Chapter 3

Lifetime Distributions

Up to this point, reliability has been defined as a constant associated with one particular instance of time. Reliability is generalized to be a function of time in this chapter, and various lifetime distribution representations that are helpful in describing the evolution of the risks to which an item is subjected over time are introduced. In particular, five lifetime distribution representations are presented: the *survivor function*, the *probability density function*, the *hazard function*, the *cumulative hazard function*, and the *mean residual life function*. These five representations apply to both continuous (for example, the lifetime of a light bulb) and discrete (for example, the lifetime of a payroll computer program that is run weekly) lifetimes. *Moments* and *fractiles* are useful ways to summarize the survival pattern of an item, although they do not completely define a distribution (as do the five lifetime distribution representations). The representations can be used to describe the lifetime distribution of systems, as well as components. Survivor functions and reliability functions are combined to give the distribution of system failure time once component failure time distributions and the arrangement of the components are known. Finally, some *distribution classes* are defined using the lifetime distribution representations that classify an item based on how it ages.

3.1 Continuous Lifetime Distribution Representations

This section introduces five functions that define the distribution of a continuous, nonnegative random variable T, the lifetime of an item. Even though the emphasis in this book will generally be in reliability engineering, and T will be the lifetime of a manufactured product, the applications are much broader. For example, the random variable T could represent

- the response time by emergency vehicles to a reported building fire in the public safety arena,
- the time for a released inmate to return to prison in a recidivism application within the criminal justice system,
- the duration of a strike in a sociology application, or
- the time between the formation of a tropical storm and the time it makes landfall in a meteorology application.

The key requirements in this setting are that T is random and nonnegative. For this reason, this field of study is often referred to as *time-to-event* modeling.

The five lifetime distribution representations presented in this chapter are not the only ways to define the distribution of T. Other methods include the moment generating function $E\left[e^{sT}\right]$, the characteristic function $E\left[e^{isT}\right]$, the Mellin transform $E\left[T^{s-1}\right]$, and the reversed failure rate $f(t)/F(t)$. The five representations used here have been chosen because of their intuitive appeal, usefulness in problem solving, and popularity in the literature. In the next section, these functions are defined for discrete distributions.

Survivor Function

The first lifetime distribution representation is the *survivor function* $S(t)$. The survivor function is a generalization of reliability. Whereas reliability is defined as the probability that an item is functioning at one particular time, the survivor function is the probability that an item is functioning at any time t:

$$S(t) = P[T \geq t] \qquad\qquad t \geq 0.$$

It is assumed that $S(t) = 1$ for all $t < 0$. A survivor function is also known as the reliability function [because $S(t)$ is the reliability at time t] and the complementary cumulative distribution function [because $S(t) = 1 - F(t)$ for continuous random variables, where $F(t) = P[T \leq t]$ is the cumulative distribution function]. All survivor functions must satisfy three conditions:

$$S(0) = 1 \qquad\qquad \lim_{t \to \infty} S(t) = 0 \qquad\qquad S(t) \text{ is nonincreasing.}$$

There are two interpretations of the survivor function. First, $S(t)$ is the probability that an individual item is functioning at time t. This is important, as will be seen later, in determining the lifetime distribution of a system from the distribution of the lifetimes of its individual components. Second, if there is a large population of items with identically distributed lifetimes, $S(t)$ is the expected fraction of the population that is functioning at time t.

The survivor function is useful for comparing the survival patterns of several populations of items. The graph in Figure 3.1 is a plot of $S_1(t)$ and $S_2(t)$, where $S_1(t)$ corresponds to population 1

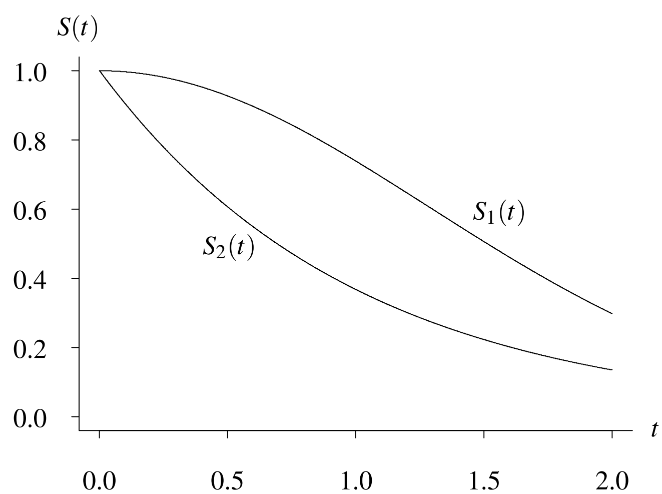

Figure 3.1: Two survivor functions.

and $S_2(t)$ corresponds to population 2. Since $S_1(t) \geq S_2(t)$ for all t values, it can be concluded that the items in population 1 are superior to those in population 2 with regard to reliability.

The conditional survivor function, $S_{T\,|\,T\geq a}(t)$, is the survivor function of an item that is known to be functioning at time a:

$$S_{T\,|\,T\geq a}(t) = \frac{P[T \geq t \text{ and } T \geq a]}{P[T \geq a]} = \frac{P[T \geq t]}{P[T \geq a]} = \frac{S(t)}{S(a)} \qquad\qquad t \geq a.$$

The conditional survivor function is assumed to be 1 for $t < a$. Figure 3.2 shows the original survivor function $S(t)$ and the conditional survivor function $S_{T\,|\,T\geq a}(t)$ for a = 0.5. Since the conditional survivor function is rescaled by the factor $S(a)$, it has the same shape as the remaining portion of the original survivor function. The conditional survivor function is useful for comparing the survival experience of a group of items that has survived to time a. Examples include manufactured items surviving a burn-in test and cancer patients surviving five years after diagnosis and treatment. The conditional survivor function is of particular interest to actuaries. For example, if a 43-year-old woman is purchasing a one-year term life insurance policy, an estimate of $S_{T\,|\,T\geq 43}(44)$ is required to determine an appropriate premium for the policy.

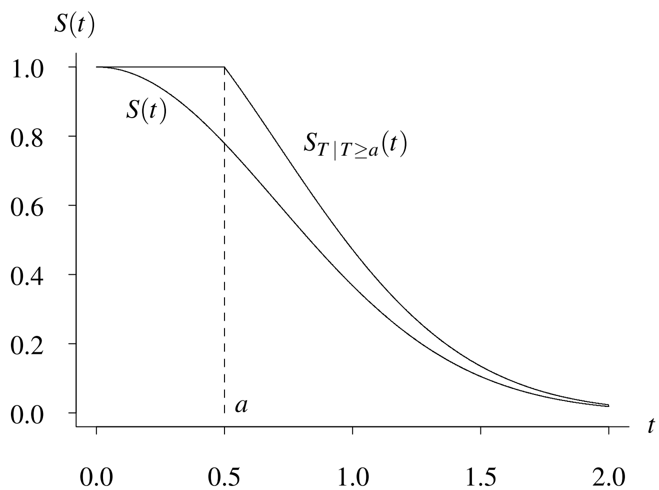

Figure 3.2: Conditional survivor function.

Probability Density Function

The second lifetime distribution representation, the *probability density function*, is defined by $f(t) = -S'(t)$ where the derivative exists, and has the probabilistic interpretation

$$f(t)\Delta t \cong P[t \leq T \leq t + \Delta t]$$

for small Δt values. Although the probability density function is not as effective as the survivor function for comparing the survival patterns of two populations, a graph of $f(t)$ indicates the likelihood of failure for any t. The probability of failure between times a and b is calculated by an integral:

$$P[a \leq T \leq b] = \int_a^b f(t)\, dt.$$

All probability density functions for lifetimes must satisfy two conditions:

$$\int_0^\infty f(t)\,dt = 1 \qquad\qquad f(t) \geq 0 \text{ for all } t \geq 0.$$

It is assumed that $f(t) = 0$ for all $t < 0$, which is consistent with our assumption that the random variable T is nonnegative. This assumption excludes distributions with negative support, such as the normal distribution. The probability density function shown in Figure 3.3 illustrates the relationship between the cumulative distribution function $F(t)$ and the survivor function $S(t)$ for a continuous lifetime. The area under $f(t)$ to the left of time t_0 is $F(t_0)$; the area under $f(t)$ to the right of t_0 is $S(t_0)$.

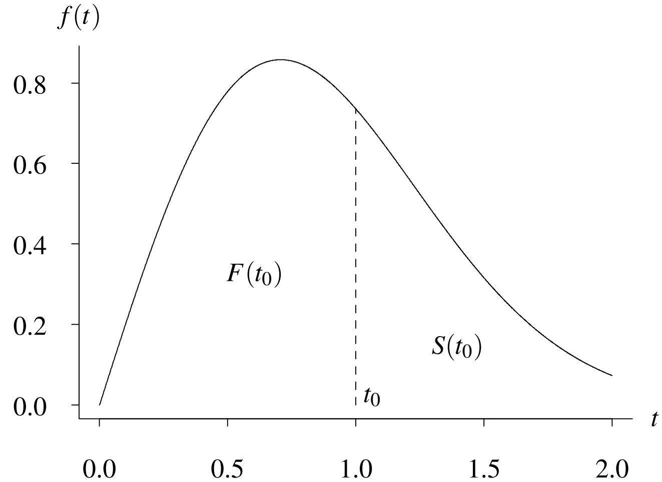

Figure 3.3: Relationship between the survivor and cumulative distribution functions.

Hazard Function

The *hazard function*, $h(t)$, is perhaps the most popular of the five representations for lifetime modeling due to its intuitive interpretation as the amount of *risk* associated with an item at time t. A second reason for its popularity is its usefulness in comparing the way risks change over time for several populations of items by plotting their hazard functions on a single set of axes. A third reason is that the hazard function is a special case of the intensity function for a nonhomogeneous Poisson process, which will be introduced in Chapter 6. A hazard function models the occurrence of one event, a failure, whereas the intensity function models the occurrence of a sequence of events over time. The hazard function goes by several aliases: in reliability it is also known as the hazard rate or failure rate; in actuarial science it is known as the force of mortality or force of decrement; in point process and extreme value theory it is known as the rate or intensity function; in vital statistics it is known as the age-specific death rate; and in economics its reciprocal is known as Mill's ratio. Referring to the hazard function as the failure rate is regrettable because that term is commonly used to describe the parameter λ in the exponential distribution.

The hazard function can be derived using conditional probability. First, consider the probability of failure between t and $t+\Delta t$:

$$P[t \le T \le t+\Delta t] = \int_t^{t+\Delta t} f(\tau)d\tau = S(t) - S(t+\Delta t).$$

Conditioning on the event that the item is working at time t yields

$$P[t \le T \le \; t+\Delta t \,|\, T \ge t] = \frac{P[t \le T \le t+\Delta t]}{P[T \ge t]} = \frac{S(t) - S(t+\Delta t)}{S(t)}.$$

If this conditional probability is averaged over the interval $[t,\; t+\Delta t]$ by dividing by Δt, an average rate of failure is obtained:

$$\frac{S(t) - S(t+\Delta t)}{S(t)\Delta t}.$$

As $\Delta t \to 0$, this becomes the instantaneous failure rate, which is the hazard function

$$\begin{aligned} h(t) &= \lim_{\Delta t \to 0} \frac{S(t) - S(t+\Delta t)}{S(t)\Delta t} \\ &= -\frac{S'(t)}{S(t)} \\ &= \frac{f(t)}{S(t)} \qquad\qquad t \ge 0. \end{aligned}$$

Thus, the hazard function is the ratio of the probability density function to the survivor function. Using the previous derivation, a probabilistic interpretation of the hazard function is

$$h(t)\Delta t \cong P[t \le T \le t+\Delta t \,|\, T \ge t]$$

for small Δt values, which is a conditional version of the interpretation for the probability density function. All hazard functions must satisfy two conditions:

$$\int_0^\infty h(t)\,dt = \infty \qquad\qquad h(t) \ge 0 \text{ for all } t \ge 0.$$

Example 3.1 Consider the Weibull distribution defined by the survivor function

$$S(t) = e^{-(\lambda t)^\kappa} \qquad\qquad t \ge 0,$$

with positive scale parameter λ and positive shape parameter κ. Find the hazard function.

By differentiating the survivor function with respect to t and negating, the probability density function is

$$f(t) = \lambda\kappa(\lambda t)^{\kappa-1} e^{-(\lambda t)^\kappa} \qquad\qquad t \ge 0,$$

so the hazard function is

$$h(t) = \frac{f(t)}{S(t)} = \lambda\kappa(\lambda t)^{\kappa-1} \qquad\qquad t \ge 0.$$

Figure 3.4 illustrates the hazard function shape for $\lambda = 1$ and several κ values. The hazard function is constant when $\kappa = 1$, increasing when $\kappa > 1$, and decreasing when $\kappa < 1$.