta < \alpha$. If α is a limit ordinal, then each set $C \in T_\alpha$ is a convex subset of X with at least two points. Hence T_α is a collection of pairwise disjoint non-degenerate convex subsets of X, and so $|T_\alpha| \leq orc(X)$. In case $\alpha = \beta + 1$, then each member $C \in T_\alpha$ is either a non-degenerate convex subset of X (and there are at most $orc(X)$ of these), or else there is some $C' \in T_\alpha$ such that $C \cup C'$ is a member of T_β. Hence $|T_\alpha| \leq orc(X)$.

To prove (c), note that any infinite branch b of T is strongly nested and the intersection of the members of b is a single point $x \in X$. The cardinality of b cannot exceed $\max\{cf(x), ci(x)\}$ and in any case $cf(x)$ and $ci(x)$ cannot exceed $orc(X)$. $\square$

A well-known result due to Todorčevic [9] is useful in recognizing branch spaces that are order complete. Todorčevic proved that if each node of a tree T is order-complete, then the branch space $\mathcal{B}(T)$ of T is also order-complete. That theorem suggests asking whether, if a branch space $\mathcal{B}(T)$ is order-complete, the nodes of T must be order-complete. The answer is "no" as our next example shows. Also, in the light of the previous proposition, Todorčevic's theorem suggests asking whether any order-complete line must be representable as the branch space of a tree with order-complete nodes. If one is willing to use trees whose nodes are as complicated as the line being represented, the answer to the second question is "yes" – one uses the trivial tree described in the introduction. But if one wants to use trees that are less complicated than the linearly ordered set being represented, then our next example shows that the answer is "no". Recall the notion of "L-non-degenerate" in the first section of this paper.

Example 3.2 *: The order-complete linearly ordered set $[0, 1]$ is isomorphic to the branch space of a countable tree (by Proposition 3.1) and yet $[0, 1]$ is not isomorphic to the branch space of any tree T that has order-complete nodes that are $[0, 1]$-non-degenerate.*

Proof: For contradiction suppose there is a tree T and a set of node orderings so that no node of T contains an order-isomorphic copy of $[0, 1]$ and that the associated branch space is order isomorphic to $[0, 1]$.

Because the branch space of T is not finite, some node N of T must have more than one point. Fix any such node N of T. Choosing a branch $b_t \in [t]_T$ for each $t \in N$ gives an order isomorphism from N into $[0, 1]$ and consequently N has a countable order-dense subset. Next, suppose there are points $s, t \in N$ with $s <_N t$ and such that no point of N lies strictly between s and t in the ordering of N. Then, because each node of T is order-complete, there is a branch $b \in [s]$ such that for each α with $lv(s) \leq \alpha < ht(b)$, the point $b(\alpha)$ is the supremum of the node to which it belongs. Similarly, there is a branch $c \in [t]$ such that $c(\beta)$ is the infimum of the node to which it belongs whenever $lv(t) \leq \beta < ht(c)$. But then b and c are adjacent points of $\mathcal{B}_T$ which is impossible because $\mathcal{B}_T$ is order isomorphic to $[0, 1]$. Therefore, the node ordering of N must be order-dense. But then the node N is non-degenerate, order-complete, dense-ordered, and has a countable order-dense subset, and that is enough to make N order isomorphic to $[0, 1]$, which is impossible. □

There is a family of order complete lines that admit branch space representations in which each node has either one or two points, as our next result shows.

Proposition 3.3 *Suppose $(X, <)$ is an order-complete linearly ordered set with the property that given $x < y$ in X, there exist points $u, v \in X$ such that $x \leq u < v \leq y$ and no point of X lies strictly between u and v. Then there is a tree T in which each node has either one or two members and whose branch space is order isomorphic to X.*

Proof: For any interval $[x, y] \subseteq X$ with at least two points, choose u, v as in the hypothesis of the proposition and let $Q[x, y] = \{[a, u], [v, b]\}$. Now use Q to build a partition tree T for X. Each node of T will have either one or two members, and the branch space of T will be order isomorphic to X. □

For another result related to Todorčevic's theorem, see the end of Section 4 where we show that a compact subset $S \subseteq \mathbb{R}$ is isomorphic to the branch space of a tree with countable, order-complete nodes if and only if S is totally disconnected.

In the remainder of this section we study the following problem: suppose that a linearly ordered set $(X, <)$ is known to be isomorphic to the branch space of some "nice" tree. For which subsets $Y \subseteq X$ can we be sure that Y can also be represented as the branch space of some similarly nice tree? The next result deals with a general situation in which a branch space representations can be found.

Proposition 3.4 *: Suppose $(X, <)$ is a linearly ordered set and that T is a tree such that, for some choice of node orderings, there is an order isomorphism f from the branch space $(\mathcal{B}_T, <_{\mathcal{B}_T})$ onto $(X, <)$. Suppose that, when X carries the open interval topology of $<$, Y is a G_δ subset of X. Then there is a subtree $S \subseteq T$ and node orderings for S such that*

> *a) there is a strictly increasing function $j : \mathcal{B}_S \to \mathcal{B}_T$ such that $g = f \circ j$ is an order isomorphism from $\mathcal{B}_S$ onto Y;*
>
> *b) each level of S is an anti-chain in T.*

Proof: In this proof, once the subtree $S \subseteq T$ is constructed we will need to carefully distinguish between the set $[t]_T = \{b \in \mathcal{B}_T : t \in b\}$ and the analogously defined $[s]_S = \{c \in \mathcal{B}_S : s \in c\}$.

Write $Y = \bigcap\{U_n : n \geq 1\}$ where each U_n is open and $U_{n+1} \subseteq U_n$. For $n \geq 1$ define

$$S_n = \{t \in T : f([t]_T) \cap Y \neq \emptyset, f([t]_T) \subseteq U_n, \text{ and if } t^{'} <_T t \text{ then } f([t^{'}]_T) \not\subseteq U_n\}.$$

Note that each S_n is an anti-chain in T. Now let

$$S = \left(\bigcup\{S_n : n \geq 1\}\right) \cup \{t \in T : f([t]_T) \subseteq Y\}$$

and partially order S by restricting the ordering of T.

Suppose that c is a branch of S. Let $c^* = \{t \in T : \exists s \in c \text{ with } t \leq_T s\}$. We claim that c^* is a branch of T. For contradiction, suppose not. Then there is some $t^* \in T - c^*$ with $t \leq_T t^*$ for each $t \in c^*$. Because c is a branch in S, $t^* \notin S$. There are three cases to consider.

<u>Case 1</u>: Suppose some $s \in c$ has $f([s]_T) \subseteq Y$. Then $s \in c^*$ so that $s \leq_T t^*$ and hence $f([t^*]_T) \subseteq f([s]_T) \subseteq Y$, whence $t^* \in S$ and that is impossible. Therefore, Case 1 can never occur.

<u>Case 2</u>: Suppose $c \subseteq \bigcup\{S_n : n \geq 1\}$ and $c \cap S_n \neq \emptyset$ for infinitely many values of n. Let $n_1 < n_2 < \cdots$ be integers such that we can choose $s_k \in c \cap S_{n_k}$. Then for each k, $s_{n_k} \leq_T t^*$ so that $f([t^*]_T) \subseteq f([s_k]_T) \subseteq U_{n_k}$. But then $f([t^*]_T) \subseteq \bigcap\{U_{n_k} : k \geq 1\} = Y$ showing that $t^* \in S$, and that is impossible. Thus Case 2 cannot occur.

<u>Case 3</u>: Suppose $c \subseteq \bigcup\{S_n : n \geq 1\}$ and $c \cap S_n \neq \emptyset$ for only finitely many integers. Because S_n is an anti-chain of T, $|c \cap S_n| \leq 1$ for each n. Hence, in Case 3, the branch c of S is finite. Let s be the maximum element of c and let m be an integer with $s \in S_m$. Then $f([s]_T) \subseteq U_m$ and $f([s]_T) \cap Y \neq \emptyset$. However, because we are not in Case 1, $f([s]_T) \subseteq Y$ is impossible. Therefore we may choose branches $b_1,\ b_2 \in [s]_T$ with $f(b_1) \in Y$ and $f(b_2) \notin Y$. Because $Y = \bigcap\{U_i : i \geq 1\}$ we may choose $k > m$ with $f(b_2) \notin U_k$. Because $f(b_1) \in Y \subseteq U_k$ there are points $r < s$ of X with $f(b_1) \in (r, s) \subseteq U_k$. Then there is some $\alpha < ht(b_1)$ with $f([b_1(\alpha)]_T) \subseteq (r, s) \subseteq U_k$. Let α_0 be the first ordinal with $f([b_1(\alpha_0)]_T) \subseteq U_k$. Then $b_1(\alpha_0) \in S_k \subseteq S$. Because s and $b_1(\alpha_0)$ both belong to the branch b_1 we know that either $b_1(\alpha_0) \leq_T s$ or else $s <_T b_1(\alpha_0)$. The first option would yield $b_1, b_2 \in [s]_T \subseteq [b_1(\alpha_0)]_T$ so that $f(b_1), f(b_2) \in f([b_1(\alpha_0)]_T) \subseteq U_k$ contrary to $f(b_2) \notin U_k$. The second option is impossible because c is a branch of S. Hence Case 3 cannot occur.

It follows that c^* must be a branch of T, as claimed. Now define $j : \mathcal{B}_S \to \mathcal{B}_T$ by the rule that $j(c) = c^*$. It is easy to see that j is 1-1.

The next step in the proof is to define a family of node orderings for the tree S. Suppose that s_1, s_2 are distinct members of S with $S_{s_1} = S_{s_2}$, where $S_{s_i} = \{s \in S : s \neq s_i, s \leq_S s_i\}$, and let N be the node of S to which the s_i belong. Then s_1, s_2 are incomparable elements of T so that $f([s_1]_T)$ and $f([s_2]_T)$ are disjoint nonempty convex subsets of X. We define $s_1 <_N s_2$ if and only if for every $x_i \in f([s_i]_T)$, $x_1 < x_2$. The relation $<_N$ linearly orders the node N.

We claim that the function j above is strictly increasing. Suppose that $c_1, c_2 \in \mathcal{B}_S$ are distinct and have $c_1 \leq_{\mathcal{B}_S} c_2$. Let δ be the first ordinal such that in the tree S we have $c_1(\delta) \neq c_2(\delta)$ where $c_i(\delta)$ is the unique point of $c \cap S_\delta$, and let N be the node of S to which the $c_i(\delta)$ both belong. Then in N we have $c_1(\delta) <_N c_2(\delta)$ which means that in $(X, <)$ we know that every point of the convex set $f([c_1(\delta)]_T)$ precedes every point of $f([c_2(\delta)]_T)$. Because $f(c_i^*) \in f([c_i(\delta)]_T)$ we see that $f(c_1^*) < f(c_2^*)$ so that, f being an order isomorphism, $c_1^* <_{\mathcal{B}_T} c_2^*$ in the branch space $\mathcal{B}_T$. Thus, $j : \mathcal{B}_S \to \mathcal{B}_T$ is strictly increasing.

Next we claim that if $c \in \mathcal{B}_S$ then $f(c^*) \in Y$. This is proved by a three case analysis as above, depending upon whether some $s \in c$ has $f([s]_T) \subseteq Y$.

Finally, we claim that if $y \in Y$ then some $c \in \mathcal{B}_S$ has $f(c^*) = y$. Fix $y \in Y$. There is a unique $b \in \mathcal{B}_T$ with $f(b) = y$ because f is onto. Let $c = S \cap b$. We claim that c is a branch of S and that $c^* = b$. There are two cases to consider.

Case 4: Suppose some $t_0 \in b$ has $f([t_0]_T) \subseteq Y$. Then $t_0 \in c$. If c is not a branch of S then there is some $s^* \in S$ having $s <_S s^*$ for each $s \in c$. In particular, $s^* \notin c$. Let t_1 be any member of b. Choose $t_2 \in b$ with $t_2 = \max(t_0, t_1)$. This is possible because t_0, t_1 both lie in the branch b and therefore are comparable elements of T. Then $t_0 \leq_T t_2$ forces $t_2 \in S \cap b$ so that $t_2 \in c$ and hence $t_2 \leq_S s^*$. Because $t_1 \leq_T t_2 \leq_T s^*$ we see that s^* is an element of T that has $t_1 \leq_T s^*$ for every $t_1 \in b$. Because b is a branch of T, we must have $s^* \in b$ and therefore $s^* \in c$, and that is impossible. Hence in Case (4) we see that c is a branch of S. It is easy to check that because $t_0 \in c$ the set c contains a cofinal subset of b. Hence $c^* = b$ as claimed.

Case 5: Suppose that no $t \in b$ has $f([t]_T) \subseteq Y$. We know that $f(b) \in Y \subseteq U_n$ for each $n \geq 1$, so there is a first ordinal α_n with $f([b(\alpha_n)]_T) \subseteq U_n$. Then $b(\alpha_n) \in S_n$ so that $b(\alpha_n) \in c$. We claim that $\{b(\alpha_n) : n \geq 1\}$ is a cofinal subset of b, because otherwise some $b(\alpha)$ has $f([b(\alpha)]_T) \subseteq \bigcap\{U_n : n \geq 1\} = Y$ which is impossible in Case (5). (From this it will follow that $c^* = b$ once we know that c is a branch of S.) For contradiction suppose that some $s^* \in S - c$ has $s \leq_S s^*$ for each $s \in c$. Then this s^* is a point of T that has $b(\alpha_n) \leq_T s^*$, so that s^* lies strictly above a cofinal subset of b and that is impossible because b is a branch of T. This completes the proof in Case 5. $\square$

If T is a very nice tree, say a countable tree, then the subtree S found in the proof of Proposition 3.4 is equally nice. But in other situations, the subtree S can be very different from the tree T, e.g., in terms of its cardinal invariants, as our next example shows.

Example 3.5 *: The linearly ordered set $X = [0, \omega_1]$ is isomorphic to the branch space of a tree T that has height ω_1 and that has every level finite (see Proposition 5.1). The subtree S of T found in the proof of Proposition 3.4 to represent the open subset $Y = [0, \omega_1)$ of X has a single level and that level has cardinality ω_1.* $\square$

It is no accident that, in the previous example, some node of S is very large. As will be seen from Corollary 5.4, any tree S whose branch space represents $[0, \omega_1)$ must have a node that contains a copy of ω_1 or of ω_1^*.

4 Branch Space Representations of Subsets of $\mathbb{R}$

The results from the previous section show that the usual two-point compactification $[-\infty, \infty]$ of $\mathbb{R}$ is order isomorphic to the branch space of a tree T with countable levels and countable branches. Furthermore, the tree T must have countable height because otherwise T would be an Aronszajn tree, and Aronszajn trees cannot have separable branch spaces (see Proposition 6.3). Then Proposition 3.4 shows that the set $\mathbb{R}$, the set $\mathbb{P}$ of all irrational numbers, and each closed subset of $\mathbb{R}$ are also representable as branch spaces of countable trees. However, it is easy, and sometimes useful, to give concrete branch space representations of the sets $\mathbb{R}$ and $\mathbb{P}$.

Example 4.1 *: The sets $\mathbb{R}$ and $\mathbb{P}$ are each representable as the branch spaces of trees with countable levels and height ω.*

Proof: To obtain $\mathbb{R}$, for any interval $[a, b) \subseteq \mathbb{R}$, let

$$P[a,b) = \{[a, a + \frac{b-a}{2}), [a + \frac{b-a}{2}, a + \frac{3(b-a)}{4}), [a + \frac{3(b-a)}{4}, a + \frac{5(b-a)}{6}), \cdots\}$$

and order each $P[a, b)$ using the precedence order from $\mathbb{R}$. Now let $T_0 = \{[n, n+1) : n \in \mathbb{Z}\}$ and let $T_{n+1} = \bigcup\{P[a, b) : [a, b) \in T_n\}$. Partially order $T = \bigcup\{T_n : 0 \leq n < \omega\}$ by reverse inclusion. The crucial property of T is that if $s \leq_T t$, then $\mathrm{cl}_{\mathbb{R}}(t) \subseteq s$ and as a result, each branch of T has non-empty intersection. Then it is easy to see that the natural injection $i(x) = \{t \in T : x \in t\}$ is the required order isomorphism from $\mathbb{R}$ onto the branch space $\mathcal{B}_T$. To represent $\mathbb{P}$, fix an indexing $\mathbb{Q} = \{q_n : n \geq 1\}$ of the set of rational numbers and let $S_0 = \{(n, n+1) : n \in \mathbb{Z}\}$. For any interval (a, b) and $n \geq 1$ let $P_n(a, b)$ be a family of pairwise disjoint open intervals of $\mathbb{R}$ each with rational endpoints, each with length less than $\frac{b-a}{2}$, none containing the rational number q_n, and such that with the natural precedence ordering form $\mathbb{R}$, the collection $P(a, b)$ is a copy of $\mathbb{Z}$. Now for $n \geq 0$ define $S_{n+1} = \bigcup\{P_{n+1}(a, b) : (a, b) \in S_n\}$ and let $S = \bigcup\{S_n : 0 \leq n < \omega\}$ be partially ordered by reverse inclusion. Order the nodes of S using the natural precedence order from $\mathbb{R}$. Once again, if $s \leq_S t$ are distinct elements of S, then $\mathrm{cl}_{\mathbb{R}}(t) \subseteq s$ so that branches of S have non-empty intersection, and the intersection contains no rational number. Hence it is easy to see that the natural injection is an order isomorphism from $\mathbb{P}$ onto the branch space of S. $\square$

Next consider which subsets of $\mathbb{R}$ are representable by nice branch spaces. The remaining results in this section will show while $\mathbb{R}$ is representable as a branch space of a tree with countable levels and countable height (indeed, with height ω as seen in Example 4.1), most subsets of $\mathbb{R}$ cannot be represented in this way. (See Corollary 4.3.) We will rely on the technical lemmas from Section 2.

Proposition 4.2 *: Suppose $X \subseteq \mathbb{R}$ is order isomorphic to the branch space of a tree T having countable nodes. Then X is an $F_{\sigma\delta}$-subset of $\mathbb{R}$ and there is a countable subtree $U \subseteq T$ such that X is order isomorphic to the branch space of U.*

Proof: According to Lemmas 2.1 and 2.2, we may assume that T has countable levels and has height ω (and then we take $U = T$). Let f be an order isomorphism from $(\mathcal{B}_T, <_{\mathcal{B}_T})$ onto X. As in Lemma 2.1 define subsets I_t and $K_b = \bigcap\{I_t : t \in b\}$ of $\mathbb{R}$ for each $t \in T$ and $b \in \mathcal{B}_T$. Note that each set I_t, being a convex subset of $\mathbb{R}$, is an F_σ-subset of $\mathbb{R}$.

Write $T = \bigcup\{T_n : n < \omega\}$ as the union of its levels. Because the collection $\{I_t : t \in T_n\}$ is pairwise disjoint for each n, we have

$$(*) \;\; X = \{f(b) : b \in \mathcal{B}_T\} \subseteq \bigcup_{b \in \mathcal{B}_t} K_b \subseteq \bigcap \left\{ \bigcup\{I_t : t \in T_n\} : n \geq 1\} \right\}.$$

Write $Y = \bigcap\{\bigcup\{I_t : t \in T_n\} : n \geq 1\}\}$ and note that Y is an $(F_\sigma)_\delta$-subset of $\mathbb{R}$.

The containment in $(*)$ is strict, provided some set $K_b = \bigcap\{I_t : t \in b\}$ has more than one element. Note that the collection $\{K_b : b \in \mathcal{B}_T, |K_b| > 1\}$ is a pairwise disjoint collection of non-degenerate convex subsets of $\mathbb{R}$ and therefore must be countable. For each K_b with $|K_b| > 1$, the set $L_b = K_b - \{f(b)\}$ is an F_σ-subset of $\mathbb{R}$ and hence so is the set $Z = \bigcup\{L_b : b \in \mathcal{B}_T \text{ and } |K_b| > 1\}$. Therefore set $\mathbb{R} - Z$ is a G_δ-subset of $\mathbb{R}$ and hence also an $(F_\sigma)_\delta$-subset of $\mathbb{R}$. From (*), we have $X = \{f(b) : b \in \mathcal{B}_T\} = Y \cap (\mathbb{R} - Z)$ so we see that X is the intersection of two $(F_\sigma)_\delta$ subsets of $\mathbb{R}$ and hence is an $(F_\sigma)_\delta$-subset of $\mathbb{R}$ as claimed. $\square$

Corollary 4.3 *: There are 2^ω subsets of $\mathbb{R}$ that are order isomorphic to the branch space of a tree with countable nodes, and there are 2^{2^ω} subsets of $\mathbb{R}$ that are not order isomorphic to a branch space of any tree with countable nodes.*

Proof: There are at most 2^ω subsets of $\mathbb{R}$ that are $F_{\sigma\delta}$-sets in $\mathbb{R}$ and 2^{2^ω} that are not. Now apply Proposition 4.2. $\square$

Proposition 4.4 *: Let T be any countable tree. Then for every choice of node orderings, the branch space of T is isomorphic to some $F_{\sigma\delta}$-subset of $\mathbb{R}$.*

Proof: For each $t \in T$ let P_t be a one or two point subset of $[t]_T$ that contains each endpoint of $[t]_T$ if any such endpoints exist. Then the set $D = \bigcup\{P_t : t \in T\}$ is a countable order-dense subset of the branch space $\mathcal{B}_T$ so that the branch space is order isomorphic to some set X of real numbers. Now apply Proposition 4.2. $\square$

Our next three results provide a negative answer to the question "Is it true that every $F_{\sigma\delta}$-subset of $\mathbb{R}$ is order-isomorphic to the branch space of some countable tree?" It uses the Banach-Mazur game (see [7]).

Recall that the Banach-Mazur game in a topological space $(X, \mathcal{T})$ is a game with players α and β in which β opens the game by specifying a non-empty open set U_0 and then the players alternately choose non-void open sets $U_0, \supseteq U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$. Player α wins if and only if $\bigcap\{U_n : n < \omega\} \neq \emptyset$. To say that the space $(X, \mathcal{T})$ is α-favorable means that player α has a winning strategy for the game, i.e., a function σ that gives, for each finite sequence $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_{2n}$ of non-void open sets, a non-void open set $U_{2n+1} = \sigma(U_0, U_1, \cdots, U_{2n})$ in such a way that α wins any play of the game where all odd-numbered sets are chosen using strategy σ. The notion of a β-favorable topological space is analogously defined.

Proposition 4.5 *Suppose that $(T, \leq_T)$ is a countable tree with some family of node orderings, and let $\mathcal{B}$ be the associated branch space. Then in the open-interval topology of the branch space ordering, either $\mathcal{B}$ has a non-void countable open set, or else $\mathcal{B}$ is α-favorable.*

Proof: Suppose that every non-void open subset of $\mathcal{B}$ is uncountable. Let $S = \{t \in T : |[t]| \leq 2\}$. Then S is countable and hence so is $C = \bigcup\{[t] : t \in S\}$. Therefore, if U is any non-void open subset of $\mathcal{B}$, we know that $U - C \neq \emptyset$.

Suppose that β begins the Banach-Mazur game by specifying a non-empty open set U_0. Choose a branch $b_1 \in U_0 - C$. Then choose $t_1 \in b_1$ with $b_1 \in [t_1] \subseteq U_0$. Because $b_1 \notin C$, we know that $t_1 \notin S$ and therefore $Int([t_1]) \neq \emptyset$ (because $[t_1]$ is a convex subset of $\mathcal{B}$ with at least three points). Player α defines $U_1 = Int([t_1])$.

Suppose that $(U_0, U_1, \cdots, U_{2n})$ is a decreasing sequence of non-void open sets where $U_{2k+1} = Int([t_{2k+1}])$ where $t_1 \leq_T t_3 \leq_T \cdots \leq_T t_{2n-1}$. Player α notes that $U_{2n} - C \neq \emptyset$ and chooses $b_{2n+1} \in U_{2n} - C$. Then $b_{2n+1} \in U_{2n} \subseteq [t_{2n-1}]$ and we may choose $t_{2n+1} \in b_{2n+1}$ with $b_{2n+1} \in [t_{2n+1}] \subseteq U_{2n}$. Because both t_{2n-1} and t_{2n+1} belong to b_{2n+1} we may assume that $t_{2n-1} \leq_T t_{2n+1}$. Because $b_{2n+1} \notin C$, we know that $t_{2n+1} \notin S$ and therefore player α may respond to β's move by defining $U_{2n+1} = Int([t_{2n+1}])$.

If $U_0, U_1, U_2, \cdots)$ is a play of the game in which α has used the above strategy, then we have $t_1 \leq_T t_3 \leq_T t_5 \leq_T \cdots$ so, by Zorn's lemma, there is some branch c of the tree that contains every t_{2k+1}. But then $c \in \bigcap\{U_n : n < \omega\}$, as required. $\square$

Corollary 4.6 *Suppose $X \subseteq \mathbb{R}$ is order-isomorphic to the branch space of some tree $(T, \leq_T)$ having countable nodes. Then in its topology as a subspace of $\mathbb{R}$, either X has a non-void countable open set or else X is α-favorable.*

Proof: We will need to distinguish carefully between the topology $\mathcal{T}$ that X inherits from $\mathbb{R}$ and the open-interval topology $\mathcal{L}$ of the linear order that X inherits from $\mathbb{R}$. In general, $\mathcal{L} \subseteq \mathcal{T}$ and the two are not the same. However, it is always true that $\mathcal{T}$ is a GO topology on $(X, <)$, where $<$ is the linear ordering that X inherits from $\mathbb{R}$. To begin the proof, suppose that $(X, \mathcal{T})$ has no countable non-void open set. Because $\mathcal{L} \subseteq \mathcal{T}$, neither does $(X, \mathcal{L})$.

According to Lemma 2.2 we may assume that the tree T is countable. Then Proposition 4.5 shows that since each non-void open subset of $(X, \mathcal{L})$ is uncountable, $(X, \mathcal{L})$ is α-favorable. Then it is easy to prove that $(X, \mathcal{T})$ must also be α-favorable: indeed, if $(Y, <_Y)$ is any linearly ordered set such that the usual open-interval topology of $<_Y$ is α-favorable, then so is $(Y, \mathcal{T})$ for any GO-topology $\mathcal{T}$ on $Y, <_Y)$. $\square$

We would like to thank Arnold Miller for suggesting the subset of $\mathbb{R}$ used in our next corollary.

Corollary 4.7 *There is a dense-in-itself $F_{\sigma\delta}$ subset of $\mathbb{R}$ that is not order isomorphic to the branch space of any tree with countable nodes.*

Proof: The product space $\mathbb{Q}^\omega$ is an absolute $F_{\sigma\delta}$ set, i.e., if $\mathbb{Q}^\omega$ is embedded in any complete metric space Y, then its image is an $F_{\sigma\delta}$ subset of Y (see Chapter III, Section 35.IV, Corollary 1 of [5]). Furthermore, each non-void open subset of $\mathbb{Q}^\omega$ is uncountable and $\mathbb{Q}^\omega$ is not a Baire space so that it is not α-favorable. However, $\mathbb{Q}^\omega$ is homeomorphic to a subset X of $\mathbb{R}$, and X is an $F_{\sigma\delta}$ in $\mathbb{R}$. In the light of Corollary 4.6, the subset X cannot be order-isomorphic to the branch space of any tree with countable nodes. $\square$

Question: Which $F_{\sigma\delta}$ subsets of $\mathbb{R}$ are order-isomorphic to the branch space of a tree with countable nodes?

Probably the most simple interesting $F_{\sigma\delta}$ subset of $\mathbb{R}$ is the set $\mathbb{Q}$ of all rational numbers. It is clear that the set $\mathbb{Q}$ of rational numbers can be represented as the branch space of a tree with countable levels and countable height: one could let $T = T_0 = \mathbb{Q}$, use equality as the partial ordering of T, and linearly order the unique node T_0 to make it a copy of $\mathbb{Q}$. What is surprising is that, in some sense, this is the only way to represent $\mathbb{Q}$ as the branch space of a tree with countable height and countable levels, as our next result shows.

Proposition 4.8 *: Let $(T, \leq_T)$ be a tree and let $\{<_N : N \in \mathcal{N}(T)\}$ be a set of node orderings such that the branch space of T is order isomorphic to $\mathbb{Q}$. Then for some node N of T, $(N, <_N)$ contains a copy of $\mathbb{Q}$, i.e., the node orderings are not $\mathbb{Q}$-non-degenerate.*

Proof: For contradiction, suppose that $(T, \leq_T)$ is a tree with a set of $\mathbb{Q}$-non-degenerate node orderings whose branch space is order isomorphic to $\mathbb{Q}$. Then each node of T must be countable (or otherwise we could choose uncountably many branches of T) so that Lemmas 2.1 and 2.2 allow is to assume that T has countable levels and height ω. For each $n < \omega$, let T_n be the n-th level of T.

<u>Claim 1</u>: Let $(N, <_N)$ be any node of T and suppose $N \subseteq T_n$. We claim that if $s <_N t$ belong to N, then the set $J = \{b \in \mathcal{B}_T : ht(b) > n \text{ and } b(n) \in N \text{ and } s \leq_N b(n) \leq_N t\}$ contains an interval of $\mathcal{B}_T$ that is order isomorphic to $\mathbb{Q}$. Choose any $b_s \in [s]_T$ and $b_t \in [t]_T$. Then $b_s <_{\mathcal{B}_T} b_t$. Consider any branch b with $b_s <_{\mathcal{B}_T} b <_{\mathcal{B}_T} b_t$. Then $ht(b) > n$ and $b_s(n) \leq_N b(n) \leq_N b_t(n)$ so $b \in J$. Therefore J above contains a non-empty interval $[b_s, b_t]_{\mathcal{B}_T}$ of $\mathcal{B}_T$. But because $\mathcal{B}_T$ is order isomorphic to $\mathbb{Q}$, it follows that J contains an interval of $\mathcal{B}_T$ that is order isomorphic to $\mathbb{Q}$.

<u>Claim 2</u>: Suppose $t \in T$ and t splits in T (i.e., has at least two immediate successors in T). Then the set $S = \{s \in T : t \leq_T s \text{ and } s \text{ splits in } T\}$ is *not* a chain. For contradiction, suppose S is a chain. Let $n = lv(t)$. Let $U = \{u \in T : t \leq_T u\}$. Then U is a subtree of T and every node of U at level $k \geq 1$ of U is a node of T

at level $n + k$. Linearly order the nodes of U exactly as they are ordered in T. We claim that every node of U is finite. Clearly the 0-th level node of U is finite – it consists of t alone. Let M be any node of U at level $k > 0$ and suppose M is infinite. Then M is a node of T at level $n + k$. Because S is a chain, $|S \cap M| \leq 1$ so we may choose $u <_M v$ in M in such a way that no point of $S \cap M$ lies between u and v in the ordering $<_M$. Then no point of M lying between u and v splits, so each $w \in M$ with $u \leq_M w \leq_M v$ is maximal in T. Choose branches $b_u \in [u]_T$ and $b_v \in [v]_T$. According to Claim 1, the set $J = \{b \in \mathcal{B}_M : ht(b) > n_0 + k \text{ and } u \leq_M b(n+k) \leq_m v\}$ contains a copy of $\mathbb{Q}$. But maximality of all points of M between u and v then tells us that $(M, <_M)$ contains a copy of $\mathbb{Q}$ and that is impossible. Therefore M must be finite. It follows from Todorčevic's theorem [9] that the branch space $\mathcal{B}_U$ is order-complete. Clearly the branch space $\mathcal{B}_U$ of the subtree U is order isomorphic to the convex subset $[t]_T$ of the branch space $\mathcal{B}_T$ of T. Because t splits, we know that $[t]_T$ is (order isomorphic to) a convex non-degenerate subset of $\mathbb{Q}$. But that is impossible because there is no order-complete convex subset of $\mathbb{Q}$ that has more than one point. We conclude that the set $S = \{s \in T : t \leq_T s \text{ and } s \text{ splits}\}$ is not a chain, i.e., must contain two elements that are not comparable in T.

Clearly some elements of the tree T must split – otherwise the unique node at the 0-th level of T would contain a copy of $\mathbb{Q}$. Let t_0 be any element of T_0 that splits in T. Let $V = \{v \in T : t_0 \leq_T v \text{ and } v \text{ splits}\}$. Apply Claim 2 recursively to show that each $v \in V$ has two incomparable successors in V. Therefore V contains a copy of the complete binary tree and consequently V has 2^ω branches, which is impossible because $|\mathbb{Q}| = \omega$. □

Remark 4.9 : *One can prove even more: if a tree T has countable nodes and if the node orderings of T are $\mathbb{Q}$-non-degenerate then the branch space $\mathcal{B}_T$ cannot be homeomorphic to the space $\mathbb{Q}$ under any mapping.*

We conclude this section on representing subsets of $\mathbb{R}$ as branch spaces by characterizing subsets of $\mathbb{R}$ that can be represented as the branch space of trees with countable, order complete nodes. Recall that a space is *totally disconnected* if $|X| > 1$ and the only connected subsets of X are singletons.

Proposition 4.10 : *Let X be a non-degenerate order-complete (i.e., compact) subset of $\mathbb{R}$. Then X can be represented as the branch space of a tree T with countable, order-complete nodes if and only if X is totally disconnected.*

Proof: Proposition 3.3 shows that if X is totally disconnected then X is representable as the branch space of a binary tree. Conversely, suppose $X \subseteq \mathbb{R}$ is order isomorphic to the branch space of a tree T with countable order-complete nodes. Now consider two branches $a <_{\mathcal{B}_T} d$ of T. Compute $\delta = \Delta_T(a, d)$, the first level of T where the two branches differ. Then $a(\delta)$ and $d(\delta)$ belong to the same node N of T so that, because N is countable and order-complete, there must exist $s, t \in N$ with $a(\delta) \leq_N s <_N t \leq_N d(\delta)$ with no point of $(N, <_N)$ lying strictly between s and t. Because each node of T is order-complete, there is a branch $b \in [s]_T$ with the property that whenever $\delta < \alpha < ht(b),\ b(\alpha)$ is the maximum of the node of T to which it belongs. Similarly there is a branch $c \in [t]_T$ such that whenever $\delta < \alpha < ht(c),\ c(\alpha)$ is the minimum of the node to which it belongs. Then $a \leq_{\mathcal{B}_T} b <_{\mathcal{B}_T} c \leq_{\mathcal{B}_T} d$ and no branch of T lies strictly between b and c. Hence the branch space of T must be totally disconnected. Hence so is X. □

5 Representing Ordinals as Branch Spaces

It follows from Proposition 3.1 that for each ordinal α, the set $[0, \alpha]$ (i.e., the ordinal $\alpha + 1$) can be represented as the branch space of a tree. However, the tree in (3.1) might have large height and large nodes and we have

Proposition 5.1 *: For each ordinal α, the linearly ordered set $[0, \alpha]$ is order isomorphic to the branch space of a tree of height α with nodes having exactly two points. (Such a tree is often called a binary tree.)*

Proof: Let T be the set $[0, \alpha) \times \{0, 1\}$. Partially order T by the rules that

a) $(\beta, 0) \leq (\gamma, 0)$ whenever $\beta \leq \gamma < \alpha$;
b) $(\beta, 0) \leq (\gamma, 1)$ whenever $\beta \leq \gamma < \alpha$;
c) there are no other relations between points of T.

The branches of T have the form $b(\beta) = \{(\gamma, 0) : \gamma \leq \beta\} \cup \{(\beta, 1)\}$ for each $\beta < \alpha$ plus the long branch $b(\alpha + 1) = \{(\beta, 0) : \beta < \alpha\}$. The nodes of T are the sets $\{(\beta, 1), (\beta + 1, 0)\}$ and $f(\beta) = b(\beta)$ is the required order isomorphism. □

A more interesting question is "For which limit ordinals λ can the set $[0, \lambda)$ be represented as a branch space?"

Proposition 5.2 *: Suppose λ is a limit ordinal that is not a regular cardinal. Then $[0, \lambda)$ is order isomorphic to the branch space of some tree whose nodes are both λ non-degenerate and λ^*-non-degenerate (where λ^* is the reversed ordering of λ).*

Proof: Compute $\kappa = cf(\lambda)$ and find a strictly increasing function $f : [0, \kappa) \to [0, \lambda)$ such that $f([0, \kappa))$ is cofinal in $[0, \lambda)$, $f(0) = 0$ and each $f(\alpha)$ is a limit ordinal. For each $\gamma < \kappa$ define $I_\gamma = [f(\gamma) + 1, f(\gamma + 1)]$ if γ is not a limit ordinal, and $I_\gamma = [f(\gamma), f(\gamma + 1)]$ if γ is a limit ordinal (including $\gamma = 0$). From Proposition 5.1 we know that each I_γ is isomorphic to the branch space of a binary tree $(T(\gamma), \leq_\gamma)$ whose height is $f(\gamma + 1)$. We may assume that γ is the root of $T(\gamma)$, i.e., $T_0(\gamma) = \{\gamma\}$, for each $\gamma < \kappa$ and that the trees $T(\gamma)$ are pairwise disjoint sets.

Define a new tree S by specifying that $S_\alpha = \bigcup\{T_\alpha(\gamma) : \gamma < \kappa\}$ and $\leq_S = \bigcup\{\leq_{T(\gamma)} : \gamma < \kappa\}$. Thus, the α-th level of S is the union of the α-th levels of the trees $T(\gamma)$. Order the node S_0 to make it a copy of κ and order all other nodes of S just as they are ordered in one of the trees $T(\gamma)$. The branch space of S is then a disjoint union of copies of the sets $I(\gamma)$ placed side by side in the natural way, so the branch space is isomorphic to $[0, \lambda)$. Note that each level of S has cardinality at most κ, and therefore the nodes of S are both λ- and λ^*-non-degenerate. □

Proposition 5.3 *: Let κ be a regular cardinal. Let $(T, \leq_T)$ be a tree with height $\leq \kappa$ and let $\{<_N : N \in \mathcal{N}(T)\}$ be a collection of node orderings that are non-degenerate with respect to both κ and κ^*. If the branch space $(\mathcal{B}_T, <_\mathcal{B})$ contains a strictly increasing (respectively decreasing) κ-sequence $K = \{b_\alpha : \alpha < \kappa\}$, then T has a branch b^* of height κ (so that $ht(T) = \kappa$) that is the supremum (respectively the infimum) of that κ-sequence in $\mathcal{B}_T$).*

Proof: Suppose there is a strictly increasing κ sequence $K = \{b_\alpha : \alpha < \kappa\}$ in $\mathcal{B}_T$. (The case where K is a strictly decreasing κ-sequence is analogous.) For each $\delta < \kappa$, we claim there is a unique point $x_\delta \in T_\delta$ with the property that $|\{\alpha < \kappa : x_\delta \in b_\alpha\}| = \kappa$. To see that there is at most one such point, note that the set $[x_\delta]_T$ is a convex subset of $\mathcal{B}_T$ and therefore that $|\{\alpha < \kappa : x_\delta \in b_\alpha\}| = \kappa$ forces $[x_\delta]_T$ to contain a final segment of K. Hence if $y_\delta \in T_\delta$ has the same property as x_δ, then $[x_\delta]_T \cap [y_\delta]_T \neq \emptyset$ and that is impossible unless $y_\delta = x_\delta$.

The argument above also shows that if x_δ exists for some δ, then $\{\alpha < \kappa : x_\delta \in b_\alpha\}$ contains a final segment of K and therefore the set $S_\delta = \{\alpha < \kappa : x_\delta \notin b_\alpha\}$ has cardinality less than κ.

To see that x_δ exists for each δ, suppose that for some level δ, the point x_δ fails to exist. Let δ_0 be the first level for which no point x_{δ_0} exists. Then x_δ exists for each $\delta < \delta_0$. We claim that if $\delta < \delta' < \delta_0$, then in the partial order of T we have $x_\delta \leq_T x_{\delta'}$. Clearly the point $x_{\delta'}$ has some predecessor, say y, in level δ of T. Then $x_{\delta'} \in b_\alpha$ implies $y \in b_\alpha$ so that y belongs to κ-many branches b_α. Because, as we showed above, x_δ is the unique member of T_δ with that property, we have $x_\delta = y \leq_T x_{\delta'}$ as claimed. Let $\rho = \{x_\delta : \delta < \delta_0\}$. Then ρ is a path in the tree T.

We claim that ρ cannot be a branch of T. As noted above, for $\delta < \delta_0,\ \{\alpha < \kappa : x_\delta \in b_\alpha\}$ is a final segment of κ, i.e., there is some $\beta_\delta < \kappa$ such that $x_\delta \in b_\alpha$ whenever $\alpha > \beta_\delta$. Because κ is regular, the ordinal $\beta^* = \sup\{\beta_\delta : \delta < \delta_0\}$ has $\beta^* < \kappa$. Then we know that if $\beta^* < \alpha < \kappa,\ x_\delta \in b_\alpha$ for every $\delta < \delta_0$. Therefore, $\rho \subseteq b_\alpha$ whenever $\beta < \alpha < \kappa$, showing that there is more than one branch of T containing ρ. Hence ρ cannot be a branch of T. Furthermore, if $\beta^* < \alpha < \kappa$ then b_α is a proper extension of ρ so that $b_\alpha \cap T_{\delta_0} \neq \emptyset$ whenever $\beta^* < \alpha < \kappa$.

Clearly, for each $\alpha < \kappa$ either for all $\delta < \delta_0,\ x_\delta \in b_\alpha$ or else for some $\delta < \delta_0,\ x_\delta \notin b_\alpha$. Let $S = \{\alpha < \kappa : \forall \delta < \delta_0,\ x_\delta \in b_\alpha\}$ and recall that $S_\delta = \{\alpha < \kappa : x_\delta \notin b_\alpha\}$. Then $\kappa = \bigcup\{S_\delta : \delta < \delta_0\} \cup S$. Also recall that $|S_\delta| < \kappa$ for each $\delta < \delta_0$ while $\delta_0 < \kappa$. Then regularity of κ yields $|\bigcup\{S_\delta : \delta < \delta_0\}| < \kappa$ so that $|S| = \kappa$.

For each α between β^* and κ, let $b_\alpha(\delta_0)$ be the unique point of $T_{\delta_0} \cap b_\alpha$ and observe that each of the points $b_\alpha(\delta_0)$ belongs to the node N of successors of the path ρ. Recall that no point of T_{δ_0} belongs to κ-many of the branches b_α. Regularity of κ shows that the set $\{b_\alpha(\delta_0) : \beta < \alpha < \kappa\}$ has cardinality κ. This allows us to choose a strictly increasing κ sequence $\{\alpha_\gamma : \gamma < \kappa\}$ of ordinals between β and κ such that $\{b_{\alpha_\gamma} : \gamma < \kappa\}$ is a strictly increasing κ-sequence in the node N, and that is impossible because of the κ-non-degeneracy hypothesis. Therefore, x_δ exists for every $\delta < \kappa$ and, as above, if $\delta < \delta' < \kappa$ then in T we have $x_\delta \leq_T x_{\delta'}$. Because the height of T is at most κ, we see that the set $b^* = \{x_\delta : \delta < \kappa\}$ is a branch of T with height κ (and that the height of T equals κ).

First we claim that $b_\alpha \leq_{\mathcal{B}_T} b^*$ for each $\alpha < \kappa$. If that is not true then for some fixed α we have $b^* <_{\mathcal{B}_T} b_\alpha$. Compute $\delta = \Delta_T(b_\alpha, b^*)$. Necessarily $\delta < ht(b^*) = \kappa$ so that x_δ is defined, and if we write $b_\alpha(\delta)$ for the unique point of $b_\alpha \cap T_\delta$ and define $b^*(\delta)$ analogously, then in the node N that contains $b^*(\delta) = x_\delta$ we must have $x_\delta = b^*(\delta) <_N b_\alpha(\delta)$. But then for any $\gamma > \alpha,\ x_\delta \notin b_\gamma$ showing that x_δ belongs to fewer than κ-many of the branches b_γ, and that is impossible. Hence b^* is an upper bound for the κ-sequence $\{b_\alpha : \alpha < \kappa\}$.

We next claim that $b^* = \sup_{\mathcal{B}_T}\{b_\alpha : \alpha < \kappa\}$. Otherwise there would be a branch b of T with $b_\alpha <_{\mathcal{B}_T} b <_{\mathcal{B}_T} b^*$. Compute $\gamma = \Delta_T(b, b^*)$. Then in the node M consisting of all successors in T_γ of $\{x_\delta : \delta < \gamma\}$ we have $b(\gamma) <_M b^*(\gamma) = x_\gamma$. Choose any of the κ-many branches b_α with $x_\gamma \in b_\alpha$ and $\alpha > \gamma$. Then we have $b(\gamma) <_M b_\alpha(\gamma)$ so that $b <_{\mathcal{B}_T} b_\alpha$ and that is impossible. Therefore $b^* = \sup_{\mathcal{B}_T}\{b_\alpha : \alpha < \kappa\}$ as claimed. □

Corollary 5.4 *: Let κ be a regular cardinal. Then $[0, \kappa)$ is not order isomorphic to the branch space of any tree $(T, \leq_T)$ that has height $\leq \kappa$ and has node orderings that are both κ and κ^* non-degenerate.*

Proof: Suppose there is an order isomorphism f from $[0, \kappa)$ onto $\mathcal{B}_T$. Write $b_\alpha = f(\alpha)$ and apply the above proposition to construct $b^* \in \mathcal{B}_T$ that lies above each b_α. Hence f is not onto. □

Corollary 5.5 *: Suppose $(T, \leq_T)$ is a tree of height ω_1 and let $\{<_N : N \in \mathcal{N}(T)\}$ be a set of node orderings that are non-degenerate with respect to both ω_1 and ω_1^*. Then the branch space $\mathcal{B}_T$ is paracompact when endowed with its usual open interval topology.*

Proof: If $\mathcal{B}_T$ is not paracompact, then there is a strictly increasing (or strictly decreasing) homeomorphism f from a stationary subset $S \subseteq [0, \kappa)$ onto a closed subset of $\mathcal{B}_T$, where κ is an uncountable regular cardinal [4]. Consequently there is a strictly increasing (or decreasing) κ-sequence $\{f(\alpha) : \alpha \in S\}$ in $\mathcal{B}_T$ that contains all of its limit points (in the open interval topology of the branch space). But according to Proposition 5.3, the branch space must also contain a branch b^* that is the supremum (or infimum) of $f(S)$ showing that $f(S)$ is *not* closed in $\mathcal{B}_T$. $\square$

6 Branch Spaces of Aronszajn Trees

An *Aronszajn tree* is a tree with height ω_1 that has countable levels and countable branches. Such trees exist in ZFC [9]. A *Souslin tree* is an Aronszajn tree in which every anti-chain is countable. Whether Souslin trees exist is undecidable in ZFC. The first result in this section sharpens Lemma 2.1 to allow it to apply to Aronszajn trees.

Lemma 6.1 *: Let $(T, \leq_T)$ be an Aronszajn tree and $\{<_N : N \in \mathcal{N}(T)\}$ a family of node orderings of T each of which is L-non-degenerate for some linearly ordered set L. Let $(\mathcal{B}_T, <_{\mathcal{B}_T})$ be the corresponding branch space. Then there is a subtree $(V, \leq_V)$ of T that is also an Aronszajn tree such that $\mathcal{N}(V) = \{M \in \mathcal{N}(T) : |M| > 1\}$ and such that, if each node of V is linearly ordered in the same way it was ordered in the construction of $\mathcal{B}_T$, then the nodes of V are L-non-degenerate and the branch space $\mathcal{B}_V$ is order-isomorphic to $\mathcal{B}_T$.*

Proof: Let V be the subtree of T found in Lemma 2.1. To complete this proof, it remains only to show that V is an Aronszajn tree.

We first show that all levels of V are countable. Let $\alpha_0 = \min\{\alpha < \omega_1 : |T_\alpha| > 1\}$. Then $V_0 = T_{\alpha_0}$ so that $|V_0| \leq \omega$. Suppose $\beta < \omega_1$ and that for each $\alpha < \beta$ we know that $|V_\alpha| \leq \omega$. Then $|\bigcup\{V_\alpha : \alpha < \beta\}| \leq \omega$. Then there is some $\gamma < \omega_1$ with $\bigcup\{V_\alpha : \alpha < \beta\} \subseteq \bigcup\{T_\alpha : \alpha < \gamma\}$. Let $\mathcal{M} = \{N \in \mathcal{N}(V) : N \subseteq V_\beta\}$ and $\mathcal{M}_0 = \{N \in \mathcal{M} : \exists \alpha \leq \gamma \text{ with } N \subseteq T_\alpha\}$ and $\mathcal{M}_1 = \mathcal{M} - \mathcal{M}_0$. Because $|\bigcup\{T_\alpha : \alpha < \gamma\}| \leq \omega$ we know that $|\mathcal{M}_0| \leq \omega$. Suppose M, N are distinct members of $\mathcal{M}_1$. Then the sets $A = \{z \in V : \exists x \in M \text{ with } z < x\}$ and $B = \{z \in V : \exists y \in N \text{ with } z < y\}$ are distinct, and each is a subset of $\bigcup\{V_\alpha : \alpha < \beta\}$ which is a subset of $\bigcup\{T_\alpha : \alpha < \gamma\}$. Because A and B differ in the set $\bigcup\{T_\alpha : \alpha < \gamma\}$, the sets $A \cap T_\gamma$ and $B \cap T_\gamma$ are subsets of different members of $\{N \in \mathcal{N}(T) : N \subseteq T_\gamma\}$ and there are only countably many members of $\{N \in \mathcal{N}(T) : N \subseteq T_\gamma\}$ because T is an Aronszajn tree. Therefore the collection $\mathcal{M}_1$ is also countable. Hence so is $\mathcal{M}$. Hence so is $V_\beta = \bigcup \mathcal{M}$, and hence the induction continues.

We next show that V has height ω_1. To do this, it is enough to show that V has cardinality ω_1 and for that it is enough to show that for each countable α, some non-singleton node of T lies at a level above α. If there were an ordinal α such that each node of T having height greater than α is a singleton, then for each $x \in T_\alpha$, the set $A(x) = \{y \in T : y > x\}$ is linearly ordered. Further, $A(x) \cap A(y) = \emptyset$ for distinct $x, y \in T_\alpha$ so that $\{z \in T : ht_T(z) > \alpha\} = \bigcup\{A(x) : x \in T_\alpha\}$ forces one of the chains $A(x)$ to be uncountable, and that is impossible because T is an Aronszajn tree. $\square$

Lemma 6.2 *: Suppose that T is an Aronszajn tree and that A is an uncountable anti-chain in T and $\beta < \omega_1$. Let $S = \{t \in T$:some $a_t \in A$ has $t \leq a_t\}$. Then S is an Aronszajn tree and there is a subset $B \subseteq \mathcal{B}_T$ such that*

a) $|B| = 2^\omega$

b) each $b \in B$ *has* $b \subset S$ *and* $b \cap A = \emptyset$.

c) Each member of B *has height* $> \beta$.

Proof: First consider the case where $\beta = 0$. Because A is uncountable, S is an Aronszajn tree. Therefore, S contains a copy W of the full binary tree of height ω [2]. Compute $\alpha_W = \sup\{lv_T(t) : t \in W\}$. Because W is a countable set, $\alpha_W < \omega_1$. Also note that each member of W has a successor in W and hence also in S. Therefore $W \cap A = \emptyset$.

Let $R = \{\rho : \rho$ is a maximal path in the subtree $W\}$. (In other words, $R = \mathcal{B}_W$.) Observe that $|R| = 2^\omega$. For each $\rho \in R$, there is a branch $b(\rho)$ of T that contains ρ. Let $R_0 = \{\rho \in R : ht_T(b(\rho)) > \alpha_W\}$. Note that if ρ_1 and ρ_2 are distinct members of R_0 then ρ_1 and ρ_2 differ below level α_W and therefore $b(\rho_1) \cap T_{\alpha_W} \neq b(\rho_2) \cap T_{\alpha_W}$. But T_{α_W} is countable and hence so is R_0.

Let $R_1 = \{\rho \in R - R_0 : b(\rho) \cap A \neq \emptyset\}$. For each $\rho \in R_1$ let $a(\rho)$ be the unique point of $b(\rho) \cap A$. We claim that for all $t \in \rho,\ t \leq_T a(\rho)$. Otherwise there is a $t_1 \in \rho$ with $a(\rho) <_T t_1$. But $t_1 \in \rho \subseteq W \subseteq S$ so that some $a \in A$ has $t_1 \leq_T a$. But then $a(\rho) <_T t_1 \leq_T a \in A$, and that is impossible because A is an anti-chain. Hence $t \leq_T a(\rho)$ for all $t \in \rho$.

Suppose ρ_1 and ρ_2 are distinct members of R_1. Considering the first level of W where ρ_1 and ρ_2 differ, we find points $t_i \in \rho_i$ such that t_1 and t_2 are incomparable in W and hence also in T. If $a(\rho_1) = a(\rho_2)$, then t_1 and t_2 would be incomparable predecessors of $a(\rho_1)$, and that is impossible. Therefore the correspondence that sends $\rho \in R_1$ to $a(\rho) \in b(\rho) \cap A$ is 1-1, and $\{a(\rho) : \rho \in R_1\} \subseteq A \cap (\bigcup\{T_\beta : \beta \leq \alpha_W\})$. But the latter set is countable, and hence so is R_1.

Let $R_2 = \{\rho \in R - (R_0 \cup R_1) : b(\rho) - S \neq \emptyset\}$. For any $\rho \in R_2$ choose $t(\rho) \in b(\rho) - S$. Then $t(\rho)$ and each point of ρ are comparable in the partially ordered set T. If there were some $t \in \rho$ with $t(\rho) \leq_T t$ then $t \in S$ would allow us to find $a \in A$ with $t \leq_T a$. But then $t(\rho) \leq_T a$ showing that $t(\rho) \in S$ and that is impossible. Therefore, for each $t \in \rho,\ t <_T t(\rho)$. It follows that if $\rho_1 \neq \rho_2$ are in R_2, then $(b(\rho_1) - S) \cap (b(\rho_2) - S) = \emptyset$. But note that each $\rho \in R_2$ has $ht(b(\rho)) \leq \alpha_W$ so that each $b(\rho) - S$ is a subset of the countable set $\bigcup\{T_\beta : \beta \leq \alpha_W\}$, and hence we have a 1-1 correspondence $\rho \to (b(\rho) - S)$ from R_2 into a family of pairwise disjoint subsets of a countable set. Hence R_2 is also countable.

Therefore the set $R_3 = R - (R_0 \cup R_1 \cup R_2)$ has 2^ω members. We let $B = \{b(\rho) : \rho \in R_3\}$ and the lemma is proved in the special case where $\beta = 0$. To establish the general case, let $\hat{T} = \{t \in T : lv_T(t) \geq \beta + 1\}$. Then $\hat{T}$ is an Aronszajn tree and $\hat{A}$ is an uncountable anti-chain in $\hat{T}$. Apply the special case proof to $\hat{T}$ and $\hat{A}$ to find a set $\hat{B}$ of 2^ω branches of $\hat{T}$ that satisfy (a) and (b) of the special case of the Lemma. For each $\hat{b} \in \hat{B}$ define $b^* = \{t \in T :$ for some $s \in \hat{b},\ , t \leq_T s\}$. Each b^* is a branch of T with height $> \beta$ and $b^* \cap A = \emptyset$. □

By an *Aronszajn line* we mean an uncountable linearly ordered set that does not contain a order isomorphic copy of ω_1 or of ω_1^*, and does not contain an order isomorphic copy of any uncountable set of real numbers [9]. Aronszajn lines also exist in ZFC; they can be obtained from lexicographic orderings of any Aronszajn tree.

Part (a) of the next proposition was used at the beginning of Section 4 and part (b) is an application of results from Section 5.

Proposition 6.3 *: Let T be any Aronszajn tree with any family of node orderings and let $(\mathcal{B}_T, <_\mathcal{B})$ be the associated branch space of T. Then:*

a) with its open interval topology, $(\mathcal{B}_T, <_\mathcal{B})$ is not separable and the branch space $(\mathcal{B}_T, <_{\mathcal{B}_T})$ has no countable order-dense subset;

b) with its open interval topology, $\mathcal{B}_T$ is Lindelöf, first-countable, and hereditarily paracompact;

c) $\mathcal{B}_T$ is not metrizable;

d) if T does not contain any Souslin subtree, then $\mathcal{B}_T$ is not perfect (i.e., $\mathcal{B}_T$ has a closed subset that is not a G_δ-subset);

e) $(\mathcal{B}_T, <_\mathcal{B})$ contains a copy of an uncountable set of real numbers and therefore is not an Aronszajn line;

f) $\mathcal{B}_T$ contains a dense subspace that is order isomorphic to an Aronszajn line.

Proof: According to Lemma 6.1 we may assume that each element of T is either maximal or splits and that no limit level of T contains any maximal elements of T.

To prove (a), suppose D is any countable subset of $\mathcal{B}_T$. Each $b \in D$ has countable height, so that the ordinal $\beta = \sup\{ht(b) : b \in D\}$ has $\beta < \omega_1$. We claim that some $t \in T_{\beta+1}$ has $|[t]_T| > 2$. If than is not the case, then there would be only a countable number of branches with height above $\beta + 1$, and therefore the overall height of T would be less than ω_1 which is not the case. Choose $t \in T_{\beta+1}$ with $|[t]_T| \geq 3$. Then when $\mathcal{B}_T$ carries its open interval topology, the convex set $[t]_T$ has non-empty interior. But $[t]_T \cap D = \emptyset$ and that shows that the countable set D cannot be topologically dense in $\mathcal{B}_T$. Hence $\mathcal{B}_T$ is not separable. Because having a countable order-dense subset is even more restrictive than having a countable topologically dense subset, we conclude that $\mathcal{B}_T$ has no countable order-dense subset.

To prove (b) observe that each node of T is countable so that Corollary 5.5 shows that when endowed with its open interval topology, the space $\mathcal{B}_T$ is paracompact. In addition, it follows from Proposition 5.3 that the branch space $\mathcal{B}_T$ is first countable. In any linearly ordered topological space, that is enough to show that the space is hereditarily paracompact [6].

It is easy to see that a paracompact space is Lindelöf if and only if it does not contain an uncountable closed discrete subset. For contradiction, suppose that $\mathcal{B}_T$ contains a closed, discrete, uncountable subset C. Because any linearly ordered topological space is collectionwise normal [6], there is a collection of pairwise disjoint open subsets $\{U_b : b \in C\}$ with the property that b is the unique point of $C \cap U_b$ for each $b \in C$. For each $b \in C$ choose $t_b \in b$ with $[t_b]_T \subseteq U_b$ and such that if $s \in b$ and $s <_T t_b$, then $[s]_t \not\subseteq U_b$. Then the set $A = \{t_b : b \in C\}$ is an uncountable anti-chain in T. Let $S = \{t \in T : \text{some } a_t \in A \text{ has } t \leq_T a_t\}$. Then S is an Aronszajn tree.

According to Lemma 6.2 above, there is a branch b^* of T with $b^* \subseteq S$ and $b^* \cap A = \emptyset$. Let G be any open neighborhood of b^* in $\mathcal{B}_T$. Then there is some $t_1 \in b^*$ with $[t_1]_T \subseteq G$. Because $t_1 \in b^* \subseteq S$, there is some $a_1 \in A$ with $t_1 \leq_T a_1$. We claim there is some $t_2 \in b^*$ with $t_1 <_T t_2$ and such that t_2 does not lie below a_1 in T. Otherwise, every element of b^* lies below a_1 so that maximality of b^* forces $a_1 \in A \cap b^* = \emptyset$. Given t_2, choose $a_2 \in A$ with $t_2 \leq a_2$. Necessarily $a_2 \neq a_1$ and we have

$$(*) \quad [a_2]_T \subseteq [t_2]_T \subseteq [t_1]_T \subseteq G \text{ and } [a_1]_T \subseteq [t_1]_T \subseteq G.$$

Recall that each of the sets $[a_i]_T$ contains a point $b_i \in C$. Because a_1 and a_2 are distinct members of the anti-chain A, $b_1 \neq b_2$. But then (*) shows that G contains at least two distinct members of the closed discrete set C showing that b^* is a limit point of C and that is impossible. Therefore, $\mathcal{B}_T$ does not contain any uncountable closed discrete subset, and therefore $\mathcal{B}_T$ is Lindelöf.

To prove (c), note that if $\mathcal{B}_T$ were metrizable, then it would be Lindelöf and metrizable, whence separable, and that contradicts (a), above.

To prove (d), suppose that T does not contain any Souslin tree. Then T contains an ω-branching Aronszajn subtree S and S cannot be a Souslin tree. Hence there is an uncountable anti-chain $B \subseteq S$. Then B is an anti-chain in T and for each $t \in B$, infinitely many branches of T belong to the convex set $[t]_T$. Therefore $int_{\mathcal{B}_T}([t]_T) \neq \emptyset$ for each $t \in B$.

Let $U = \bigcup\{int_{\mathcal{B}_T}([t]_T) : t \in B\}$. Because $\mathcal{B}_T$ is perfect, there are closed subsets $F_n \subseteq \mathcal{B}_T$ such that $U = \bigcup\{F_n : n \geq 1\}$. Then for some $n_0 \geq 1$ the set $A = \{t \in B : int_{\mathcal{B}_T}([t]_T) \cap F_{n_0}\} \neq \emptyset$ is uncountable. Choose $b_t \in int_{\mathcal{B}_T}([t]_T) \cap F_{n_0}$ Then $\{b_t : t \in A\}$ is an uncountable, closed discrete subset of $\mathcal{B}_T$. But, as established in the proof of (b), the branch space $\mathcal{B}_T$ contains no such subsets. Thus (d) is proved.

To prove (e), we need to recall a lemma from [2] guaranteeing that any Aronszajn tree contains a copy W of the complete binary tree of height ω. Then $\mathcal{B}_W$ is the usual Cantor set and is an uncountable set of real numbers. For each branch $\rho \in \mathcal{B}_W$ there is at least one branch $b(\rho) \in \mathcal{B}_T$ with $\rho \subseteq b(\rho)$. Then the correspondence that sends ρ to $b(\rho)$ is an order isomorphism that embeds an uncountable set of real numbers into $\mathcal{B}_T$, and hence $\mathcal{B}_T$ is not an Aronszajn tree.

To prove (f) we need to use a kind of linear ordering not yet seen in this paper, namely the lexicographic ordering of the tree T itself. We will use the same node orderings used to define the ordering of $\mathcal{B}_T$ to define the lexicographic ordering of T. According to [9], any Aronszajn tree with a lexicographic ordering is an Aronszajn line.

We now define a function f from an uncountable subset $D \subseteq T$ into $\mathcal{B}_T$ that is strictly increasing and has the property that for each $t \in T,\ f(D) \cap [t]_T \neq \emptyset$. That will be enough to show that the subspace $f(D)$ is dense is $\mathcal{B}_T$ and is an Aronszajn line.

We define f and D recursively. For each $t \in T_0$, let $f(t)$ be any element of $[t]_T$ and let $D_0 = T_0$. Suppose $\alpha < \omega_1$ and that for each $\beta < \alpha$ we have defined sets $D_\beta \subseteq \bigcup\{T_\gamma : \gamma \leq \beta\}$ and a strictly increasing function $f_\beta : D_\beta \to \mathcal{B}_T$ in such a way that if $\gamma < \beta < \alpha$ then f_β extends f_γ and such that for each $s \in T_\beta$, some $t \in D_\beta$ has $f_\beta(t) \in [s]$. Write $E_\alpha = \bigcup\{D_\beta : \beta < \alpha\}$ and let $D_\alpha = E_\alpha \cup \{t \in T_\alpha : \forall s \in E_\alpha,\ f(s) \notin [t]_T\}$. Define $f_\alpha(t)$ to be $f_\beta(t)$ if t is in some D_β with $\beta < \alpha$ and define $f_\beta(t)$ to be any member of $[t]_T$ otherwise. We let $D = \bigcup\{D_\alpha : \alpha < \omega_1\}$ and $f = \bigcup\{f_\alpha : \alpha < \omega_1\}$. The only remaining question is whether D is uncountable. If it is not, then $f(D)$ is a countable set of branches of T with $f(D) \cap [t]_T \neq \emptyset$ for each $t \in T$, and that makes $\mathcal{B}_T$ separable, contrary to (a). $\Box$

Remark 6.4 *: Assertion (d) of Proposition 6.3 can be sharpened somewhat. The precise hypothesis needed in (d) is that the subtree $S = \{t \in T : int_{\mathcal{B}_T}([t]_T) \neq \emptyset\}$ is not a Souslin tree. It would also be enough to know that the subtree $U = \{t \in T : |[t]_T| > 2\}$ is not a Souslin tree.*

7 Some Topological Properties of Branch Spaces

We can characterize certain other topological properties that the branch spaces of trees might or might not have. Recall that a *π-base* for a topological space X is a collection $\mathcal{P}$ of non-empty open subsets of X such that if $G \neq \emptyset$ is open then some $P \in \mathcal{P}$ has $P \subseteq G$. We will say that a tree T is *semi-special* if there is a sequence $\{A_n : n \geq 1\}$ of anti-chains in T such that for each $t \in T$ there is some $a \in \bigcup\{A_n : n \geq 1\}$ having $t \leq a$. If it happens that $T = \bigcup\{A_n : n \geq 1\}$ for some sequence of anti-chains, then we say that T is *special*.

Proposition 7.1 *: Let T be any tree. If there is a family of node orderings such that $\mathcal{B}_T$ has a σ-disjoint π-base, then there is a subtree S of T that is semi-special and has $\mathcal{B}_S$ order isomorphic to $\mathcal{B}_T$.*

Proof: If necessary, use Lemma 2.1 to replace T by a subtree that satisfies 2.1. Therefore, we may assume that T satisfies Lemma 2.1.

Let $\mathcal{P} = \bigcup\{\mathcal{P}(n) : n \geq 1\}$ be a π-base for $\mathcal{B}_T$ where each $\mathcal{P}(n)$ is a disjoint collection of non-empty open sets. For $n \geq 1$, let $A_n = \{t \in T : [t]_T \subseteq$ some member of $\mathcal{P}(n)$ and if $s <_T t$ then $[s]_T$ is not a subset of any member of $\mathcal{P}(n)\}$. Let $A_0 = \{t \in T : t$ is a maximal element of $T\}$. Then each A_n is an anti-chain.

Fix any $t \in T$. If t or some successor of t is a maximal element of T then either $t \in A_0$ of some successor of t belongs to A_0. Hence assume that t is not maximal and that no successor of t in T is maximal. Then $int_{\mathcal{B}_T}([t]_T) \neq \emptyset$, so there is some $n \geq 1$ and some $P \in \mathcal{P}(n)$ with $P \subseteq [t]_T$. Choose $b \in P$ and then choose the minimal $t_n \in b$ with $b \in [t_n]_T \subseteq P$. Then $t_n \in A_n$ and because $[t_n]_T \subseteq P \subseteq [t]_T$ we see that $t \leq_T t_n$ as required. $\square$

Without some additional hypotheses, the converse of Proposition 7.1 is false: take any linearly ordered $(X, <)$ whose open interval topology does not have a σ-disjoint π-base. Let $T = T_0 = X$ with T_0 being ordered as a copy of $(X, <)$. Clearly T is a special tree and because $\mathcal{B}_T$ is exactly X, the branch space has no σ-disjoint π-base. However, one can prove

Proposition 7.2 *: Suppose T is a tree with a family of node orderings such that for each $t \in T$, $int_{\mathcal{B}_T}([t]_T) \neq \emptyset$. Then $\mathcal{B}_T$ has a σ-disjoint π-base if and only if T is semi-special.*

Proof: Half of the proposition follows from Proposition 7.1. For the other half, if $\{A_n : n \geq 1\}$ is the sequence of anti-chains in the definition of semi-special and if $\mathcal{P}(n) = \{int_{\mathcal{B}_T}([t]_T) : t \in A_n\}$, then $\bigcup\{\mathcal{P}(n) : n \geq 1\}$ is the required π-base. $\square$

Recall that among first-countable regular spaces, the existence of a σ-disjoint π-base is equivalent to the existence of a dense metrizable subspace [10]. In particular, this equivalence holds for any branch space of a semi-special Aronszajn tree.

A property that is stronger than the existence of a σ-disjoint π-base is the existence of a σ-disjoint base.

Proposition 7.3 *: Suppose T is a tree that, for some node ordering, $\mathcal{B}_T$ has a σ-disjoint base. Then there is a subtree $S \subseteq T$ such that*

a) S is special;

b) for each branch b of T, $b \cap S$ is cofinal in b;

c) if nodes of S are ordered consistently with the ordering of $\mathcal{B}_T$, then the branch space of S is order isomorphic to the branch space of T;

d) if T is an Aronszajn tree, then so is S.

Proof: If necessary, we replace T by a subtree that satisfies Lemma 2.1. This allows us to assume that T itself satisfies 2.1. Let $\mathcal{B}(n)$ be a disjoint collection of open sets such that $\mathcal{B} = \bigcup\{\mathcal{B}(n) : n \geq 1\}$ is a base for $\mathcal{B}_T$. Let $A(n) = \{t \in T : [t]_T \subseteq$ some member of $\mathcal{B}(n)$ and no strict predecessor of t has this property $\}$. Then each $A(n)$ is an anti-chain in T. Let $S = \bigcup\{A(n) : n \geq 1\}$ and partially order S as a subtree of T.

Let b be any branch of T and let $\{n_1, n_2, \cdots\}$ be the set of all integers n such that some member of $\mathcal{B}(n)$ contains b. For each n_k let $B(n_k)$ be the unique member of $\mathcal{B}(n_k)$ that contains b. There is some $t_k \in b$ that is the first member of b with $[t_k]_T \subseteq B(n_k)$. Then $t_k \in A(n_k) \subseteq S$. For contradiction, suppose $\{t_k : k \geq 1\}$ is not cofinal in b. Then there is some $t^* \in b$ with $t_k <_T t^*$ for each k. Then $b \in [t^*]_T \subseteq [t_k]_T \subseteq B(n_k)$ so that $[t^*]$ is a subset of every member of the base that contains b. Hence $[t^*]_T = \{b\}$ so that t^* is a maximal member of T (because each member of T is either maximal or splits in T) and $b = \{t \in T : t \leq_T t^*\}$. There are two cases to consider. If t^* has an immediate predecessor t^{**} in T, then $t_k \leq_T t^{**}$ for each k. Hence $[t^{**}]_T \subseteq [t_k]_T$ for each k so that $[t^{**}]$ is a subset of every member of the base $\mathcal{B}$ that contains b, showing that $[t^{**}] = \{b\}$ and that makes t^{**} maximal in T, which is false. Therefore, t^* has no immediate predecessor in T, and therefore $lv_T(t^*)$ is a limit ordinal. It follows from part (b) of Lemma 2.1 that the node of T containing t^* must also contain some element $u^* \neq t^*$. Let $c \in [u^*]$. Then $c \neq b$ so we may choose an element $B_0 \in \mathcal{B}$ with $b \in B_0$ and $c \notin B_0$. Because $b \in B_0$ there is an n_k with $B_0 \in \mathcal{B}(n_k)$ and then we have $b \in [t_k] \subseteq B_0$. Because t^* and u^* have exactly the same set of predecessors, $t_k \leq_T u^*$ showing that $c \in [t_k] \subseteq B_0$, and that is false. Therefore $b \cap S$ is cofinal in b, so that assertion (b) holds, and assertion (c) now follows directly.

To prove (d), suppose that T is an Aronszajn tree. In the light of (b), $|S| = \omega_1$. Clearly S has no uncountable branches, so that it will be enough to show that each level of S is countable. For contradiction, suppose there are uncountable levels in S and let α be the first ordinal such that S_α is uncountable. Then the set $A = S_\alpha$ is an uncountable anti-chain in T. Consider the set $U = \{s \in T :$ for some $a \in A,\ s \leq_S a\}$. The subset $E = \bigcup\{S_\gamma : \gamma < \alpha\}$ of U is countable so that $\beta = \sup\{lv_T(s) : s \in E\}$ is a countable ordinal. Apply Lemma 6.2 to find a branch b of T with height $> \beta$, $b \subseteq U$, and $b \cap A = \emptyset$. Choose $t_0 \in b$ with $lv_T(t_0) > \beta$ and then use (b) to find some $s_0 \in S$ with $t_0 <_T s_0$. Then $s_0 \notin E$ shows that $lv_S(s_0) \geq \alpha$. But $s_0 \in b \subseteq U$ so that $lv_S(s_0) \leq \alpha$ from which it follows that $lv_S(s_0) = \alpha$. But then $s_0 \in b \cap A = \emptyset$ and that is impossible. Therefore, every level of S is countable, as claimed. □

Remark 7.4 *: The proof of assertion (d) in Proposition 7.3 shows that if S is a subtree of an Aronszajn tree T and has the property that $b \cap S$ is cofinal in b for every $b \in \mathcal{B}_T$, then S is also an Aronszajn tree.*

Example 7.5 *: Topological types of branch spaces of Aronszajn trees.*

In this example, all Aronszajn trees satisfy Lemma 6.1. Starting with an Aronszajn tree T, one can obtain many different topological types as branch spaces of T. Each node of T is a countable set and if each node is ordered to make it order-complete, the resulting branch space is compact by Todorčevic's theorem. It is not separable in the light of Proposition 6.3, and is not metrizable in the light of part (c) of the same proposition. If T contains no Souslin subtrees, then the branch space is not perfect. An impressive use of a compact branch space of an

Aronszajn tree appears in [8] where W.X. Shi constructs a compact linearly ordered topological space that is not metrizable and yet every subspace of it has a σ-minimal base.

Put countably many pairwise disjoint copies of that compact branch space side by side, obtaining a Lindelöf linearly ordered topological space Y that is not compact. To obtain Y as the branch space of an Aronszajn tree, put countably many copies of T side by side, one above each integer.

In a model of ZFC that contains Souslin trees, if one starts with a Souslin tree (which is certainly an Aronszajn tree), then for any choice of node orderings one obtains a branch space that is hereditarily Lindelöf but not separable. This branch space would be a Souslin line and would satisfy the topological countable chain condition (= every pairwise disjoint collection of non-empty open sets is countable), a weaker relative of separability. However, if T is an ω-branching Aronszajn tree that is not Souslin and we order each non-limit node so that it is a copy of $\mathbb{Z}$, the resulting branch space is not hereditarily Lindelöf and does not satisfy the topological countable chain condition. Furthermore, if we start with a special Aronszajn tree that is ω-branching and order each node at non-limit levels to make it a copy of $\mathbb{Z}$, the resulting branch space has a σ-disjoint base, namely $\{[t]_T : t \in T\}$, and some closed subset of the branch space is not a G_δ-set of the branch space. □

8 Open Questions

a) For which subsets $X \subseteq \mathbb{R}$ is there a tree T and node orderings that are X-non-degenerate and have the property that $\mathcal{B}_T$ is order isomorphic to X? (According to Proposition 4.8, the set $\mathbb{Q}$ is not representable in this way, while both $\mathbb{R}$ and $\mathbb{P}$ are.)

b) Which $F_{\sigma\delta}$-subsets of $\mathbb{R}$ are order isomorphic to the branch space of some countable tree?

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