Present and Future Values of Income Streams

Present Value of a Single Future Payment: What is today's value of a promise to pay at some future date? If the promise is completely trustworthy, and if just one future payment is involved, then we answer that question by using $A = P_0e^{rt}$ to compute how much we would need to invest today (P_0) in order to end up with the promised amount A in the specified number of years. That P_0 value is the *present value of the single future payment*. Note that we are not considering the possibility that there might be a question about whether the payment will actually be made.

Example How much should we pay today for a trustworthy promise to pay $1,000 in five years, assuming that today's interest rate is four percent? We solve

$$1000 = P_0 * e^{.04*5}$$

to get $P_0 = 1000/e^{.0r*5} = 1000 * e^{-.04*5} = 818.73$, an amount less that $1,000$. That computation is called *discounting to present value*.

Income Streams: A sequence of future payments is called an *income stream*. What is the present value of an income stream? We could compute the present value of each of the individual payments in the stream and add them up. For long streams, that leads to a lot of messy arithmetic.

There is an easy calculus-based approach that gives an approximate answer to the present value question for an income stream. The answer is only approximate because it assumes that the payments arrive continuously, rather than in discrete lumps, much in the same way that water flows out of a hose into a swimming pool.

Present Value of an Income Stream – General Example: We are to receive a stream of payments at the constant rate of K dollars per year, spread over the time interval between now (time = 0) and some ending date $E = EndTime$. We divide the time interval $[0,E]$ into millions of subintervals, each of length Δt. Consider any one of those future intervals, stretching from time t to time $t+\Delta t$. That interval's length is Δt years, and during the interval money flows at the rate of K dollars per year, so the amount of money that we receive during that time interval is $K * \Delta t$. Think of $K * \Delta t$ as being a single payment arriving t years in the future. Its present value is essentially $(K * \Delta t) * e^{-rt}$ where r is today's interest rate. If we add up all of those millions of present values we get

$$\Sigma K * e^{-rt} * \Delta t.$$

We note that this sum is just an approximation because it assumes that the future amount $K * \Delta t$ arrived at time t while in fact its arrival was spread out over the entire time interval $[t, t+\Delta t]$. However, this error tends to disappear if we use more and more subintervals because then $\Delta t \to 0$. Our next realization is that our sum of present values is actually a Riemann sum for a definite integral $\int_0^E K * e^{-rt}\, dt$ and that as $\Delta t \to 0$, the present value sums approach the value of that integral. Therefore we have

$$Present\ value\ of\ stream = \int_0^E K * e^{-rt}\, dt.$$

Numerical Example We win a million dollar lottery and are scheduled to receive payments of $50,000 per year for twenty years. What is the present value of that stream, assuming a prevailing interest rate of 4% ? We compute

$$\int_0^{20} 50000 * e^{-0.04*t}\, dt = 688,338.79.$$

We have used the fact that the antiderivative of e^{kt} is $\frac{e^{kt}}{k}$, where $k = -0.04$.

Future Value of a Stream: What if we receive an income stream and deposit the payments directly into a savings account at the moment they arrive. How large will our account be at the end of the stream? That ending value is called the *future value of the stream*.

The tricky part is that different payments sit in the account, gaining interest, for different lengths of time. The payments near the beginning of the stream (time $t = 0$) collect interest for a long time, while the payments received near time E = End time will earn interest for only a little while.

Once again, let's think of the payment stream as arriving continuously, and divide the time interval $[0, E]$ into millions of subintervals of equal length Δt. We look at the amount of money $K * \Delta t$ that arrives during one of the time subintervals.

Question: How long does that amount sit in the bank, earning interest?
Answer: For $(E - t)$ years.
Question: To what will that payment grow by time E?
Answer: to $(K * \Delta t) * e^{r*(E-t)}$
Question: What will the sum of all those future values look like?
Answer: $\Sigma K * e^{r*(E-t)} \Delta t$
Question: Is that also a Riemann sum?
Answer: Yes, for the definite integral $\int_0^E K * e^{r*(E-t)}\, dt$.

That is a formidable integral. You will soon learn a change of variables technique using $u = r * (E - t)$ but there is a shortcut. Note that $e^{r(E-t)} = e^{rE} * e^{-rt}$ and that e^{rE} is a constant. Therefore

$$\int_0^E K * e^{r*(E-t)}\, dt = \int_0^E K * e^{-rt} * e^{rE}\, dt = e^{rE} * \int_0^E K * e^{-rt}\, dt = (Present\ Value\ of\ stream) * e^{rE}.$$

In other words, to compute the future value of the stream we first compute its present value and then ask "To what amount would that present value grow if invested at rate r for E years?"

Homework

1) What is the future value of the \$50,000 per year lottery income stream lasting 20 years described above, assuming a prevailing interest rate of four percent?

2) You borrow \$100,000 from the bank today and agree to pay it back in equal payments over 20 years. Assume the interest rate is four percent. What size will your payments be? [Hint: The bank gets an income stream of unknown size K, spread over the next twenty years. Decide whether the \$100,000 figure is the present or future value of the stream. Then solve for K.

3) We found that the present value of a \$50,000 per year income stream lasting twenty years is \$688,338.79. What if the stream lasts 40 years? What if it lasts 80 years? [Answers – The present values will be about \$1 million for the 40 year stream and about \$1.2 million for the 80 year stream.]

4) Find the present and future values of a \$50,000 per year income stream that lasts 20 years if the prevailing interest rate is 5 percent rather than 4 percent. Think about the impact of changes in the prevailing interest rate on present and future values.

5) Text, page 363, number 21 and 25.