Chapter 2

Coherent Systems Analysis

This chapter examines techniques for expressing the arrangement of components in a system and for determining its reliability. We assume that an item consists of n components, arranged into a *system*. Two key decisions that are always a part of the modeling process are (1) which elements of the system are excluded and included and (2) the level of detail to be represented by the components. If the item under study is an automobile, for example, the modeler needs to decide whether the entire electrical system should be modeled as a component, or whether the individual assemblies, such as the alternator or ignition, should be modeled as components. The first two sections in this chapter survey *structural properties* associated with a system of components, and the next two sections survey *probabilistic properties* associated with a system of components. *Structure functions* are used to relate the states of the individual components to the state of the system. A block diagram is often helpful in visualizing the way components are arranged. *Minimal path sets* and *minimal cut sets* are two other ways of specifying the arrangement of components in a system. *Reliability functions* are used to determine the system reliability at a particular point in time, given the component reliabilities at that time. *Reliability bounds* are used to obtain an interval that contains the system reliability when there are many components and the exact system reliability is difficult to compute.

2.1 Block Diagrams and Structure Functions

A *structure function* is a useful tool in describing the way in which n components are interrelated to form a system. The structure function defines the system state as a function of the component states. A *system* is assumed to be a collection of *n components*. In addition, it is assumed that both the components and the system can either be functioning or failed. Although this *binary assumption* may be unrealistic for certain types of components or systems, it makes the mathematics involved more tractable. The assumption implies that component failure due to catastrophic causes is more readily modeled than component failure due to gradual degradation. An electrical component that undergoes *drift* or a mechanical component that undergoes *wear* must have a threshold value defined that will place the state of the item into the functioning or failed states in order to analyze the item using these techniques. A discussion of techniques to overcome the binary assumption is given at the end of this chapter in Section 2.5.

The failed and functioning states for both components and systems will be denoted by 0 and 1, respectively, as shown in the following definitions.

Definition 2.1 The state of component i, denoted by x_i, is

$$x_i = \begin{cases} 0 & \text{if component } i \text{ has failed} \\ 1 & \text{if component } i \text{ is functioning} \end{cases}$$

for $i = 1, 2, \ldots, n$.

These n values can be written as a system state vector, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. Since there are n components, there are 2^n different values that the system state vector can assume, and $\binom{n}{j}$ of these vectors correspond to exactly j functioning components, $j = 1, 2, \ldots, n$. The structure function, $\phi(\boldsymbol{x})$, maps the system state vector $\boldsymbol{x}$ to 0 or 1, yielding the state of the system.

Definition 2.2 The *structure function* ϕ is

$$\phi(\boldsymbol{x}) = \begin{cases} 0 & \text{if the system has failed when the state vector is } \boldsymbol{x} \\ 1 & \text{if the system is functioning when the state vector is } \boldsymbol{x}. \end{cases}$$

The most common system structures are the series and parallel systems, which are described by the two definitions that follow.

Definition 2.3 A *series system* functions if and only if all of its components function.

This definition implies that $\phi(\boldsymbol{x})$ assumes the value 1 when $x_1 = x_2 = \cdots = x_n = 1$, and 0 otherwise for a series system. Therefore, the structure function can be expressed as

$$\begin{aligned} \phi(\boldsymbol{x}) &= \begin{cases} 0 & \text{if there exists an } i \text{ such that } x_i = 0 \\ 1 & \text{if } x_i = 1 \text{ for all } i = 1, 2, \ldots, n \end{cases} \\ &= \min\{x_1, x_2, \ldots, x_n\} \\ &= \prod_{i=1}^{n} x_i. \end{aligned}$$

These three different ways of expressing the value of the structure function are equivalent, although the third is preferred because of its compactness. Systems that function only when all their components function should be modeled as series systems. If a calculator, for example, is modeled as a system of four components (keyboard, display, microprocessor, and battery), a series arrangement is appropriate because failure of any component results in system failure.

Block diagrams are useful to visualize a system of components. The block diagram corresponding to a series system of n components is shown in Figure 2.1. Although there are similarities, a block diagram is not equivalent to an electrical wiring diagram, a blueprint, or a work breakdown structure. A block diagram is a graphic device for expressing the arrangement of the components to form a system. If a path can be traced through functioning components from the far left to the far right on a block diagram, then the system functions. The boxes represent the system components, and either component numbers or component reliabilities are typically placed inside the boxes.

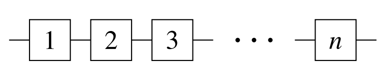

Figure 2.1: Series system block diagram.

The series arrangement of components is probably, though somewhat unfortunately, the most common in practice. The human body, for example, has the circulatory, nervous, digestive, and endocrine components that must all function for the system to function. Likewise, an automobile must have the starter, alternator, ignition switch, fuel pump, exhaust system, engine, transmission, brakes, and tires operational for the system to function. Unfortunately, a series system has the *worst* possible arrangement of components. At the opposite extreme is a *parallel* system, which has the *best* possible arrangement of components.

Definition 2.4 A *parallel system* functions if and only if one or more of its components function.

The definition implies that $\phi(\boldsymbol{x})$ assumes the value 0 when $x_1 = x_2 = \cdots = x_n = 0$, and 1 otherwise for a parallel system. Therefore,

$$\begin{aligned}\phi(\boldsymbol{x}) &= \begin{cases} 0 & \text{if } x_i = 0 \text{ for all } i = 1, 2, \ldots, n \\ 1 & \text{if there exists an } i \text{ such that } x_i = 1 \end{cases} \\ &= \max\{x_1, x_2, \ldots, x_n\} \\ &= 1 - \prod_{i=1}^{n}(1 - x_i).\end{aligned}$$

As in the case of the series system, the three ways of defining $\phi(\boldsymbol{x})$ are equivalent. The block diagram of a parallel arrangement of n components is shown in Figure 2.2. A parallel arrangement of components is appropriate when all components must fail for the system to fail. Numerous examples of parallel systems exist. Kidneys are an example of a two-component parallel system, since many people live normal lives with a single kidney. Another two-component system is the brake system on an automobile that contains two reservoirs for brake fluid.

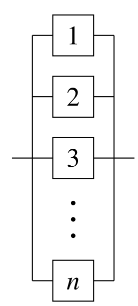

Figure 2.2: Parallel system block diagram.

Definition 2.5 A *k-out-of-n system* functions if and only if k or more of its n components function.

Series and parallel systems are special cases of k-out-of-n systems: a series system is an n-out-of-n system; a parallel system is a 1-out-of-n system. A suspension bridge that needs only k of its n cables to support the bridge, and an automobile engine that needs only k of its n cylinders to run are examples of k-out-of-n systems. The structure function for a k-out-of-n system is

$$\phi(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \sum_{i=1}^{n} x_i < k \\ 1 & \text{if } \sum_{i=1}^{n} x_i \ge k. \end{cases}$$

The block diagram for a k-out-of-n system is difficult to draw in general, but for specific cases it can be drawn by repeating components in the diagram.

Example 2.1 The block diagram for a 2-out-of-3 system, for example, is shown in Figure 2.3. The block diagram indicates that if all three, or exactly two out of three components (in particular 1 and 2, 1 and 3, or 2 and 3) function, then the system functions. Find the associated structure function.

The structure function for a 2-out-of-3 system is

$$\begin{aligned}\phi(\boldsymbol{x}) &= 1-(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)\\ &= x_1x_2+x_1x_3+x_2x_3-x_1^2x_2x_3-x_1x_2^2x_3-x_1x_2x_3^2+(x_1x_2x_3)^2.\end{aligned}$$

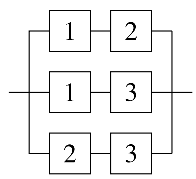

Figure 2.3: Two-out-of-three system block diagram.

Some systems require a more complex arrangement of components than k-out-of-n systems. The next two examples illustrate how to combine series and parallel arrangements to determine the appropriate structure function for a more complex system.

Example 2.2 An airplane has four engines, two on each wing. The airplane will fly (function) if at least one engine on each wing functions. Although many issues other than engine failure could cause the airplane to fail, this example only considers engine failure. Find the structure function for this system.

In this case, the four engines are denoted by components 1, 2, 3, and 4, with components 1 and 2 being on the left wing and components 3 and 4 being on the right wing. For the moment, if the plane is considered to consist of two wings (not considering individual engines), then the wings are arranged in series, since failure of the propulsion on either wing results in system failure. Each wing can be modeled as a two-component parallel subsystem of engines because only one engine on each wing is required to function. The appropriate block diagram for the system is shown in Figure 2.4. The structure function is the product of the structure functions of the two parallel subsystems:

$$\phi(\boldsymbol{x}) = \left[1-(1-x_1)(1-x_2)\right]\left[1-(1-x_3)(1-x_4)\right].$$

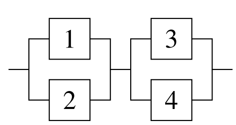

Figure 2.4: Four-engine block diagram.