Areas and Antiderivatives

A lot of the new chapter is devoted to working the "find a derivative" problem backwards. We will start with $f(x) =$ the derivative of a function $F(x)$ and try to determine what the function $F(x)$ must have been. (We will say that we are finding the *antiderivative* of $f(x)$.) For example we will start with $y' = 2x$ (the formula for the derivative) and ask what the function y must be. No doubt, you recognize this as a differential equation and you can guess that the solution is $y = x^2 + C$ where C is any constant. You will learn a lot of rules for anti-derivatives, but I want to introduce this chapter by looking at the question "Why would anyone want to know an antiderivative?"

Finding areas of objects with curved edges was, with a few exceptions, something that the great Greek geometers could not do. Let's start with something simple. Suppose we have a continuous and increasing function $y = f(x)$ whose graph lies above the x-axis for $a \le x \le b$. We want the numerical area of the figure whose edges are the x axis, the vertical lines $x = c$ and $x = d$ and the curve $y = f(x)$. (Pictures in class.)

Our approach is a devious one. Instead of looking for a single number, we invent a whole new function, called the *cumulative area function*. For $a \le x \le b$, let $A(x)$ be the area of our region from a to x. What we want is $A(d)$. You might ask "If we don't know how to find the single number $A(d)$, how can we hope to know the formula $A(x)$?" That's a reasonable question, but as it happens, we can figure out what $A'(x)$ is, and then us the antiderivative process to find $A(x)$.

What is $A'(x)$? We recall the official definition

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}.$$

This is the area under our curve between x and $x+h$, divided by h. Let's look at the case where $h > 0$. (Relevant pictures in class.)

We see that this curved-top area is trapped between two rectangular areas, each having base = h. The inside rectangle has

$$area1 \ = h \times (\ the\ minimum\ height\ of\ the\ curve\ f(x)\ between\ x\ and\ x+h) = h \times f(x)$$

and the outside rectangle has

$$area2 \ = h \times (\ the\ maximum\ height\ of\ the\ curve\ f(x)\ between\ x\ and\ x+h) = h \times f(x+h)$$

so that we have

$$hf(x) \le A(x+h) - A(x) \le hf(x+h).$$

Dividing through by h gives

$$f(x) \le \frac{A(x+h) - A(x)}{h} \le f(x+h).$$

Now ask what happens as $h \to 0$. Because f is a continuous function, we have $\lim_{h \to 0} f(x+h) = f(x)$, so we get

$$f(x) \le \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \le f(x)$$

which says

$$A'(x) = f(x).$$

In English, "the bounding curve is the derivative of the cumulative area function."

In our new terminology, "the cumulative area function is one of the antiderivatives of the bounding curve." But which one?

Let's start with a simple example where we know the area geometrically and can check our answer. Suppose the bounding curve is $y = 2x$ and we want to know the area bounded by that curve and the x-axis for $0 \le x \le 4$. (picture in class) Because that are is a right triangle, we can find its area by computing $\frac{1}{2} * base * height = \frac{1}{2} * 4 * 8 = 16$. Compare that to our cumulative area approach. We know that the cumulative area function $A(x)$ is one of the antiderivatives of the bounding curve $y = 2x$, so we conclude that $A(x) = x^2 + C$. To find C we need a data point and we can cleverly find one: as it happens, when $x = 0$, $A(0)$ is trying to find the total area under the bounding curve from 0 on the left to 0 on the right. But certainly that area is just zero, so we have $0 = A(0) = 0^2 + C$ and we conclude that $C = 0$. Therefore, $A(x) = x^2$ and the answer that we want is $A(4) = 4^2 = 16$ which we know geometrically to be the right answer.

Will it always happen that the constant $C = 0$? Unfortunately, no. To understand why, let's change the problem just a little. Suppose we want the area under $y = 2x$ and above the x-axis from $x = 1$ to $x = 4$. (You can compute this area by Greek geometry, taking a large triangle area minus a smaller triangle area.) We now have a new cumulative area function, which we'll call $B(x)$, that computes the area under our bounding curve between 1 and x. Once again we know that $B' = 2x$ and so again $B(x) = x^2 + C$. What is C? We need a data point, and we can cleverly find one: we know that the area starting and ending at $x = 1$ has value zero, so we have $0 = B(1) = 1^2 + C$, showing that $C = -1$. Therefore, the cumulative area function in this new case is $B(x) = x^2 - 1$, and the answer that we want is $B(4) = 4^2 - 1 = 15$.

Much of this chapter is devoted to finding clever shortcuts to simplify the ideas above, but for now we don't have such tools. Therefore we will work things out the long way.

Homework: Use cumulative area functions to find the following areas. In each case the first step is to find the cumulative area function. It will include a $+C$ term, and then we must cleverly find a data point that will allow us to pin down the constant C.

1) bounded by the curve $y = 2x$ and the x-axis, for $3 \le x \le 5$. (Ans = 16)

2) bounded by the curve $y = 3x^2$ and the x-axis for $0 \le x \le 4$ (Ans = 64) (Archimedes could have worked this problem using his own version of calculus, but it was essentially the most difficult example he could work out.)

3) bounded by the curve $y = e^x$ and the x-axis for $0 \le e \le 1$ (ans = $e - 1$)

4) bounded by $y = \frac{1}{x}$ and the x-axis for $2 \le x \le 5$ (ans = $\ln(2.5)$ although your answer might not look like that at first)

5) bounded by $y = \frac{3}{2}x^{\frac{1}{2}}$ for $1 \le x \le 4$ (ans = 7)