Example 8.4 A life test with Type I censoring at $c = 5000$ hours was conducted on an electronic device used in undersea digital repeaters. The device will be used at the bottom of the Atlantic Ocean at a temperature of approximately 10°C. Because of the high reliability of the device and the limited testing time, temperature is used as an accelerating factor. One specific measure of interest in the accelerated life test is the probability of survival to 10,000 hours at 10°C. The life tests were conducted at four temperature levels: 10°, 40°, 60°, and 80°C. All $n_1 = 30$ of the devices placed on test at 10° survived 5000 hours. Not surprisingly, heavy right censoring occurred at this lowest stress level. There was heavy censoring of the $n_2 = 100$ devices placed on test at 40°, where only $r_2 = 10$ of the devices failed at times

1298, 1390, 3187, 3241, 3261, 3313, 4501, 4568, 4841, 4982,

and the other $n_2 - r_2 = 100 - 10 = 90$ devices were right censored at 5000 hours. There were $n_3 = 20$ devices placed on test at 60°, and $r_3 = 9$ of the devices failed at times

581, 925, 1432, 1586, 2452, 2734, 2772, 4106, 4674,

and the other $n_3 - r_3 = 20 - 9 = 11$ devices were right censored at time 5000. Finally, $n_4 = 15$ devices were placed on test at 80°, and $r_4 = 14$ of the devices failed at times

283, 361, 515, 638, 854, 1024, 1030, 1045, 1767, 1777, 1856, 1951, 1964, 2884,

and the other $n_4 - r_4 = 15 - 14 = 1$ device was right censored at 5000 hours.

The next two sections illustrate the parametric analysis of complete and right-censored data sets with no covariates associated with the exponential and Weibull distributions. Chapter 9 presents parameter estimation techniques for models that explicitly account for covariates.

8.2 Exponential Distribution

The exponential distribution is popular due to its tractability for parameter estimation and inference. It can be used to model a lifetime that exhibits the memoryless property. The general likelihood theory developed in Chapter 7 is used to estimate the single ($p = 1$) parameter. The exponential distribution can be parameterized by either its failure rate λ or its mean $\mu = 1/\lambda$. Using the failure rate to parameterize the distribution, the survivor, density, hazard, and cumulative hazard functions are

$$S(t, \lambda) = e^{-\lambda t} \qquad f(t, \lambda) = \lambda e^{-\lambda t} \qquad h(t, \lambda) = \lambda \qquad H(t, \lambda) = \lambda t$$

for $t \geq 0$. Note that the unknown parameter λ has been added as an argument in these lifetime distribution representations because it is now also an argument in the likelihood function and is estimated from data.

All the analysis in this and subsequent sections assumes that a *random sample* of n items from a population has been placed on a test and subjected to typical field operating conditions. Equivalently, $t_1, t_2, \ldots, t_n$ are independent and identically distributed random lifetimes from a particular population distribution (exponential in this section). As with all statistical inference, care must be taken to ensure that a random sample of lifetimes is collected. Consequently, random numbers should be used to determine which n items to place on test. Laboratory conditions should adequately mimic field conditions. Only representative items should be placed on test because items manufactured using a previous design may have a different failure pattern than those with the current design.

Four classes of data sets (complete, Type II right censored, Type I right censored, and randomly right censored) are considered in separate subsections, followed by subsections on comparing two exponential populations and life testing.

Complete Data Sets

A complete data set consists of failure times $t_1, t_2, \ldots, t_n$. Although lowercase letters are used to denote the failure times here to be consistent with the notation for censoring times, the failure times are nonnegative random variables. The likelihood function can be written as a product of the probability density functions evaluated at the failure times:

$$L(\lambda) = \prod_{i=1}^{n} f(t_i, \lambda).$$

Note that the $\boldsymbol{t}$ argument has been left out of the likelihood expression for compactness. Using the last expression for the log likelihood function (adapted for a complete data set) from Section 7.5,

$$\ln L(\lambda) = \sum_{i=1}^{n} \left[\ln h(t_i, \lambda) - H(t_i, \lambda) \right].$$

For the exponential distribution, this is

$$\ln L(\lambda) = \sum_{i=1}^{n} \left[\ln \lambda - \lambda t_i \right] = n \ln \lambda - \lambda \sum_{i=1}^{n} t_i.$$

To determine the maximum likelihood estimator for λ, the single-element score vector

$$U(\lambda) = \frac{\partial \ln L(\lambda)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} t_i,$$

also known as the *score statistic*, is equated to zero and solved for λ, yielding

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i},$$

where the denominator is often referred to as the *total time on test*. Not surprisingly, the maximum likelihood estimator $\hat{\lambda}$ is the reciprocal of the sample mean (see Example 7.7).

Example 8.5 Fit the exponential distribution to the complete data set of $n = 23$ ball bearing failure times from Example 8.1.

For this particular data set, the total time on test is $\sum_{i=1}^{n} t_i = 1661.16$ million revolutions, yielding the maximum likelihood estimate

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i} = \frac{23}{1661.16} = 0.01385$$

failure per 10^6 revolutions. The number of significant digits reported in the point estimate matches the number of digits in the data set. The value of the log likelihood function at the maximum likelihood estimate is $\ln L(\hat{\lambda}) = -121.435$, which will be used later in this chapter when comparing the exponential and Weibull fits to this data set.

Figure 8.1 displays a graph of the *empirical survivor function* for the ball bearing failure times, which takes a downward step of $1/n = 1/23$ at each data value, along with the fitted exponential survivor function $S(t) = e^{-\hat{\lambda}t}$. Empirical and fitted distributions are traditionally compared by plotting the two survivor functions on the same set of axes because the probability density function and hazard function suffer from the drawback of requiring the data to be divided into cells to plot the empirical distribution. A close match between the two indicates a good fit. Although more details are given later concerning formal statistical goodness-of-fit tests, it is apparent from this figure that the exponential distribution is a very poor fit. This particular data set was chosen for this example to illustrate one of the shortcomings of using the exponential distribution to model any data set without assessing the adequacy of the fit. Extreme caution must be exercised when using the exponential distribution since, as indicated in Figure 8.1, the exponential distribution is not an adequate probability model for this data set.

There are two clues that the exponential distribution would perform poorly in this setting. First, we neglected to plot a histogram of the ball bearing failure times prior to fitting the exponential distribution. The histogram in Figure 8.2 indicates a nonzero mode to the population probability density function, implying that the exponential distribution is probably not going to be an adequate model. Second, knowing the physics of failure can be helpful in this case. Ball bearings typically fail by wearing out. When a ball bearing's diameter falls outside of a prescribed range, it is considered to be failed. This indicates that the hazard function for a ball bearing will probably increase over time, so a distribution with a monotone increasing hazard function from the IFR class would be a better choice than the exponential distribution. As shown in the next section, the Weibull distribution provides a much better approximation to this particular data set. Since the exponential distribution can be fitted to any data set that has at least one observed failure, the adequacy of the model must always be assessed. The point and interval estimators for λ associated with the exponential distribution are legitimate

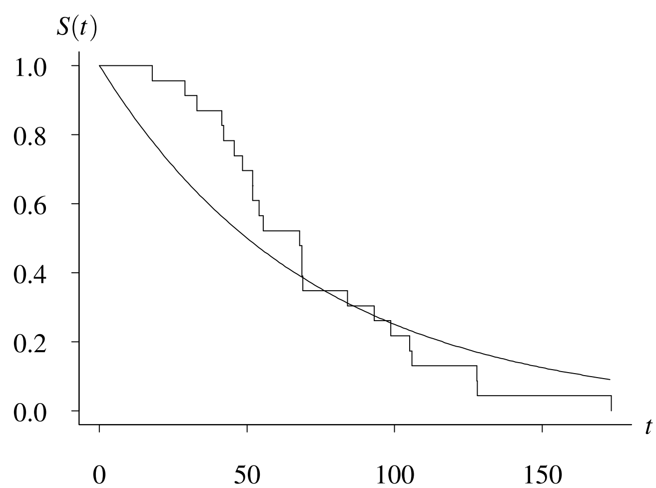

Figure 8.1: Empirical and fitted exponential survivor functions for the ball bearing data set.

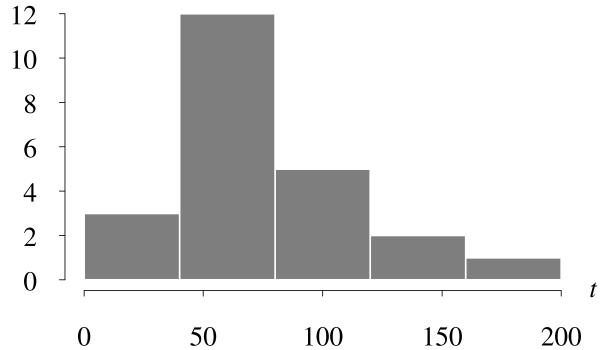

Figure 8.2: Histogram of the ball bearing failure times.

only if the data set is a random sample drawn from an exponential population. That is clearly *not* the case for this particular data set.

Information matrices. To find the Fisher and observed information matrices associated with a complete data set from an exponential(λ) population, the derivative of the score statistic is required:

$$\frac{\partial^2 \ln L(\lambda)}{\partial \lambda^2} = -\frac{n}{\lambda^2}.$$

Taking the expected value of the negative of this quantity yields the 1×1 Fisher information matrix

$$I(\lambda) = E\left[\frac{-\partial^2 \ln L(\lambda)}{\partial \lambda^2}\right] = E\left[\frac{n}{\lambda^2}\right] = \frac{n}{\lambda^2}.$$

If the maximum likelihood estimator $\hat{\lambda}$ is used as an argument in the negative of the second partial derivative of the log likelihood function, the 1×1 observed information matrix is obtained:

$$O(\hat{\lambda}) = \left[\frac{-\partial^2 \ln L(\lambda)}{\partial \lambda^2}\right]_{\lambda=\hat{\lambda}} = \frac{n}{\hat{\lambda}^2} = \frac{\left(\sum_{i=1}^n t_i\right)^2}{n}.$$

Confidence interval for λ. Asymptotic confidence intervals for λ based on the likelihood ratio statistic or the observed information matrix are unnecessary for a complete data set because the sampling distribution of $\sum_{i=1}^n t_i$ is tractable. In particular, from Property 4.7 of the exponential distribution,

$$2\lambda \sum_{i=1}^n t_i = \frac{2n\lambda}{\hat{\lambda}}$$

has the chi-square distribution with $2n$ degrees of freedom. Therefore, with probability $1-\alpha$,

$$\chi^2_{2n,\,1-\alpha/2} < \frac{2n\lambda}{\hat{\lambda}} < \chi^2_{2n,\,\alpha/2},$$

where $\chi^2_{2n,p}$ is the $(1-p)$th fractile of the chi-square distribution with $2n$ degrees of freedom. Performing the algebra required to isolate λ in the middle of the inequality yields the exact two-sided $100(1-\alpha)\%$ confidence interval

$$\frac{\hat{\lambda}\chi^2_{2n,\,1-\alpha/2}}{2n} < \lambda < \frac{\hat{\lambda}\chi^2_{2n,\,\alpha/2}}{2n}.$$