4.2 Exponential Distribution

Just as the normal distribution plays a pivotal role in classical statistics because of the central limit theorem, the exponential distribution plays a pivotal role in reliability and lifetime modeling because it is the only continuous distribution with a constant hazard function. The exponential distribution has often been used to model the lifetimes of electronic components and is appropriate when a used component that has not failed is statistically as good as a new component in terms of its remaining time to failure. This is a rather restrictive assumption. Moreover, the exponential distribution is presented first because of its simplicity. The exponential distribution has a single positive scale parameter λ, often called the *failure rate*, measured in failures per unit time. The five lifetime distribution representations are

$$S(t) = e^{-\lambda t} \qquad f(t) = \lambda e^{-\lambda t} \qquad h(t) = \lambda \qquad H(t) = \lambda t \qquad L(t) = \frac{1}{\lambda}$$

for $t \geq 0$, and are plotted in Figure 4.3 for $\lambda = 1$ and $\lambda = 2$. Two-parameter distributions, which are more complex but can model a wider variety of situations, are presented in subsequent sections.

The centrality, tractability, and importance of the exponential distribution make it a key probability distribution to know well. In that light, this section surveys 15 probabilistic properties of the exponential distribution that are useful in understanding how it is unique and when it should be applied. In all the properties, it is assumed that the nonnegative lifetime T has the exponential distribution with parameter λ, which denotes the number of failures per unit time, which could be seconds, hours, or years (or even miles or cycles). The symbol $\sim$ means "is distributed as." The shorthand $T \sim$ exponential(λ) is read as "the random variable T is distributed as an exponential random variable with parameter λ." Some brief comments follow the statement of each of the properties; proofs of the properties are contained in Appendix A.

Property 4.1 (memoryless property) If $T \sim$ exponential(λ), then

$$P[T \geq t] = P[T \geq t + s \,|\, T \geq s] \qquad t \geq 0;\ s \geq 0.$$

As shown in Figure 4.4 for $\lambda = 1$ and $s = 0.5$, this result indicates that the conditional survivor function for the lifetime of an item that has survived to time s is identical to the survivor function for the lifetime of a brand new item. This used-as-good-as-new assumption is very strong. Consider, for example, whether the exponential distribution should be used to model the lifetime of a candle with an expected burning time of 5 hours. If several candles are sampled and burned, we could imagine a bell-shaped histogram for candle lifetimes, centered around 5 hours. The exponential lifetime model is certainly *not* appropriate in this case, because a candle that has burned for 4 hours does not have the same remaining lifetime distribution as that of a brand new candle. The exponential distribution would only be appropriate for candle lifetimes if the remaining lifetime of a used candle is identical to the lifetime of a new candle. A consumer would certainly prefer a new candle to a used candle in terms of its longevity. This same argument can be used to reason that the exponential lifetime model should not be applied to mechanical components that undergo wear (for example, bearings) or fatigue (structural supports), or electrical components that contain an element that burns away (filaments) or degrades with time (batteries). An electrical component for which the exponential lifetime assumption might be justified is a fuse. A fuse is designed to fail when there is a power surge that causes the fuse to fail, resulting in a blown fuse which must be replaced. Assuming that the fuse does not undergo any weakening or degradation over time and that power surges that cause failure occur at a constant rate over time, the exponential lifetime assumption is appropriate, and a

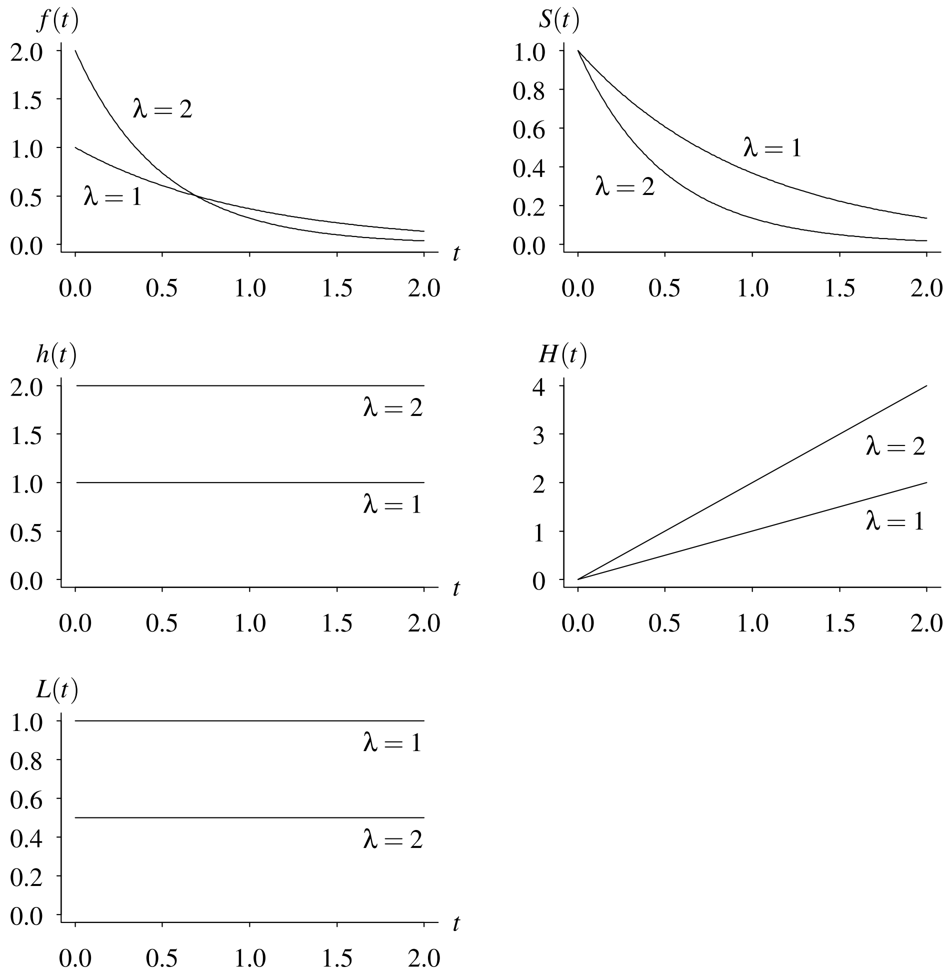

Figure 4.3: Lifetime distribution representations for the exponential distribution.

used fuse that has not failed is therefore as good as a new one in terms of longevity.

The exponential distribution should be judiciously applied because the memoryless property restricts its applicability. It can easily be misapplied for the sake of simplicity because the statistical techniques for the exponential distribution are particularly tractable, or because small sample sizes do not support more than a one-parameter distribution.

Property 4.2 The exponential distribution is the only continuous distribution with the memoryless property.

This result indicates that the exponential distribution is the only continuous lifetime distribution for which the conditional lifetime distribution of a used item is identical to the original lifetime distribution. The only discrete distribution with the memoryless property is the geometric distribution.

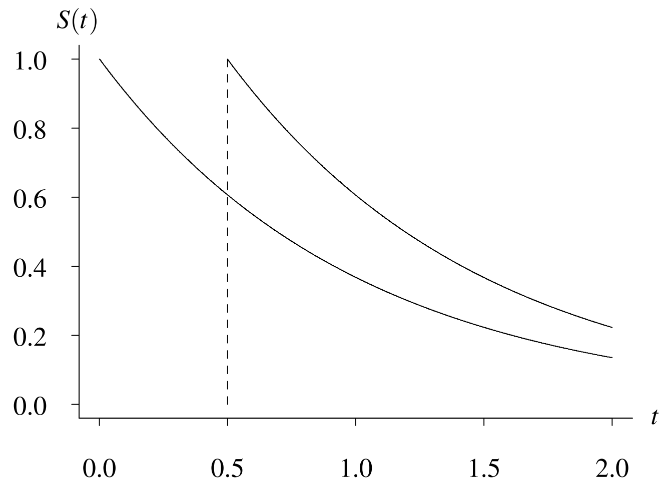

Figure 4.4: The memoryless property of the exponential distribution.

Property 4.3 If $T \sim$ exponential(λ), then $\lambda T \sim$ exponential(1).

An exponential random variable with $\lambda = 1$ is often called a *unit* exponential random variable. This particular exponential distribution is important for random variate generation, as indicated in the next property.

Property 4.4 If T is a continuous nonnegative random variable with cumulative hazard function $H(t)$, then $H(T) \sim$ exponential(1).

This property is mathematically equivalent to the probability integral transformation, which states that $F(T) \sim U(0, 1)$, resulting in the inverse-cdf technique for generating random variates for Monte Carlo simulation: $T \leftarrow F^{-1}(U)$, where $U \sim U(0, 1)$. Using Property 4.4, random lifetime variates are generated by

$$T \leftarrow H^{-1}\big(-\ln(1-U)\big)$$

because $-\ln(1-U)$ is a unit exponential random variate. Random lifetimes generated in this fashion are generated by the *cumulative hazard function technique*.

Example 4.2 Assuming that the failure time of an item has the Weibull distribution with survivor function

$$S(t) = e^{-(\lambda t)^{\kappa}} \qquad t \ge 0$$

for positive scale parameter λ and positive shape parameter κ, find an equation to convert $U(0, 1)$ random numbers to Weibull random variates.

The cumulative hazard function for the Weibull distribution is

$$H(t) = -\ln S(t) = (\lambda t)^{\kappa} \qquad t \ge 0,$$

which has inverse

$$H^{-1}(y) = \frac{y^{1/\kappa}}{\lambda} \qquad y \ge 0.$$

Weibull random variates can be generated by

$$T \leftarrow \frac{1}{\lambda}\left[-\ln(1-U)\right]^{1/\kappa},$$

where U is uniformly distributed between 0 and 1.

Figure 4.5 illustrates the geometry associated with generating a variate using the cumulative hazard function technique. The value of $-\ln(1-U)$, the unit exponential random variate, is indicated on the vertical axis, and the corresponding random variate T is indicated on the horizontal axis.

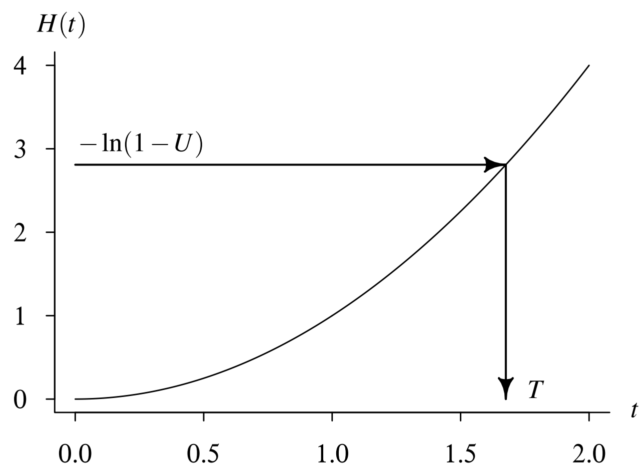

Figure 4.5: Generating a random variate by the inverse cumulative hazard function technique.

The next result gives a general expression for the sth moment of an exponential random variable.

Property 4.5 If $T \sim \text{exponential}(\lambda)$, then

$$E[T^s] = \frac{\Gamma(s+1)}{\lambda^s} \qquad s > -1,$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$.

Additional information on the gamma function, $\Gamma(\alpha)$, and other related functions is given in Appendix B. When s is a nonnegative integer, this expression reduces to $E[T^s] = s!/\lambda^s$. By setting $s = 1, 2, 3$, and 4, the mean, variance, coefficient of variation, skewness, and kurtosis can be obtained:

$$E[T] = \frac{1}{\lambda} \qquad V[T] = \frac{1}{\lambda^2} \qquad \gamma = 1 \qquad \gamma_3 = 2 \qquad \gamma_4 = 9.$$

Since the coefficient of variation of an exponential random variable is 1, a quick check for exponentiality for a data set is to see if the ratio of the sample standard deviation to the sample mean is approximately 1. The histogram of the sample values should also have the appropriate shape, although it will be subject to random sampling variability, which is more pronounced for small sample sizes.

Property 4.6 (self-reproducing) If $T_1, T_2, \ldots, T_n$ are independent, $T_i \sim \text{exponential}(\lambda_i)$, for $i = 1, 2, \ldots, n$, and $T = \min\{T_1, T_2, \ldots, T_n\}$, then

$$T \sim \text{exponential}\left(\sum_{i=1}^{n} \lambda_i\right).$$

This result indicates that the minimum of n independent exponential random lifetimes also has the exponential distribution. This is important in two applications. First, if n components, each with independent exponential times to failure, are arranged in series, the distribution of the system failure time is also exponential with a failure rate equal to the sum of the component failure rates. When the n components have the same failure rate λ, the system lifetime is exponential with failure rate $n\lambda$. Second, when there are several independent, exponentially distributed *causes* of failure competing for the lifetime of an item (for example, failing by open or short circuit for an electronic item or death by various diseases for a human being), the lifetime can be modeled as the minimum of the individual lifetimes from each cause of failure. This second application will be expanded upon in Section 5.1.

Property 4.7 If $T_1, T_2, \ldots, T_n$ are independent and identically distributed exponential(λ) random variables, then

$$2\lambda \sum_{i=1}^{n} T_i \sim \chi^2(2n),$$

where $\chi^2(2n)$ denotes the chi-square distribution with $2n$ degrees of freedom.

This property is useful for determining a confidence interval for λ based on a data set of n independent exponential lifetimes. With probability $1 - \alpha$,

$$\chi^2_{2n,1-\alpha/2} < 2\lambda \sum_{i=1}^{n} T_i < \chi^2_{2n,\alpha/2},$$

where the left- and right-hand sides of this inequality are the $\alpha/2$ and $1 - \alpha/2$ fractiles of the chi-square distribution with $2n$ degrees of freedom. (This subscript convention differs from that presented in Section 3.3 for fractiles.) This notation is illustrated in Figure 4.6, with the three areas under the probability density function of the chi-square random variable plotted on the graph. Rearranging this expression yields an exact two-sided $100(1-\alpha)\%$ two-sided confidence interval for λ:

$$\frac{\chi^2_{2n,1-\alpha/2}}{2\sum_{i=1}^{n} T_i} < \lambda < \frac{\chi^2_{2n,\alpha/2}}{2\sum_{i=1}^{n} T_i}.$$

Property 4.8 If $T_1, T_2, \ldots, T_n$ are independent and identically distributed exponential(λ) random variables, $T_{(1)}, T_{(2)}, \ldots, T_{(n)}$ are the corresponding order statistics (the observations sorted in ascending order), the ith gap is $G_i = T_{(i)} - T_{(i-1)}$ for $i = 1, 2, \ldots, n$, and if $T_{(0)} = 0$, then

(a) $P[G_i \ge t] = e^{-(n-i+1)\lambda t}$ $\qquad t \ge 0;\ i = 1, 2, \ldots, n$

(b) $G_1, G_2, \ldots, G_n$ are independent.