

Math 108: Introduction to \mathbb{R}^3

Pictures for this handout will be drawn in class.

I. Coordinates, special planes, and surfaces in \mathbb{R}^3

Recall two dimensional graphing. We start with two mutually perpendicular axes (called the x and y axes) that meet at origin, denoted $(0,0)$. We locate a point in the plane by giving two coordinates telling us how to get to the point from origin. Consequently, $(2,3)$ is the point that we reach by going 2 units in the positive direction along the x axis and then 3 units in the positive direction vertically. The plane is often called \mathbb{R}^2 . The x axis is the set of all points $(x,0)$ where x is allowed to be any real number. Similarly, the y axis is the set of all points $(0,y)$ where y is allowed to be any real number.

In three dimensional space, we have three mutually perpendicular axes called the x , y , and z axes that meet at origin, denoted $(0,0,0)$. We locate a point in space by giving three coordinates telling us how to get to the point from origin. Consequently, we reach the point $(2,3,4)$ by going 2 units in the positive direction along the x -axis, then 3 units in the positive y direction, then 4 units in the positive z direction. Three dimensional space is often called \mathbb{R}^3 . The x axis is the set of all points $(x,0,0)$ where x is allowed to be any real number. The y axis is the set of all points $(0,y,0)$ where y is any real number. The z axis is the set of all points $(0,0,z)$ where z is any real number. The x,y plane is the set of all points $(x,y,0)$ where x and y are allowed to be any real numbers. The x,z plane is the set of all points $(x,0,z)$ where x and z are allowed to be any real numbers. The y,z plane is similarly defined.

Drawing things in \mathbb{R}^3 requires perspective drawing, which is a challenge.

In \mathbb{R}^2 we graph functions $y = f(x)$ whose domains are (pieces of) the x axis. We get curves floating above or below the x axis, and the formula $y = f(x)$ tells us the height of the curve above or below the point $(x,0)$ on the x axis.

In \mathbb{R}^3 we graph functions $z = f(x,y)$ whose domains are (pieces of) the x,y plane. We get surfaces floating above or below the x,y plane, and the formula $z = f(x,y)$ tells us the height of the surface above or below the point $(x,y,0)$ in the x,y plane.

The plane \mathbb{R}^2 contains special lines. For a constant a , the line $x = a$ is a line through the point $(a,0)$ on the x -axis that is parallel to the y axis and contains all points (a,y) for all possible values of y . Therefore the graph of $x = 0$ is the y axis. For a constant b , the line $y = b$ is a line through $(0,b)$ that is parallel to the x axis and contains all points (x,b) for all possible values of x . Therefore the graph of $y = 0$ is the x axis.

Three dimensional space \mathbb{R}^3 contains special planes. For a constant a , the graph of $x = a$ is a plane through $(a,0,0)$ containing all points (a,y,z) for all possible values of y and z . Therefore the graph of $x = 0$ is the plane containing origin and the entire y and z axes. (This is the y,z plane.) For a constant b the graph of $y = b$ is the plane consisting of all points (x,b,z) for all possible values of x and z . In particular, the graph of $y = 0$ is the plane containing origin and the x and z axes, namely the x,z plane. The graph of $z = c$ is the set of all points (x,y,c) and is parallel to the x and y axes.

Homework 1: In \mathbb{R}^3 draw the special planes $x = 2$ and $y = 3$. Describe the intersection of these two special planes.

II. Slicing surfaces with special planes

General Principle #1: If we intersect a surface S with a plane P , we get a curve lying in P . We study the surface by studying the intersections of the surface S with the special planes $x = \text{constant}$ and $y = \text{constant}$ and $z = \text{constant}$. The process of intersecting the surface S with the plane P is called “slicing the surface S with the plane P ”.

Imagine the surface $z = x^2 + y^2 + 1$ floating above the x, y plane. We slice it with special plane $x = 3$ and the points we get must have both $x = 3$ and $z = x^2 + y^2 + 1$, so we get points satisfying $z = 9 + y^2 + 1 = 10 + y^2$. We recognize this curve as being a parabola opening upwards. In fact, for every slice $x = a$ we get a curve $z = (1 + a^2) + y^2$ that is a parabola opening upwards. Next, imagine slicing that same surface with the special plane $y = 2$. We get the curve $z = x^2 + 4 + 1$ which is a parabola opening upwards. In fact, slicing with any special plane $y = b$, we get the curve $z = x^2 + b^2 + 1$ and that is a parabola opening upwards.

We could also slice with the horizontal plane $z = c$ where c is a constant. We would get $x^2 + y^2 + 1 = c$ which is $x^2 + y^2 = c - 1$. In case $c < 1$ no points satisfy that equation because $x^2 + y^2$ is always positive while $c - 1 < 0$. In case $c = 1$, only $x = 0, y = 0$ satisfy the equation, and we conclude that the surface hits the slicing plane $z = 1$ in the single point $(0, 0, 1)$. In case $c > 1$ we get $x^2 + y^2 + 1 = c$ which is $x^2 + y^2 = (c - 1)$. Because $c - 1 > 0$ this is a circle with radius $\sqrt{c - 1}$.

Now the problem is to put together all of those slices to get the surface $z = x^2 + y^2 + 1$. This is not easy at first. We will draw the picture in class.

Homework 2: Describe the curves obtained by slicing the surface $z = 3 - x^2 - y^2$ with the special planes $x = a$ where a is a constant. What if we slice with the special planes $y = b$? What if we slice with the special planes $z = \text{constant}$?

Homework 3: Describe what you get if you slice the surface $z = 2x + 3y$ with the special planes $x = a$ where a is any constant. What if you slice with the plane $y = b$?

Pasting together slices was all the ancients could do. Today we have software that can graph in \mathbb{R}^3 , so we can get rough ideas of what the surface $z = f(x, y)$ will look like.

Probably the surface $z = f(x, y)$ will have hilltops and valley bottoms (a.k.a local maxima or local minima or relative maxima or relative minima). Our job is to find them exactly.

General Principle #2: If the surface $z = f(x, y)$ has a hilltop directly above the point $(a, b, 0)$ and if we slice the surface with the plane $y = b$ then we get a curve $z = f(x, b)$ with variables z and x , and the value $x = a$ must give a local maximum for that function. Similarly, if we slice our surface with $x = a$ we get a curve $z = f(a, y)$ which must have a local maximum at $y = b$ (There is an analogous principle for valley bottoms on the surface.)

Homework 4 Consider the surface $z = x^2 + 2x - y^2 + 4y$ and the point $(-1, 2)$.

- Slice the surface with $x = -1$. What curve do you get? Does it have a local maximum where $y = 2$?
- Slice the surface with the point $y = 2$. What curve do you get? Does it have a local maximum where $x = -1$?
- Could the given surface have a hilltop directly above $(-1, 2, 0)$?

General Principle #3: If the surface $z = f(x, y)$ has a hilltop directly above the point $(a, b, 0)$ and if we slice the surface with the plane $y = b$ then we get a curve $z = f(x, b)$ with variables z and x , and the derivative of $z = f(x, b)$ with respect to x must have value zero when $x = a$. (There is an analogous principle for valley bottoms on the surface.)

Homework 5: State an analog of General Principle #3 if the surface $z = f(x, y)$ has a valley bottom directly above the point $(a, b, 0)$ and we slice the surface with the special plane $x = a$.

III. Partial derivatives

Consider the surface $z = f(x, y) = x^3 + xy + y^2$ and the point $(a, b, 0)$ of its domain. Slicing that surface with the plane $x = a$ gives us the formula $z = a^3 + ay + y^2$. That is the formula of a curve, and we know how to find $\frac{dz}{dy} = 0 + a + 2y$. Alternatively we may slice with the vertical plane $y = b$ to get the curve $z = f(x, b) = x^3 + bx + b^2$ and we know how to find $\frac{dz}{dx} = 3x^2 + b$. As noted in General Principal #3, if a hill top or valley bottom of the surface lies directly above $(a, b, 0)$, then both of these derivatives, when evaluated at $x = a$ and $y = b$, must have value zero.

But what if we do not know what point $(a, b, 0)$ to consider? Suppose our job is to find the point $(a, b, 0)$. We are now ready to talk about “partial derivatives of $f(x, y)$.” We will do two artificial-looking things that are usually described as

“Hold x constant and find the derivative of $f(x, y)$ with respect to the variable y ”

or

“Hold y constant and find the derivative of $f(x, y)$ with respect to the variable x .”

The first is denoted by $\frac{\partial f}{\partial y}$ or $\frac{\partial z}{\partial y}$ and the second by $\frac{\partial f}{\partial x}$ or $\frac{\partial z}{\partial x}$. We already know all of the derivative rules that we will ever need. The new and tricky part is to be able to say to ourselves “Even though x looks like a variable, it is held constant” or “Even though y looks like a variable, it is really constant.” Consider some examples.

With $z = x^3 + xy + y^2$, hold y constant. Then $\frac{\partial z}{\partial x} = 3x^2 + y + 0$. Next, hold x constant in the z formula. Then $\frac{\partial z}{\partial y} = 0 + x + 2y$.

Let $z = x + e^{2y}$. Then $\frac{\partial z}{\partial x} = 1$ because y is constant, and therefore so is e^{2y} . Also, $\frac{\partial z}{\partial y} = 0 + 2e^{2y}$ because with x constant, its derivative with respect to y is zero.

Let $z = \frac{x^3}{y}$. Then $\frac{\partial z}{\partial x} = \frac{1}{y} * 3x^2$ because with y held constant, $\frac{1}{y}$ is also constant so that z is just the constant $\frac{1}{y}$ times the variable part x^3 . Also $\frac{\partial z}{\partial y} = x^3 * (-1)y^{-2}$ because with x held constant, z is just the constant x^3 times the variable part y^{-1} .

Let $z = x^2 e^{xy}$. Hold x constant. Then the only variable part of z is $e^{constant*y}$ so we have $\frac{\partial z}{\partial y} = x^2 * e^{xy} * x = x^3 e^{xy}$. Next hold y constant. Then z is the product of two functions of the variable x , namely x^2 and $e^{constant*x}$ so we need the product rule and we have $\frac{\partial z}{\partial x} = 2x * e^{xy} + x^2 * e^{xy} * y$.

Suppose $z = xy \ln(x + 1)$. Then $\frac{\partial z}{\partial x}$ has $y = constant$ so $z = constant * x * \ln(x + 1)$ and we use the product rule to get

$$\frac{\partial z}{\partial x} = y * \left(x * \frac{1}{x + 1} + 1 * \ln(x + 1) \right).$$

With x held constant we get $z = (the\ constant\ x \ln(x + 1)) * y$ so that

$$\frac{\partial z}{\partial y} = x \ln(x + 1).$$

Relation to slicing surfaces with vertical planes: The process “Hold y constant and treat x as if it’s the only variable” is the same as slicing surfaces with special vertical planes $y = constant$ to get curves and then taking derivatives of the resulting curves with respect to x . Look at the following example. Start with $z = x^3 y^2$. First compute $\frac{\partial z}{\partial x} = 3x^2 y^2$ because we hold y constant. Evaluating $\frac{\partial z}{\partial x}$ at the point $x = a, y = b$ gives $\frac{\partial z}{\partial x}(a, b) = 3a^2 b^2$. Second, slice the given surface with the vertical plane $y = b$ to get the curve

$z = x^3 * b^2$. Its derivative with respect to the variable x is $3x^2 * b^2$ and when we evaluate that derivative at $x = a$ we get $3a^2b^2$ which is what we got before.

An economic application: Often the number z of gizmos produced is determined by the amount of labor used and the amount of capital invested, so $z = f(x, y)$ where x is the number of hours of labor used and y is the amount of capital used. You have heard your economics professors talk about “the marginal productivity of labor.” They mean “the rate of change of gizmo production if we hold capital expenditure constant and allow the labor variable to change.” In other words, then mean $\frac{\partial z}{\partial x}$. The economic term “marginal productivity of capital” is analogously defined.

Homework 6: p 405 # 1,3,5,7,9 , 11,19, plus “If $z = f(x, y)$ gives the number z of gizmos produced if we use x hours of labor and y units of capital, what do we mean by the marginal productivity if capital?”

IV. Candidates for high and low points on surfaces

Theorem: Suppose $z = f(x, y)$. If (a, b) is a high or low point on the surface, then $\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 0$. Therefore, to find candidates for hilltops or valley bottoms on the surface, solve the system of equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Example 4-1: Let $z = x^2 - y^2 - 4x + 6y - 5$. Then we must solve $0 = \frac{\partial z}{\partial x} = 2x - 4$ and $0 = \frac{\partial z}{\partial y} = -2y + 6$. Solving $2x - 4 = 0$ and $-2y + 6 = 0$ shows that the point $(2, 3)$ is the only possible place where a hilltop or a valley bottom might occur.

As in single variable calculus, it can happen that “not every candidate is elected.” In particular, we can see that the point $(2, 3)$ gives neither a hilltop nor a valley bottom for the surface. Slicing the surface with the vertical plane $x = 2$ gives the curve $z = 4 - y^2 - 8 + 6y - 5 = -9 + 6y - y^2$, which is a parabola opening downwards. Therefore, if $(2, 3)$ gives a local max or min of the surface, it must be a maximum point. Next slice with $y = 3$. We get $z = x^2 - 9 - 4x + 18 - 5 = x^2 - 4x + 4$ which is a parabola opening upwards, so if $(2, 3)$ is a local max or min, it must be a local minimum. Those two conclusions contradict each other, and we are forced to conclude that $(2, 3)$ does not give a maximum or a minimum on the surface. The fact that different slices have different concavity is the problem.

Example 4-2: If $z = x^2 + y^2 + xy + x + 2y$ then the candidates for hill tops and valley bottoms on the surface are found by solving the two equations $0 = \frac{\partial z}{\partial x} = 2x + y + 1$ and $0 = \frac{\partial z}{\partial y} = 2y + x + 2$. The first gives $y = -2x - 1$ which we substitute into the second to get

$$0 = 2y + x + 2 = 2(-2x - 1) + x + 2 = -4x - 2 + x + 2 = 3x$$

and so $x = 0$. Putting $x = 0$ into $y = -2x - 1$ gives $y = -1$. Therefore our only candidate for max or min of x is $(0, -1)$.

Is the candidate $(0, -1)$ elected? Slicing the surface with the vertical planes $x = 0$ and $y = -1$ gives the curves $z = y^2 + 2y$ and $z = x^2 + 1 - x + x - 2$ and both are parabolas opening upwards, so we do not have the same contradiction the we discovered in Example 4-1.

But there are other vertical planes through $(0, -1)$ that we could have used to slice the surface, for example the vertical plane $x - y = 1$. Is it true that every possible vertical slice of the surface gives a curve opening upwards? That is for tomorrow.

Homework 7: page 414, # 1,3,5,7,9, plus find all candidates for 17, 19, 21.

V. Second partial derivatives

Recall from single variable calculus that for a curve $y = f(x)$ we used $f'(x)$ to study the rate of change of y and we used $f''(x)$ to study concavity of the curve. Something similar happens with $z = f(x, y)$ but there are complications.

Recall that the symbol $\frac{\partial}{\partial x}$ means “Hold y constant and take the derivative with respect to x ,” and the symbol $\frac{\partial}{\partial y}$ has an analogous meaning. Recall that $\frac{\partial z}{\partial x}$ is a formula, often involving both x and y . That allows us to ask about both

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) \text{ and } \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right).$$

Similarly $\frac{\partial z}{\partial y}$ is a formula involving x and y so we can ask about both

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) \text{ and } \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right).$$

In other words, the function $z = f(x, y)$ has two first derivatives and four second derivatives.

We compress the notations for second partial derivatives, e.g.,

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) \text{ and } \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right).$$

The second partial derivatives $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ are called pure second partials while $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ are called mixed second partials.

Example 5-1: Let $z = \frac{x^3}{y}$. Then $\frac{\partial z}{\partial x} = \frac{1}{y} * 3x^2$ so that

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{1}{y} * 3x^2 \right) = \frac{6x}{y}$$

and

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{1}{y} * 3x^2 \right) = (-1)y^{-2} * 3x^2.$$

Example 5-2: Still with $z = \frac{x^3}{y}$ we have $\frac{\partial z}{\partial y} = (-1) * x^3 * y^{-2}$ so $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} ((-1) * x^3 * y^{-2}) = -3x^2 y^{-2}$.

Note that for the function $z = \frac{x^3}{y}$, the two mixed second partials were equal. That is no accident. The following theorem is proved in Math 403:

Theorem: Suppose $f(x, y)$ has continuous second partial derivatives. Then the mixed partials of f are equal.

What good are second partials? Recall that $\frac{\partial z}{\partial x}$ is the derivative of the curve obtained by slicing the surface $z = f(x, y)$ with the vertical plane $y = \text{constant}$. Therefore the pure second partial $\frac{\partial^2 z}{\partial x^2}$ is just the second derivative of that curve, so presumably $\frac{\partial^2 z}{\partial x^2}$ allows us to study concavity of the curve. A similar comment applies to the other pure second partial. The use of the mixed second partials is harder to explain, but they will appear in the next section.

Homework 8: page 406 #19, 23 plus “ Consider the surface $z = f(x, y) = x^2 + 4x - 3y^2 + 12y$. How do we know that $(-2, 2)$ is the only point where $f(x, y)$ might have a high or low point? Now slice the surface with the vertical plane $x = -2$. Discuss the concavity of the resulting curve at the point where

$x = -2, y = 2$. Finally slice the surface with the vertical plane $y = 2$ and discuss the concavity of the resulting curve at the point where $x = -2, y = 2$.”

VI. Second partial derivative tests

Recall the three basic shapes of (parts of) curves $y = f(x)$ from single variable calculus. In each of the following pictures, $f'(a) = 0$ so that $x = a$ is a candidate for max or min.

There are three basic shapes of (parts of) surfaces $z = f(x, y)$ that we will encounter:

We will need some way to distinguish between them. The second partial derivatives will be the key.

Theorem: Suppose $f(x, y)$ has continuous partial derivatives and that (a, b) is known to be a candidate for max/min of $z = f(x, y)$, i.e., suppose (a, b) is a solution of $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$. Find the formula

$$Dval = \frac{\partial^2 z}{\partial x^2} * \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 .$$

Compute $Dval(a, b)$.

- i) if $Dval(a, b) < 0$ then neither max nor min occurs at (a, b) .
- ii) if $Dval(a, b) = 0$ this test will not determine what happens at (a, b) .
- iii) if $Dval(a, b) > 0$ then all slices by vertical planes through (a, b) have the same concavity so that the surface has either a max or a min at (a, b) , and you determine which is it by looking at the concavity of the curve obtained by slicing the surface with either $x = a$ or $y = b$, i.e., by looking at the sign of $\frac{\partial^2 z}{\partial x^2}(a, b)$. In this case (iii), a positive pure second partial derivative gives a concave up slice curve (and therefore the surface has a valley bottom above (a, b)) and a negative pure second partial derivative gives concave down slice curve (and therefore the surface has a hilltop above (a, b)).

Example 6-1: Consider $z = 6xy^2 - 2x^3 - 3y^4$. Suppose we have used the first partial derivative test to determine that the points $(1, 1), (1, -1)$ and $(0, 0)$ are candidates for max/min of the surface. Use $Dval$ to determine what really happens at these candidates.

$$\begin{aligned} \text{We have } \frac{\partial z}{\partial x} &= 6y^2 - 6x^2; \quad \frac{\partial^2 z}{\partial x^2} = -12x; \quad \frac{\partial^2 z}{\partial y \partial x} = 12y; \\ \frac{\partial z}{\partial y} &= 12xy - 12y^3; \quad \frac{\partial^2 z}{\partial y^2} = 12x - 36y^2; \\ \text{and } Dval &= (-12x)(12x - 36y^2) - (12y)^2 \end{aligned}$$

Therefore $Dval(1, 1) = (-12)(12 - 36) - 144 = 288 - 144 > 0$ so that $(1, 1)$ gives either a max or a min. To determine which, we note that $\frac{\partial^2 z}{\partial x^2}(1, 1) = -12$ so our slices at $(1, 1)$ are all concave down and we have a hilltop at $(1, 1)$

At the candidate $(1, -1)$, $Dval(1, -1) = (-12)(12 - 36) - (-12)^2 > 0$ so we have either a max or min, and since $\frac{\partial^2 z}{\partial y^2}(1, -1) = 12 - 36(-1)^2 < 0$ we have another hilltop at $(1, -1)$.

Because $Dval(0, 0) = 0$ our test does not tell us what happens at $(0, 0)$.

Example 6-2: Consider $z = x^4 - 4xy + y^4$. To find candidates for max/min, we solve $0 = \frac{\partial z}{\partial x} = 4x^3 - 4y \Rightarrow y = x^3$ and $0 = \frac{\partial z}{\partial y} = -4x + 4y^3 \Rightarrow y^3 = x$. Therefore $y = y^9$ so that either $y = 0$ or else $y^8 = 1$. Therefore y is one of $0, -1, 1$. From $x = y^3$ we see that our candidates are $(0, 0), (1, 1), (-1, -1)$. We must check each one. We have:

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 4; \frac{\partial^2 z}{\partial y \partial x} = -4; \frac{\partial^2 z}{\partial y^2} = 12y^2 \text{ so that } Dval = (12x^2 - 4)(12y^2) - (-4)^2 = (12x - 4)(12y^2) - 16.$$

$Dval(0, 0) = (-4) * (0) - 16 < 0$ so neither max nor min occurs at $(0, 0)$.

$Dval(1, 1) = (12 - 4)(12) - 16 > 0$ so we have max or min at $(1, 1)$. Because $\frac{\partial^2 z}{\partial x^2}(1, 1) = 12 - 4 > 0$, we have a valley bottom at $(1, 1)$.

$Dval(-1, -1) = (12 - 4)(12) - 16 > 0$ so we have a max or a min at $(-1, -1)$. Because $\frac{\partial^2 z}{\partial y^2}(-1, -1) = 12(-1)^2 > 0$ we have another valley bottom above $(-1, -1)$.

Homework 9: p 414. #11,13,15,17,19,21,23,29