The β-space Property in Monotonically Normal Spaces and GO-Spaces

by

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Abstract

In this paper we examine the role of the β-space property (equivalently of the MCM-property) in generalized ordered (GO-)spaces and, more generally, in monotonically normal spaces. We show that a GO-space is metrizable iff it is a β-space with a G_δ-diagonal and iff it is a quasi-developable β-space. That last assertion is a corollary of a general theorem that any β-space with a σ-point-finite base must be developable. We use a theorem of Balogh and Rudin to show that any monotonically normal space that is hereditarily monotonically countably metacompact (equivalently, hereditarily a β-space) must be hereditarily paracompact, and that any generalized ordered space that is perfect and hereditarily a β-space must be metrizable. We include an appendix on non-archimedean spaces in which we prove various results announced without proof by Nyikos.

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1 Introduction

To say that a space (X, τ) is a β-space [14] means that there is a function g from $\{1, 2, 3 \cdots\} \times X$ to τ such that each $g(n, x)$ is a neighborhood of x and if $y \in g(n, x_n)$ for each n, then the sequence $\langle x_n \rangle$ has a cluster point in X. The function $g(n, x)$ in that definition is said to be a *β-function for X*. Many types of spaces have β-functions, e.g., semi-stratifiable spaces [8], wΔ-spaces, strict p-spaces, and countably compact spaces [12].

Recently the β-space property re-emerged in a completely different context, namely the study of monotone modifications of topological properties. The following definition appears in [11].

Definition 1.1 *A topological space is* monotonically countably metacompact *(MCM) if for each decreasing sequence $D = \{D_n : n < \omega\}$ of closed sets with $\bigcap\{D_n : n < \omega\} = \emptyset$, there is a sequence $\{U(n, D) : n < \omega\}$ of open sets satisfying:*

a) for each $n < \omega,\ D_n \subseteq U(n, D)$;

b) $\bigcap\{U(n, D) : n < \omega\} = \emptyset$;

c) if $C = \{C_n : n < \omega\}$ is a decreasing sequence of closed sets with empty intersection, and if $C_n \subseteq D_n$ for each n, then $U(n, C) \subseteq U(n, D)$ for each n.

Note that the open sets $U(n, D)$ in that definition depend on an entire decreasing sequence $D = \langle D_n \rangle$ of closed sets with empty intersection. In a subsequent paper, Ge Ying and Chris Good [23] proved:

Lemma 1.2 *For a T_1-space X, the following are equivalent:*

a) X is MCM;

b) for each point $x \in X$ there is a sequence $\{g(n, x) : n < \omega\}$ of open neighborhoods of x such that if $\{D_n : n < \omega\}$ is a decreasing sequence of closed sets with empty intersection, then the sets $G(n, D_n) = \bigcup\{g(n, x) : x \in D_n\}$ satisfy $\bigcap\{G(n, D_n)) : n < \omega\} = \emptyset$;

c) X is a β-space. $\square$

In the light of Lemma 1.2 we will use the terms "β-space" and "MCM-space" interchangeably in this paper, depending upon which one sounds better in a given context.

It is well-known that every GO-space is hereditarily normal and has the much stronger property called monotone normality ([13]). By way of contrast, it is also well-known that every GO-space is hereditarily countably metacompact, but familiar examples show that GO-spaces may fail to have the monotone countable metacompactness property. The following examples were announced in [11].

Example 1.3 Neither the Sorgenfrey line nor the Michael line nor the lexicographic product $\mathbb{Z}^{\omega_1}$ (a LOTS that is a topological group) is MCM (equivalently, none of the three is a β-space).

Those examples immediately lead one to ask which GO-spaces are MCM and what is the role of the MCM property (equivalently, the β-space property) among GO-spaces.

Section 2 contains metrization theorems for GO-spaces that involve the β-space property. The first provides another solution of the equation

$$\text{GO-space} + G_\delta\text{-diagonal} + (?) = \text{metrizable.}$$

We prove:

Theorem 1.4 *A GO-space is metrizable if and only if it is a β-space with a G_δ-diagonal.*

Theorem 1.5 *A GO-space is metrizable if and only if it is a quasi-developable β-space.*

Theorem 1.5 is a corollary of a general theorem that asserts:

Theorem 1.6 *Any regular β-space with a σ-point-finite base is developable.*

In Section 3 we investigate the hereditary β-space property. We begin by using a theorem of Balogh and Rudin [2] and a stationary set argument to show that:

Proposition 1.7 *Any monotonically normal space (and in particular, any GO-space) that is hereditarily a β-space is hereditarily paracompact.*

The rest of Section 3 is devoted to proving metrization theorems that depend on the hereditary β-property (equivalently, the hereditary MCM property). We first show that a GO-space with a σ-closed discrete set is metrizable if and only if each of its subspaces is a β-space and then investigate what happens if "X has a σ-closed discrete dense set" is weakened to "X is perfect." Normally, one expects that metrization theorems for GO-spaces with σ-closed-discrete dense sets will not generalize to perfect spaces because normally one runs into Souslin space problems when one considers perfect GO-spaces that do not, a priori, have σ-closed-discrete dense sets. We get around this problem using results of Qiao and Tall, coupled with some results about non-archimedean spaces that were announced many years ago by Peter Nyikos. Based on those results, we prove:

Theorem 1.8 *A GO-space is metrizable if and only if it is perfect and each of its subspaces is a β-space.*

Because the required results of Nyikos have never been published, we include our proofs of them in Section 4 of this paper.

Recall that a *generalized ordered space* (GO-space) is a triple $(X, <, \tau)$ where $<$ is a linear ordering of X and τ is a Hausdorff topology on X that has a base of order-convex subsets (possibly including singletons). Probably the best-known GO-spaces are the Sorgenfrey line and the Michael line. If τ is the usual open interval topology of the ordering, then $(X, <, \tau)$ is a *linearly ordered topological space* (LOTS). Čech proved that the GO-spaces are exactly those spaces that embed topologically in some LOTS.

In this paper we reserve the symbols $\mathbb{Z},\ \mathbb{Q}$, and $\mathbb{R}$ for the sets of all integers, rational, and real numbers, respectively. For any ordinal $x,\ \mathrm{cf}(x)$ denotes the cofinality of x. We will need to distinguish between subsets of X that are *relatively discrete* (i.e., are discrete when topologized as subspaces of X) and sets that are both closed and discrete subsets of X. We will also need to distinguish between dense sets that are σ-relatively discrete subsets of X (i.e., that are unions of countably many relatively discrete subsets of X) and those that are σ-closed-discrete (i.e., countable unions of closed discrete subsets of X)..

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2 Metrization and the β-space property

In this section, we investigate how the β-space property interacts with other topological properties to provide metrization theorems. Recall that any LOTS with a G_δ-diagonal is metrizable [15] while GO-spaces with G_δ-diagonals may fail to be metrizable (e.g., the Sorgenfrey and Michael lines). The β-space property is exactly what is missing, and we have:

Proposition 2.1 *A GO-space is metrizable if and only if it is a β-space with a G_δ-diagonal.*

Proof: Half of the proposition is trivial. To prove the other half, recall that a space X is paracompact if it is a GO-space with a G_δ-diagonal [15]. Also recall that any space X is semi-stratifiable if it is a β-space with a G^*_δ-diagonal (see Theorem 7.8(ii) of [12]) and that any paracompact space with a G_δ-diagonal has a G^*_δ-diagonal. Therefore X is semi-stratifiable. Hence X is metrizable [15]. $\square$

Our next result is a general theorem – it is not restricted to GO-spaces.

Proposition 2.2 *A T_3, β-space with a σ-point-finite base is developable. A T_3 space is metrizable if and only if it is a collectionwise normal β-space with a σ-point-finite base.*

Proof: The second assertion of the proposition follows from the first because any collectionwise normal developable space is metrizable.

The first assertion is already known: Hodel [14] noted that any space with a σ-point-finite base is a γ-space and proved (in his Proposition 4.2) that any T_1-space that is both a γ-space and a β-space must be developable. An alternate approach begins with $\bigcup\{\mathcal{B}(n) : n \geq 1\}$, a σ-point-finite base for X. We may modify that base if necessary so that $\mathcal{B}(2k)$ is the collection of all singleton isolated points of X for each $k \geq 1$. We may also assume that each $\mathcal{B}(n)$ is closed under finite intersections so that, if $x \in \bigcup \mathcal{B}(i)$ then there is a member $B(i,x) \in \mathcal{B}(i)$ that is the smallest member of $\mathcal{B}(i)$ that contains x. Now let $g(n,x)$ be a β-function for X. Because X is first-countable, we may assume that $\{g(n,x) : n \geq 1\}$ is a decreasing local base at x for each point $x \in X$ and that if x is isolated, then $g(n,x) = \{x\}$ for each n. One first proves that for any fixed $x \in X$ and $n \geq 1$, there is some $m \geq n$ with $x \in \bigcup \mathcal{B}(m)$ and $B(m,x) \subseteq g(n,x)$, where $B(m,x)$ is the smallest member of $\mathcal{B}(m)$ that contains x. Then for each fixed x and n we may define $\phi(n,x)$ to be the first integer $m \geq n$ having $x \in \bigcup \mathcal{B}(m)$ and $B(m,x) \subseteq g(n,x)$. Observe that for each fixed x, $\phi(n,x) \leq \phi(x,n+1)$. Now define $h(n,x) = \bigcap\{B(i,x) : x \in \bigcup \mathcal{B}(i) \text{ and } i \leq \phi(x,n)\}$. Then $h(n+1,x) \subseteq h(n,x)$ and $h(n,x) \subseteq g(n,x)$ so that h is also a β-function for X and $\{h(n,x) : n \geq 1\}$ is a local base at x for each point of X. Verify that if $p \in h(n,x_n)$ for each $n \geq 1$, then $\langle x_n \rangle$ clusters to p. Now apply a theorem of Aull [1] to show that X, having a σ-point-finite base, must be quasi-developable. To complete the proof, all we need to show is that X is perfect. For any closed set C, let $G_n = \bigcup\{h(n,x) : x \in C\}$. Then G_n is an open set and in the light of Claim 2, $\bigcap\{G_n : n \geq 1\} = C$. $\square$

We do not know whether the previous proposition can be generalized to quasi-developable spaces. (That would be a generalization, because Aull has proved that any space with a σ-point-finite base is quasi-developable.) A recent paper [16] claimed that any quasi-developable β-space must be developable, but some details of the proof are unclear.

Whether or not each quasi-developable β-space is developable, we have the following equivalence for GO-spaces:

Corollary 2.3 *A GO-space is metrizable if and only if it is a quasi-developable β-space.*

Proof: To prove the non-trivial half of the corollary, suppose X is a GO-space that is quasi-developable and a β-space. Then by [3], [15] X has a σ-point-finite base and is collectionwise normal. Now apply Proposition 2.2. $\square$

3 The hereditary β-space property

In our paper [6] we proved that any GO-space that is hereditarily a β-space must be hereditarily paracompact. The key to the proof was a pressing down lemma argument that showed:

Lemma 3.1 *No stationary subset of a regular uncountable cardinal can be hereditarily a β-space in its relative topology.* $\square$

In [6], we then combined Lemma 3.1 with a characterization of paracompactness in generalized ordered spaces from [9] to get the desired result. Since the time of that earlier paper, Balogh and Rudin [2] have significantly generalized the result from [9], showing that a monotonically normal space fails to be paracompact if and only if it contains a closed subspace that is homeomorphic to a stationary set in a regular uncountable cardinal. Combining that result with Lemma 3.1 gives:

Corollary 3.2 *A monotonically normal space that is hereditarily a β-space is hereditarily paracompact.* $\square$

In the remainder of this section we prove that the hereditary MCM property is a natural component of metrizability in GO-spaces. We begin by recalling the following lemma [10], [4].

Lemma 3.3 *Suppose X is a GO-space with a dense subset that is σ-closed-discrete. Then:*

a) X is perfect (i.e., each closed set is a G_δ-set) and first-countable;

b) there is a sequence $\{\mathcal{H}(n) : n \geq 1\}$ of open covers of X such that for each $p \in X$, $\bigcap\{St(p, \mathcal{H}(n)) : n \geq 1\}$ has at most two points.

c) (Faber's Metrization Theorem [10]) the GO-space X is metrizable if and only if the sets $R = \{x \in X : [x, \rightarrow) \in \tau\}$, $L = \{x \in X : (\leftarrow, x] \in \tau\}$ and $I = \{x \in X : \{x\} \in \tau\}$ are each σ-closed-discrete in X. $\square$

Theorem 3.4 *Let $(X, <, \tau)$ be a GO-space. Then X is metrizable if and only if X has a σ-closed-discrete dense set and is hereditarily a β-space.*

Proof: Any metric space has a σ-closed-discrete dense set and is hereditarily a β-space. To prove the converse, suppose X has a σ-closed-discrete dense subset E and is hereditarily a β-space. We will apply Faber's metrization theorem in part (c) of 3.3. The set of isolated points, being a subset of E, is σ-closed-discrete. We prove that the set $R = \{x \in X : [x, \rightarrow) \in \tau\}$ is σ-closed-discrete; the proof for the set $L = \{x \in X : (\leftarrow, x] \in \tau\}$ in Faber's theorem (see Lemma 3.3(c)) is analogous.

By Lemma 3.3, X is perfect so that each relatively discrete subset of X is σ-closed-discrete. Therefore, it will be enough to show that the set R is the union of countably many relatively discrete, but perhaps not closed, subsets. To that end, for each $x \in R$, find a sequence $\{g(n,x) : n \geq 1\}$ of sets that satisfy Lemma 1.2 for the subspace (R, τ_R). Replacing those sets by smaller sets if necessary, we may assume:

a) $\{g(n,x) : n \geq 1\}$ is a decreasing local base at x in the subspace (R, τ_R);

b) each set $g(n,x)$ is contained in some member of the cover $\mathcal{H}(n)$ described in Lemma 3.3(b);

c) $g(n,x) \subseteq [x, \rightarrow)$ for each $x \in R$ and each n;

d) if $a < b < c$ are points of R with $a, c \in g(n,x)$ then $b \in g(n,x)$.

Let $\mathcal{G}(n) = \{g(n,x) : x \in R\}$ and define $R(n) = \{x \in R : St(x, \mathcal{G}(n)) \subseteq [x, \rightarrow)\}$. Then $R(n) \subseteq R(n+1)$ for each n. Let $R^* = \bigcup\{R(n) : n \geq 1\}$. For each $x \in R^*$ there is some n with $x \in R(n)$. Then $St(x, \mathcal{G}(k)) \subseteq [x, \rightarrow)$ for each $k \geq n$. But then $g(k,x)$ is the unique member of $\mathcal{G}(k)$ that contains x for each $k \geq n$. For suppose $y \in R$ and $x \in g(k,y)$ where $k \geq n$. Then $y \in g(k,y) \subseteq St(x, \mathcal{G}(k)) \subseteq [x, \rightarrow)$ yields $x \leq y$. But if $x < y$, then $x \in g(k,y)$ makes $g(k,y) \subseteq [y, \rightarrow)$ impossible, contrary to (c) in the description of how the sets $g(k,z)$ are chosen for $z \in R$.

Therefore the sets $St(x, \mathcal{G}(k)) = g(k,x)$ form a neighborhood base for x in the space (R, τ_R) so that the subspace (R^*, τ_{R^*}) is developable and hence is metrizable. Applying Faber's metrization theorem, we see that the set R^* is the countable union of subspaces that are relatively discrete.

We claim $R = R^*$. If not, then there is a point $y \in R - R^*$. Then $y \notin R(n)$ for each n so there must be some points $x_n \in R$ with $y \in g(n, x_n) \not\subseteq [y, \rightarrow)$. Because $g(n, x_n) \subseteq [x_n, \rightarrow)$ we must have $x_n < y$.

There cannot be an infinite sequence $n_1 < n_2 < \cdots$ with $x_{n_1} = x_{n_2} = \cdots$ because then $y \in g(n_k, x_{n_k}) = g(n_k, x_{n_1})$ would make it impossible for the sets $g(n, x_{n_1})$ to be a decreasing local base at the point x_{n_1}. We claim that there cannot be a sequence $m_1 < m_2 < \cdots$ with $x_{m_1} > x_{m_2} > \cdots$. For suppose such a decreasing subsequence exists. Then for each $j \geq m_2$ we have

$$\{x_{m_2}, x_{m_1}, y\} \subseteq [x_{m_j}, y] \cap R \subseteq g(m_j, x_{m_j}) \subseteq g(j, x_{m_j}) \in \mathcal{G}(j)$$

which shows that

$$\{x_{m_2}, x_{m_1}, y\} \subseteq St(y, \mathcal{G}(j)) \subseteq St(y, \mathcal{H}(j))$$

for each $j \geq 2$ and that is impossible in the light of the special properties of the covers $\mathcal{H}(n)$ described in part (b) of Lemma 3.3.

Therefore the sequence $\langle x_n \rangle$ has no constant subsequences and no strictly decreasing subsequences, so there must be a strictly increasing subsequence $x_{n_1} < x_{n_2} < \cdots$. Let $A_k = \{x_{n_i} : i \geq k\}$ and observe that A_k has no limit points in R. Hence $\{A_k : k \geq 1\}$ is a decreasing sequence of closed sets with empty intersection. However, with $G(k, A_k)$ defined as in Lemma 1.2, we have $y \in \bigcap\{G(k, A_k) : k \geq 1\}$ and that is impossible. Hence $R = R^*$, so R is the union of countably many subspaces, each being relatively discrete. The same is true of the subset L and we may now apply Faber's metrization theorem to complete the proof. □

Experience has shown that many results proved for GO-spaces having σ-closed-discrete dense sets become axiom-sensitive when stated for the broader class of perfect GO-spaces. It is somewhat surprising that the Theorem 3.4 is not of this type. We begin with a result about dense metrizable subspaces. Then, by combining Theorem 3.4 with some known results about non-archimedean spaces (i.e., spaces with a base that is a tree under reverse inclusion) we obtain a new metrization theorem for perfect GO-spaces.

Proposition 3.5 *Let X be a first-countable GO-space that is hereditarily a β-space. Then X has a dense metrizable subspace.*

Proof: We need two results from the literature.

> a) Any first-countable GO-space contains a dense non-archimedean subspace, i.e., a dense subspace having a base of open, convex sets that is a tree under reverse inclusion.
>
> b) Any first-countable, non-archimedean β-space is metrizable.

The first is due to Qiao and Tall [20] (who proved the result for first-countable LOTS, but a slight modification of their proof establishes the result for first-countable GO-spaces). The second is due to Nyikos [17]. No proof of the second result has appeared in print and we include a proof and relevant definitions in the final section of this paper.

Now suppose X is a first countable GO-space. Let Y be a dense non-archimedean subspace of X. If X is hereditarily a β-space, then Y is a β-space. Now apply assertion b) above to show that Y is metrizable. □

Theorem 3.6 *: Suppose X is a GO-space. Then X is metrizable if and only if X is perfect and hereditarily MCM.*

Proof: To prove the non-trivial part of the theorem, suppose that X is perfect and hereditarily MCM. Apply Proposition 3.5 to find a dense metrizable subspace Y of X. Then Y contains a dense subset D that is the union of countably many subsets $D(n)$, each being relatively discrete. But then, X being perfect, each $D(n)$ is the union of countably many subsets $D(n, k)$ where each $D(n, k)$ is a closed discrete subspace of X. Now apply Theorem 3.4 to conclude that X is metrizable. □

As noted in the Introduction, any compact or countably compact space is β-space because every sequence in a countably compact space has a cluster point. Hence the lexicographic square is a β-space, as is the ordinal space $[0, \omega_1)$. However, the *hereditary* β-space property is another matter, and we have the following question.

Question 3.7 Is there a compact, first-countable LOTS X that is hereditarily a β-space and not metrizable?

Note that, in the light of Corollary 3.5, if X is a first-countable compact LOTS that is hereditarily a β-space, then X has a dense metrizable subspace, as does each subspace of X. Also note that by Theorem 3.6 and assertion (b) in the proof of (3.6), many kinds of subspaces of such an X will be metrizable. These include perfect subspaces (a class that includes all separable subspaces and, more generally, all subspaces with a σ-closed discrete dense subset), non-archimedean subspaces, and subspaces with a point-countable base (because, according to a result of Chaber [7] (see also Theorem 7.9 of [12]) any first-countable, paracompact β-space with a point-countable base must be metrizable). Other results in the literature suggest that one place to look for the required example is in the branch spaces of certain trees ([21], [22]).

4 Appendix on non-archimedean spaces

A regular space X is *non-archimedean* if it has a base that is a tree under reverse inclusion. Basic topological results about such spaces were announced by Peter Nyikos in [17], [18], and [19] but, Nyikos has informed us, no proof of the one result needed in this paper (namely that a first-countable non-archimedean β-space is metrizable) has ever been published. The goal of this appendix is to provide the required proof. Our approach is as follows. First we will show that any non-archimedean space is paracompact. Next we will show that any first-countable non-archimedean β-space is developable and then, from general metrization theory, we will conclude that any first-countable, non-archimedean β-space is metrizable. It happens to be true that any non-archimedean space is a GO-space, but we will not use that fact in our proofs.

Lemma 4.1 *Suppose $\mathcal{B}$ is a tree-base for the non-archimedean space X. Then:*

a) each member of $\mathcal{B}$ is clopen;

b) each subspace of X is ultraparacompact;

c) if $p \in X$ and if $\mathcal{C} \subseteq \mathcal{B}$ has $p \in \bigcap \mathcal{C}$, then either $\bigcap \mathcal{C}$ is a neighborhood of p or else $\bigcap \mathcal{C} = \{p\}$ and $\mathcal{C}$ is a neighborhood base at p.

Proof: For any $p \in X$, let $\mathcal{B}(p) = \{B \in \mathcal{B} : p \in B\}$. Then $\mathcal{B}(p)$ is well-ordered by reverse inclusion and if $p \in B_1 \cap B_2$ (where $B_i \in \mathcal{B}$) then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

To prove a), let $B \in \mathcal{B}$. Let p be any limit point of B and suppose $p \notin B$. Choose $q \in B$. Because $p \neq q$, we may choose $B' \in \mathcal{B}$ with $p \in B' \subseteq X - \{q\}$. Then $B \cap B' \neq \emptyset$ and $B \subseteq B'$ is impossible, so that $p \in B' \subseteq B$, contrary to $p \notin B$. Hence B is clopen.

To prove b), recall that a space Y is *ultraparacompact* if each open cover of Y has a pairwise disjoint open refinement. Let $\mathcal{U}$ be any collection of open subsets of X. Let $\mathcal{D}$ be the collection of all $B \in \mathcal{B}$ that are contained in some member of $\mathcal{U}$. Let $\mathcal{V}$ be the collection of all minimal members of $\mathcal{D}$ with respect to the tree ordering (i.e., reverse inclusion) of $\mathcal{B}$. Then $\mathcal{V}$ refines $\mathcal{U}$, is pairwise disjoint, and has $\bigcup \mathcal{V} = \bigcup \mathcal{U}$. It follows that every open subspace of X, and hence every subspace of X, is ultraparacompact.

To prove c), suppose $\mathcal{C} \subseteq \mathcal{B}$ and $p \in \bigcap \mathcal{C}$ and suppose that $\bigcap \mathcal{C}$ is not a neighborhood of p. We claim that $\bigcap \mathcal{C} = \{p\}$. For suppose there are at least two points p, q in $\bigcap \mathcal{C}$ and choose any member $B_0 \in \mathcal{B}$ with $p \in B_0 \subseteq X - \{q\}$. Then B_0 meets each $C \in \mathcal{C}$ and B_0 cannot contain any member of $\mathcal{C}$. Hence B_0 is a subset of each member of $\mathcal{C}$ and therefore $B_0 \subseteq \bigcap \mathcal{C}$. But that makes $\bigcap \mathcal{C}$ a neighborhood of p which is impossible. Hence $\bigcap \mathcal{C} = \{p\}$. Let B_1 be any member of $\mathcal{B}$ that contains p. Because $\bigcap \mathcal{C}$ is not a neighborhood of p, we must have $B_1 \not\subseteq \bigcap \mathcal{C}$ so that for some $C \in \mathcal{C},\ B_1 \not\subseteq C$. Hence $C \subseteq B_1$ as required to show that $\mathcal{C}$ is a local base at p.$\square$

Proposition 4.2 *If X is a non-archimedean β-space in which points are G_δ-sets, then X is metrizable.*

Proof: Part c) of Lemma 4.1 shows that a non-archimedean space in which points are G_δ-sets must be first-countable.

Let $\mathcal{B}$ be a tree-base for the space X and let $g(n, x)$ be a β-function for X as described in Section 1. Because we can replace each $g(n, x)$ by a smaller neighborhood of x and still have a β-function, we may assume that $g(n, x) \in \mathcal{B}$ and that $\{g(n, x) : n \geq 1\}$ is a local base at x. We may also assume that $g(n + 1, x)$ is a proper subset of $g(n, x)$ unless x is isolated and that $g(n, x) = \{x\}$ for each n if x is isolated.

We now describe a partition process that will be applied to various sets $g(n, x)$. If x is isolated, then $g(n, x) = g(n+1, x) = \{x\}$ and we let $\mathcal{W}(g(n, x)) = \{g(n+1, x)\}$. If x is not isolated, then the set $S = g(n, x) - g(n + 1, x)$ is not empty and, by part a) of Lemma 4.1, S is open. Let the members of $\mathcal{W}(g(n, x))$ be $g(n + 1, x)$ together with all members of the collection $\{g(k, y) : k \geq n + 1 \text{ and } g(k, y) \subseteq S\}$ that are minimal in the ordering of the tree $(\mathcal{B}, \supseteq)$. Then $\mathcal{W}(g(n, x))$ is a pairwise disjoint open cover of $g(n, x)$ by sets of the form $g(k, y) \in \mathcal{B}$ where $k \geq n + 1$. Note that if $g(k, y) \in \mathcal{W}(g(n, x))$ with $y \neq x$, then $x \notin g(k, y)$. For each set $W \in \mathcal{W}(g(n, x))$ choose one point $y(W) \in W$ such that $W = g(k, y(W))$ and $k \geq n + 1$, making sure that if $W = g(n + 1, x)$, then $y(W) = x$. Let $C(g(n, x)) = \{y(W) : W \in \mathcal{W}(g(n, x))\}$.

Let $\mathcal{H}(0) = \{X\}$. Given $\mathcal{H}(n)$ for some n, define

$$\mathcal{H}(n + 1) = \bigcup \{\mathcal{W}(g(m, x)) : g(m, x) \in \mathcal{H}(n)\}.$$

Let $C(n) = \{y(W) : W \in \mathcal{H}(n)\}$. Each $\mathcal{H}(n)$ is a pairwise disjoint cover of X by members of $\mathcal{B}$ that have the form $g(y, k)$ for exactly one $y \in C(n)$ and $k \geq n$.

We claim that the sequence $\mathcal{H}(1), \mathcal{H}(2), \cdots$ is a development for X. Fix any $p \in X$. Then p belongs to exactly one member of $\mathcal{H}(n)$ so that $\mathrm{St}(p, \mathcal{H}(n))$ is a member of $\mathcal{H}(n)$ and has the form $g(k_n, y_n)$ where $y_n \in C(n)$ and $k_n \geq n$. Furthermore, $g(k_{n+1}, y_{n+1}) \subseteq g(k_n, y_n)$ because of the way that the collections $\mathcal{H}(n)$ were constructed. Because $k_n \geq n$ we have $p \in g(n, y_n)$ and therefore the sequence $y_1, y_2, \cdots$ must cluster at some point $q \in X$. Because $g(k_{n+1}, y_{n+1}) \subseteq g(k_n, y_n)$ we see that each $g(k_n, y_n)$ contains $\{y_m : m \geq n\}$ and therefore the point q is a point of the closure of each $g(k_n, y_n)$. But $g(k_n, y_n)$ is clopen, being a member of $\mathcal{B}$, so that $\{p, q\} \subseteq \bigcap \{g(k_n, y_n) : n \geq 1\}$.

If infinitely many terms in sequence $y_1, y_2, \cdots$ are the same, say $y_n = y_N$ for each n in the infinite set I, then because $k_n \geq n$ the sets $g(k_n, y_n)$ form a local base at y_N so that $p, q \in \bigcap\{g(k_n, y_n) : n \geq 1\} = \{y_N\}$ forces $p = q = y_N$ and hence $\{St(p, \mathcal{H}(n)) : n \geq 1\}$ is a local base at $\{p\}$.

If the sequence $y_1, y_2, \cdots$ has no constant subsequences, then there is a subsequence of distinct terms. For notational simplicity, assume that $y_i \neq y_j$ whenever $i \neq j$. Then we know that $y_n \notin g(k_{n+1}, y_{n+1})$ so that the set $T = \bigcap\{g(k_n, y_n) : n \geq 1\}$ contains no point y_k. Hence T cannot be a neighborhood of q even though $q \in T$. But by part c) of Lemma 4.1 we know that since T is not a neighborhood of q, it must be true that $T = \{q\}$ and $\{g(k_n, y_n) : n \geq 1\}$ is a neighborhood base at q. But $\{p, q\} \subseteq T$ then forces $p = q$ so that, once again, $\{\mathrm{St}(p, \mathcal{H}(n)) : n \geq 1\}$ is a local base at p.

At this stage of the proof, we know that X is developable and paracompact (by part (b) of Lemma 4.1) and therefore metrizable. □

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