Finding f' Using Secants and Limits

Suppose we have a function $f(x) = x^2 + 3x + 4$ and we want to know the formula for the slope of the tangent line at a generic point $x = a$. The rules for derivatives tell us that the answer will be $f'(a) = 2a + 3$. How would we use secants and limits to verify that answer? The process described below is informally called the "**sliding secants method**".

The point of tangency corresponding to $x = a$ is the point

$$P = (a, f(a)) = (a, a^2 + 3a + 4).$$

A nearby point Q on the curve has $x = a + h$ where h is a small non-zero distance. Therefore

$$Q = (a+h, f(a+h)) = (a+h, (a+h)^2 + 3(a+h) + 4)$$

so that the secant line PQ has

$$\textit{Secant Slope} = \frac{f(a+h) - f(a)}{(a+h) - a}.$$

Substituting the formulas for $f(a)$ and $f(a+h)$ gives

$$\textit{Secant Slope} = \frac{(a+h)^2 + 3(a+h) + 4) - (a^2 + 3a + 4)}{h} = \frac{(a^2 + 2ah + h^2 + 3a + 3h + 4) - (a^2 + 3a + 4)}{h}.$$

We collect terms in the numerator, being careful of signs, and then factor out an h to get

$$\textit{Secant Slope} = \frac{2ah + h^2 + 3h}{h} = \frac{h(2a + h + 3)}{h} = 2a + h + 3.$$

At this point, we know that if we step off h units from $x = a$ to get a new point Q on the curve, then the secant line PQ has $\textit{Secant Slope} = 2a + h + 3$. Now ask what happens to the value $\textit{Secant Slope} = 2a + h + 3$ as $h \to 0$. We see that *Secant Slope* gets closer and closer to $2a + 3$.

But we see that as $h \to 0$, the secant line approximates the tangent line more and more closely, so *Secant Slope* must be getting closer and closer to the desired tangent line slope. Therefore, we really do have

$$f'(a) = \textit{Tangent Line Slope at } (a, f(a)) = 2a + 3,$$

just as the power rule predicted.

In our course we will always use various rules to start with $f(x)$ and write down the formula for $f'(x)$ unless the problem says something like "Use secant slopes and limits to show that the derivative of $f(x) = x^2 + 3x + 4$ is $f'(x) = 2x + 3$" and in that case you must use a calculation such as the one given above.