More on Compact Spaces

1 Products of compact spaces, junior grade

In HW you proved the following "tube lemma":

Lemma 1.1 *Suppose $\mathcal{U}$ is an open cover of $X \times Y$ where Y is compact. Let $x \in X$. Then there is an open neighborhood $G(x)$ of x and a finite subcollection $\mathcal{U}(x) \subseteq \mathcal{U}$ such that $\mathcal{U}(x)$ covers $G(x) \times Y$.*

Theorem 1.2 *Suppose X and Y are compact spaces. Then so is $X \times Y$.*

Proof: Let $\mathcal{U}$ be any open cover of $X \times Y$. For each $x \in X$ find an open set $G(x)$ that contains x and a finite $\mathcal{U}(x) \subseteq \mathcal{U}$ that covers $G(x) \times Y$. Then $\{G(x) : x \in X\}$ is an open cover of X, so we may choose $x_i \in X$ with $1 \leq i \leq n$ such that $X = \bigcup\{G(x_i) : 1 \leq i \leq n\}$. Letting $\mathcal{U}_0 = \bigcup\{\mathcal{U}(x_i) : 1 \leq i \leq n\}$, we obtain a finite subcollection of $\mathcal{U}$.

To show that $\mathcal{U}_0$ covers $X \times Y$, let $(x,y) \in X \times Y$. Find i with $x \in G(x_i)$. Then $(x,y) \in G(x_i) \times Y \subseteq \bigcup \mathcal{U}(x_i) \subseteq \bigcup \mathcal{U}_0$. □

2 The Baire Category Theorem

We have already seen the Baire Category Theorem for complete metric spaces. One version asserts that if $G_1, G_2, \cdots$ are dense open sets in a complete metric space X, then $\bigcap\{G_n : n \geq 1\}$ is a dense subset of X. That result also holds in any compact Hausdorff space.

Theorem 2.1 *(Baire Category Theorem) If $G_1, G_2, \cdots$ is a sequence of dense open subsets of a compact Hausdorff space X, then $\bigcap\{G_n : n \geq 1\}$ is dense in X.*

Proof: Let U be any non-empty open set. We will show that some point of U lies in each set G_n.

We know that $U \cap G_1 \neq \emptyset$ because G_1 is dense. Choose $x_1 \in U \cap G_1$ and then choose an open set V_1 with $x_1 \in V_1 \subseteq \text{cl}(V_1) \subseteq U \cap G_1$. This is possible because X is regular and $U \cap G_1$ is open.

Given V_1, we know that $V_1 \cap G_2 \neq \emptyset$, because G_2 is dense. Choose $x_2 \in V_1 \cap G_2$ and then find an open set V_2 with $x_2 \in V_2 \subseteq \text{cl}(V_2) \subseteq V_1 \cap G_2$.

In general, given a nonempty open V_n, note that the set $V_n \cap G_{n+1} \neq \emptyset$ because G_{n+1} is dense in X. Choose $x_{n+1} \in V_n \cap G_{n+1}$ and then find an open set V_{n+1} with

$$x_{n+1} \in V_{n+1} \subseteq \text{cl}(V_{n+1}) \subseteq V_n \cap G_{n+1}.$$

Then the sets $K_n = \text{cl}(V_n)$ form a decreasing sequence of non-empty compact sets in the Hausdorff space X, so from a HW problem we know that there is some point $y \in \bigcap\{K_n : n \geq 1\}$. Then $y \in K_1 = \text{cl}(V_1) \subseteq U$ and for each n we have $y \in K_n = \text{cl}(V_n) \subseteq G_n$. Hence U meets the set $\bigcap\{G_n : n \geq 1\}$, as required. □

Corollary 2.2 *The Baire Category Theorem holds in any locally compact Hausdorff space.*

Proof: Let X be a locally compact Hausdorff space. If X is actually compact, then apply the previous theorem. If X is locally compact but not compact, then find a compact Hausdorff space X^* that contains X as a dense open subspace. Let $\langle G_n \rangle$ be any sequence of dense relatively open subsets of X. Then each G_n is open and dense in X^*, so the intersection of all sets G_n must be dense in X^*, in the light of the previous theorem. But that intersection is a subset of X, so it must be dense in X. $\square$