5.4 Proportional Hazards Model

One common feature of the accelerated life and proportional hazards lifetime models is that they are both used to account for the effects of covariates on a random lifetime. All the notation developed in Section 5.3 applies to proportional hazards models as well as accelerated life models. Whereas accelerated life models modify the rate that an item moves through time based on the values of the covariates, proportional hazards models modify the hazard function by the factor $\psi(\boldsymbol{z})$.

The *proportional hazards* model can be defined by

$$h(t) = \psi(\boldsymbol{z}) h_0(t) \qquad t \geq 0.$$

It is often known as the *Cox proportional hazards model* because it was devised by British statistician Sir David Cox. The covariates increase the hazard function when $\psi(\boldsymbol{z}) > 1$ or decrease the hazard function when $\psi(\boldsymbol{z}) < 1$. As before, a popular choice for the link function is the log-linear form $\psi(\boldsymbol{z}) = e^{\boldsymbol{\beta}'\boldsymbol{z}}$, where $\boldsymbol{\beta}$ is a $q \times 1$ vector of regression coefficients corresponding to the q covariates.

The other lifetime distribution representations can be determined from this definition. For example,

$$\begin{aligned} H(t) &= \int_0^t h(\tau)\, d\tau \\ &= \int_0^t \psi(\boldsymbol{z}) h_0(\tau)\, d\tau \\ &= \psi(\boldsymbol{z}) H_0(t) \qquad t \geq 0. \end{aligned}$$

Table 5.2 gives the various lifetime distribution representations for the accelerated life and proportional hazards models. This table allows a modeler to determine any of the first four lifetime distribution representations for either model once the baseline distribution and link function are specified, as illustrated in Example 5.11.

	Accelerated life	Proportional hazards
$S(t)$	$S_0\big(t\psi(\boldsymbol{z})\big)$	$[S_0(t)]^{\psi(\boldsymbol{z})}$
$f(t)$	$\psi(\boldsymbol{z}) f_0\big(t\psi(\boldsymbol{z})\big)$	$\psi(\boldsymbol{z}) f_0(t) [S_0(t)]^{\psi(\boldsymbol{z})-1}$
$h(t)$	$\psi(\boldsymbol{z}) h_0\big(t\psi(\boldsymbol{z})\big)$	$\psi(\boldsymbol{z}) h_0(t)$
$H(t)$	$H_0\big(t\psi(\boldsymbol{z})\big)$	$\psi(\boldsymbol{z}) H_0(t)$

Table 5.2: Lifetime distribution representations for regression models.

Example 5.11 Consider the case of a Weibull baseline distribution in a proportional hazards model. Find the hazard and survivor functions.

The Weibull baseline hazard function with parameters λ and κ is

$$h_0(t) = \kappa \lambda^{\kappa} t^{\kappa - 1} \qquad t \geq 0.$$

So the hazard function for an item with covariates $\boldsymbol{z}$ is

$$h(t) = \psi(\boldsymbol{z})h_0(t) = \psi(\boldsymbol{z})\kappa\lambda^{\kappa}t^{\kappa-1} \qquad t \geq 0.$$

Using Table 5.2, the appropriate formula for determining the survivor function is

$$S(t) = [S_0(t)]^{\psi(\boldsymbol{z})} \qquad t \geq 0.$$

Using the usual baseline survivor function for the Weibull distribution,

$$S(t) = \left[e^{-(\lambda t)^{\kappa}}\right]^{\psi(\boldsymbol{z})} = e^{-(\lambda t)^{\kappa}\psi(\boldsymbol{z})} \qquad t \geq 0.$$

This can be recognized as a Weibull lifetime with scale parameter $\lambda\psi(\boldsymbol{z})^{1/\kappa}$ and shape parameter κ. Example 5.9 showed that a Weibull baseline model under the accelerated life assumption also produced a Weibull time to failure. Thus, if a modeler specifies a Weibull baseline distribution in either model, the time to failure distribution will also have the Weibull distribution. The Weibull distribution is the only baseline distribution in which the accelerated life and the proportional hazards models coincide in this manner. The values of the regression coefficients $\boldsymbol{\beta}$ will differ, however, because the functional form of $S(t)$ is not identical for the accelerated life and proportional hazards models.

The final two examples in this section present two figures which are designed to enhance your intuition associated with the accelerated life and proportional hazards models.

Example 5.12 To illustrate the difference between the accelerated life and proportional hazards models graphically, consider the baseline hazard function

$$h_0(t) = \begin{cases} 1 & 0 \leq t < 1 \\ t & t \geq 1 \end{cases}$$

and a single binary covariate z. Assume that when $z = 0$ (the control case), the link function is $\psi(z) = 1$, and when $z = 1$ (the treatment case), the link function is $\psi(z) = 2$ for both models. Plot the hazard functions for the baseline and treatment cases for both models.

Using the formulas for $h(t)$ given in Table 5.2, the hazard functions for these cases are plotted in Figure 5.3. The baseline hazard function $h_0(t)$ is given by the solid line and applies to both the accelerated life (AL) and proportional hazards (PH) models in the control case. The dashed line is the hazard function for the proportional hazards model in the treatment case. The hazard function for $z = 1$ is always twice the baseline case because

$$h_{\text{PH}}(t) = \psi(z)h_0(t) \qquad t \geq 0.$$

The dotted line is the hazard function for the accelerated life model in the treatment case ($z = 1$) in which

$$h_{\text{AL}}(t) = \psi(z)h_0\big(t\psi(z)\big) \qquad t \geq 0.$$

So in addition to doubling the risk that the item is subject to, the item also moves through time at twice the rate of an item in the control case.

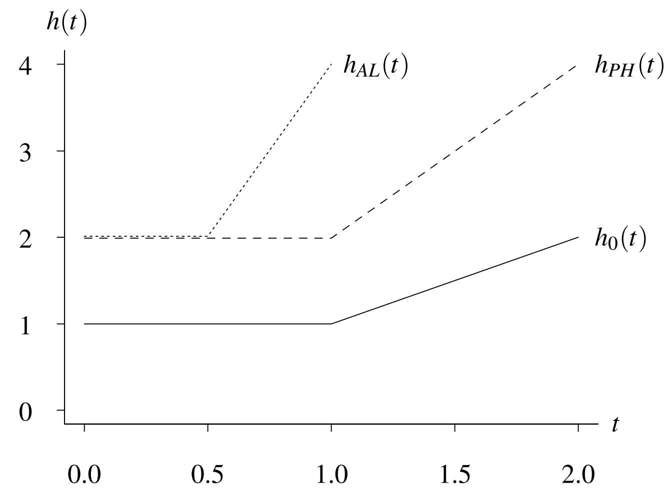

Figure 5.3: Hazard functions for a piecewise-continuous baseline hazard function.

A final example further illustrates the geometry associated with the accelerated life and proportional hazards models. This example focuses on the cumulative hazard functions for these two models.

Example 5.13 As in Example 5.12, consider a single binary covariate z, and assume that $\psi(0) = 1$ and $\psi(1) = 2$. The baseline cumulative hazard function is

$$H_0(t) = 0.1t^2 \qquad t \geq 0.$$

Plot the cumulative hazard functions for the baseline and treatment cases for both the accelerated life and proportional hazards models.

Table 5.2 gives the cumulative hazard function for the accelerated life (AL) and proportional hazards (PH) models as

$$H_{\text{AL}}(t) = H_0\big(t\psi(z)\big) \qquad \text{and} \qquad H_{\text{PH}}(t) = \psi(z)H_0(t)$$

for $t \geq 0$. The multiplicative influence of $\psi(z)$ on time in the accelerated life model and the multiplicative influence of $\psi(z)$ on the cumulative hazard function in the proportional hazards model are illustrated in Figure 5.4. The baseline cumulative hazard function is plotted as a solid line. This is a special case of the Weibull distribution. For any vertical line, such as the one at $t = 1.9$ (arbitrarily chosen),

$$H_{\text{PH}}(t) = 2H_0(t) \qquad t \geq 0.$$

For any horizontal line, such as the one at $H(t) = 0.256$ (also arbitrarily chosen),

$$H_{\text{AL}}(t) = H_0(2t) \qquad t \geq 0.$$