Answers to selected exercises

Chapter 1

1.1 Answers will vary. Example: How many sequences of "heads" and "tails" are possible in 32 flips of a coin?

1.3 (a) 1296
(b) 126

1.5 120

1.7 52

1.9 (a) 676,000
(b) 626,000

1.11 7200

1.13 (a) 350
(b) 230
(c) 14,850

1.15 4200

1.19 (a) 2^n
(b) $\sum_{i=j+1}^{n} \binom{n}{i}$ $\quad j = 0, 1, \ldots, n-1$

1.21 Use the binomial theorem with $x = 1$ and $y = 1$

1.23 Consider the two cases in which the first six and last six digits are identical and non-identical

1.25 (a) 55
(b) 28
(c) 10

1.27 $\binom{n_1+n_2-1}{n_2-1} = \binom{n_1+n_2-1}{n_1}$

1.29 60

1.31 $324,000$

1.33 2730

1.35 28

1.37 190

1.39 11

1.41 Use the definition of $\binom{n}{r}$

1.43 111

1.45 47

1.47 126

1.49 70

1.51 2250

1.53 945

1.55 (a)

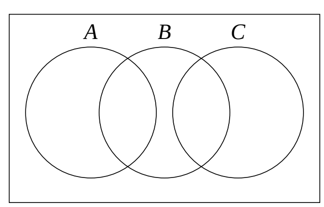
n	absolute error	relative error
8	417.6	0.01036
16	$1.087 \cdot 10^{11}$	0.00519
32	$6.843 \cdot 10^{32}$	0.00226
64	$1.651 \cdot 10^{86}$	0.00130
128	$2.509 \cdot 10^{212}$	0.00065

(b) $\lim_{n\to\infty} \frac{(n+1)!}{n!} \cong n+1$

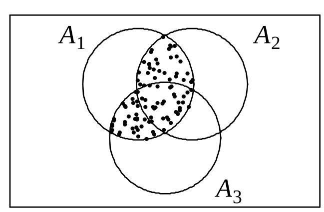
(c) Exact: 402387260077...0000000000
Approximate: $0.4023872601 \cdot 10^{2568}$

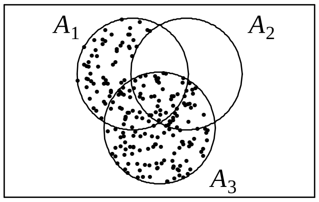
1.57 4917

1.59

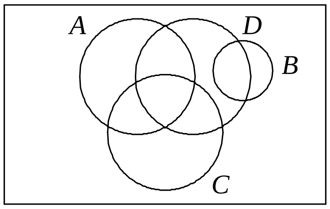


1.61

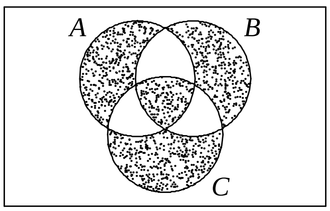


1.63 

1.65 15

1.67 

1.69 $A \cap (B' \cup C) = \{1, 2, 4, 5\}$

1.71 

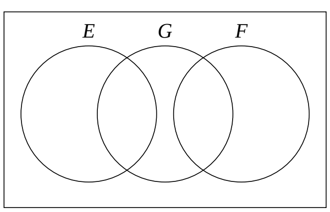
1.73 83

Chapter 2

2.1 (a) 81
(b) 3, 24, 18, 36

2.3 0.7

2.5 Use probability axioms and associated results

2.7 

2.9 $\frac{4}{15} \leq P(A \cap B) \leq \frac{3}{5}$

2.11 $P(A' \cup B') = 1$ and $P(A' \cap B') = 0$

2.13 0.1

2.15 $2^n - 1$

2.17 $1/16$

2.19 $\frac{5394}{9{,}860{,}459} \cong 0.0005470$

2.21 $\frac{5}{432} \cong 0.01157$

2.23 $\frac{1}{57} \cong 0.01754$

2.25 (a) $\frac{3150}{12{,}209} \cong 0.2580$
(b) $\frac{4}{15} \cong 0.2667$

2.27 $\frac{2109}{9520} \cong 0.2215$

2.29 $1/4$

2.31 (a) 28
(b) 0.18
(c) $\frac{128{,}665{,}503}{751{,}870{,}280} \cong 0.1711$

2.33 $\frac{48}{95} \cong 0.5053$

2.35 $\frac{155}{323} \cong 0.4799$

2.37 $1/3$

2.39 (a) $1/1024$
(b) $45/1024$
(c) $1/4$

2.41 $P(\text{product even}) = 1 - \binom{8}{n} / \binom{15}{n}$
for $n = 1, 2, \ldots, 8$
$P(\text{product even}) = 1$
for $n = 9, 10, \ldots, 15$

2.43 $\frac{\binom{7}{x}\binom{8}{y}\binom{9}{6-x-y}}{\binom{24}{6}}$ for $x = 0, 1, 2, \ldots, 6$;
$y = 0, 1, 2, \ldots, 6$; and $x + y \leq 6$

2.45 $\frac{2475}{9044} \cong 0.2737$

2.47 $\frac{1}{42} \cong 0.02381$

2.49 $P(\text{product is even}) = 1 - \frac{1}{2^n}$
for $n = 1, 2, 3, \ldots$

2.51 $\frac{1}{81} \cong 0.01235$

2.53 $\frac{19{,}513}{156{,}849} \cong 0.1244$

2.55 $\frac{3}{10}$

2.57 $\frac{15}{128} \cong 0.1172$

2.59 $\frac{7{,}496{,}455{,}078{,}125}{8{,}796{,}093{,}022{,}208} \cong 0.8522$

2.61 $\frac{1}{e} \cong 0.3679$

2.63 (a) $\frac{25}{54} \cong 0.4630$

(b) $\frac{25}{108} \cong 0.2315$

(c) $\frac{25}{162} \cong 0.1543$

2.65 $\frac{\binom{20}{i}\binom{80}{4-i}}{\binom{100}{4}}$ for $i = 0, 1, 2, 3, 4$

2.67 $\frac{1{,}242{,}904}{11{,}388{,}335} \cong 0.1091$

2.69 $\frac{7}{2592} \cong 0.002701$

2.71 $P(\text{one seed wins tournament}) \cong 0.52$

2.73 (a) $P(E \mid F') = \frac{27}{77} \cong 0.3506$

(b) $P(E \cap F \mid G) = \frac{2}{9} \cong 0.2222$

(c) $P(E \cap F' \mid F \cap G) = 0$

2.75 If $P(A \mid B) = P(B \mid A) = 0$, then it can be concluded that the events A and B are disjoint. If $P(A \mid B) = P(B \mid A) > 0$, then it can be concluded that $P(A) = P(B)$.

2.77 $\frac{19{,}282}{38{,}675} \cong 0.4986$

2.79 $\frac{4}{105} \cong 0.03810$

2.81 $\frac{3}{16} = 0.1875$

2.83 $\frac{4}{95} \cong 0.04211$

2.85 $\frac{9}{49} \cong 0.1837$

2.87 (a) $\frac{32}{663} \cong 0.04827$

(b) $\frac{15{,}104}{162{,}435} \cong 0.09298$

2.89 $\frac{553}{858} \cong 0.6445$ and $\frac{31{,}956{,}543{,}629}{51{,}227{,}404{,}613} \cong 0.6238$

2.91 Use the definition of conditional probability

2.93 (a) $\frac{5}{72} \cong 0.06944$

(b) $\frac{7}{72} \cong 0.09722$

2.95 $\frac{3}{392} \cong 0.007653$

2.97 If one of the inmates sees that the others hats are the same color, then he should guess the opposite color.

2.99 $\frac{1}{18} \cong 0.05556$

2.101 (e)

2.103 $\frac{p^2}{1-2p(1-p)}$

2.105 $\frac{2i}{n(n+1)}$ for $i = 1, 2, \ldots, n$

2.107 $\frac{2}{3}$

2.109 0.58

2.111 $\frac{26}{45} \cong 0.5778$

2.113 $\frac{1}{24} \cong 0.04167$

2.115 (a) $\frac{3m+2}{6m}$

(b) $\frac{2m}{3m+2}$

(c) $\left(\frac{3m+2}{6m}\right)^n$

(d) $\binom{n}{x}\left(\frac{3m+2}{6m}\right)^x\left(1-\frac{3m+2}{6m}\right)^{n-x}$ for $x = 0, 1, 2, \ldots, n$

2.117 $\left(\frac{1}{2}\right)^{n-1}\frac{1}{2} + \left(\frac{3}{4}\right)^{n-1}\frac{1}{4} + \left(\frac{7}{8}\right)^{n-1}\frac{1}{8} + \left(\frac{15}{16}\right)^{n-1}\frac{1}{16} + \left(\frac{31}{32}\right)^{n-1}\frac{1}{32} + \left(\frac{31}{32}\right)^{n-1}\frac{1}{32}$

2.119 Use an algebraic approach or induction

2.121 (a) $P(A \mid B) = 0$

(b) $P(A \mid B) = P(A)$

(c) $P(A \mid B) = P(A)/P(B)$

(d) $P(B \mid A) = 1$

(e) $P(B) > 0$

2.123 Use complementary probability

2.125 $560 p_D^2 p_I^3 p_R^4$

2.127 $(0.99)^{2131}$

2.129 17

2.131 $\frac{27}{64} \cong 0.4219$

2.133 $\left(\frac{23}{24}\right)^{11} \cong 0.6262$

2.135 $(1 + m - mp)p^n$

2.137 $p = \frac{1}{3}$

2.139 (a) false

(b) true

(c) false

(d) true

(e) false

2.141 840

2.143 0.994

2.145 $\frac{49}{648} \cong 0.07562$

2.147 $2^n - n - 1$

2.149 Mark should place one red ball in the first urn and all of the remaining $2n-1$ balls in the second urn.

Chapter 3

3.1 $f(x) = \frac{\binom{r}{2}\binom{w}{x-3}}{\binom{r+w}{x-1}} \cdot \frac{r-2}{r+w-(x-1)}$

for $x = 3, 4, \ldots, w+3$

3.3 $P(X \leq 11) = \frac{457}{7776} \cong 0.05877$

3.5 $f(x) = \begin{cases} 1/6 & x=0 \\ 1/3 & x=1 \\ 1/3 & x=2 \\ 1/6 & x=3 \end{cases}$

3.7 $f(x) = \frac{5-x}{15}$ for $x = 0, 1, 2, 3, 4$

3.9 $f(x) = \begin{cases} 1/16 & x=0 \\ 7/16 & x=1 \\ 5/16 & x=2 \\ 1/8 & x=3 \\ 1/16 & x=4 \end{cases}$

3.11 $f(x) = \frac{6-x}{15} \qquad x = 1, 2, 3, 4, 5$

3.13 $P(X=5) = \frac{5}{36} \cong 0.1389$

3.15 $f(x) = (1-p)^{x-1}p + p^{x-1}(1-p)$

for $x = 2, 3, \ldots$

3.17 $f(x) = \begin{cases} 1/1024 & x=0 \\ 143/1024 & x=1 \\ 360/1024 & x=2 \\ 269/1024 & x=3 \\ 139/1024 & x=4 \\ 64/1024 & x=5 \\ 28/1024 & x=6 \\ 12/1024 & x=7 \\ 5/1024 & x=8 \\ 2/1024 & x=9 \\ 1/1024 & x=10 \end{cases}$

3.19 (a) $0 < a \leq 1$

(b) $c = \frac{3}{2(3a-a^3)}$

3.23 $g(2)$

3.25 $F(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$

3.27 $x_{0.64} = 0.8$

3.29 Use the cumulative distribution function technique:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(F_X(X) \leq y) \\ &= P(F_X^{-1}(F_X(X)) \leq F_X^{-1}(y)) \\ &= P(X \leq F_X^{-1}(y)) \\ &= F_X(F_X^{-1}(y))) \\ &= y \qquad 0 < y < 1 \end{aligned}$$

3.31 $\frac{7.25}{(0.05)^{4/5}} \cong 79.65$

3.33 $2^{-1/\theta}$

3.35 $f(x) = \frac{1}{n^m}\sum_{i=0}^{m-1}\binom{m}{m-i}(x-1)^i$

for $x = 1, 2, \ldots, n$, or, equivalently,

$f(x) = \left(\frac{x}{n}\right)^m - \left(\frac{x-1}{n}\right)^m$

for $x = 1, 2, \ldots, n$

3.37 $a = 1/10$, $b = \sqrt{99}/10 \cong 0.9950$

3.39 $f_Y(y) = \frac{1}{2}f_X\left(\frac{y}{2}\right) \qquad y > 0$

3.41 $x_{0.95} = 20^{1/\theta}$

3.43 $f_Y(y) = \frac{1}{6} \qquad y = \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \ldots, 1$

3.45 (a) $\mathcal{B} = \{y \,|\, 3 \leq y < 7\}$

(b) $y_{0.75} = 5$

3.47 $f(4) = 0.3$

3.49 Fix: $F_Y(y) = \frac{y^{3/2}+2y^{1/2}}{3} \qquad 0 \leq y < 1$

3.51 Fix: $f_Y(y) = \frac{e^{-y/2}}{\sqrt{2\pi y}} \qquad y > 0$

3.53 $\mu = E[X] = 2p-1$

$\sigma^2 = V[X] = 4p - 4p^2$

3.55 (a) ∞

(b) Yes

3.57 $\mu = E[X] = 1/2$

3.59 $E[1.1X] = 49.5$ and $V[1.1X] = 121$

3.61 (a) $c = 1/5$

(b) $E[X] = 26/15$

3.63 (a) $x_{0.95} = \sqrt{-\ln(0.05)}$

(b) $1/\sqrt{2}$

3.65 $E\left[X^3\right] = 2p - 1$

3.67 Use the definition of a moment generating function

3.69 (a) $f(x) = \dfrac{1}{(n-1)(x_{i+1} - x_i)}$ for $x_i < x < x_{i+1}$; $i = 1, 2, \ldots, n-1$

(b) $E[X] = \frac{x_1 + 2(x_2 + x_3 + \cdots + x_{n-1}) + x_n}{2(n-1)}$

3.71 (a) $E\left[X^n\right] = \begin{cases} 0 & n = 1, 3, 5, \ldots \\ 1 & n = 2, 4, 6, \ldots \end{cases}$

(b) $f_X(x) = 1/2 \qquad x = -1, 1$

3.73 $E[X] = 2$

3.75 $\frac{6^{\pi+1} - 2^{\pi+1}}{4(\pi+1)} \cong 99.7572$

3.77 $E[X] = 28/3$

3.79 $E[X] = 4$

3.81 $P(X = 7) = 0.7$

3.83 $-\frac{14}{495} \cong -0.02828$
(a loss of about 2.8 cents)

3.85 $41/2, 297/4$

3.87 2

3.89 (a) $\sqrt{2}$

(b) A Monte Carlo simulation in R for 10,001 replications:

```
t1 = runif(10001, 0, 2 * pi)
t2 = runif(10001, 0, 2 * pi)
x1 = cos(t1)
y1 = sin(t1)
x2 = cos(t2)
y2 = sin(t2)
chord = sqrt((x1 - x2) ^ 2 +
             (y1 - y2) ^ 2)
sort(chord)[5001]
```

3.91 $16\frac{2}{3}$ minutes

3.93 $E[X] = \frac{2\ln 2 - 4\ln 3 + 8\ln 5 - \ln 7}{\ln 10} \cong 3.4402$

3.95 $m = \begin{cases} 0 & 0 < p < 1 - \frac{1}{\sqrt[3]{2}} \\ 1 & 1 - \frac{1}{\sqrt[3]{2}} < p < \frac{1}{2} \\ 2 & \frac{1}{2} < p < \frac{1}{\sqrt[3]{2}} \\ 3 & \frac{1}{\sqrt[3]{2}} < p < 1 \end{cases}$

3.97 $E[X] = 3$

3.99 (a) $E[X] = 7$

(b) $E[X] = \frac{213}{31} \cong 6.8710$

3.101 (a) $E\left[|X - m|\right] = \frac{b-a}{4}$

(b) $E\left[|X - m|\right] = \begin{cases} 2p & 0 < p < 1 - \frac{1}{\sqrt{2}} \\ 2p^2 - 2p + 1 & 1 - \frac{1}{\sqrt{2}} < p < \frac{1}{\sqrt{2}} \\ 2 - 2p & \frac{1}{\sqrt{2}} < p < 1 \end{cases}$
and undefined for $p = 1 - \frac{1}{\sqrt{2}}$ and $p = \frac{1}{\sqrt{2}}$

3.103 $1717/18 \cong 95.3889$

3.105 $E[X] = \frac{\theta}{\theta+1}$

3.107 (a) $f(x) = \frac{\binom{80}{2}\binom{20}{x}}{\binom{100}{2+x}} \cdot \frac{78}{98-x}$
for $x = 0, 1, 2, \ldots, 20$

(b) Maple code:

```
total := 0;
for x from 0 to 20 do
  total := total + x * 80 * 79 *
      78 * 20! * (100 - x - 3)! *
      (x + 2)! / (x! * 2 * 100! *
      (20 - x)!):
od:
total;
```

which yields $E[X] = 20/27 \cong 0.7407$

3.109 $\frac{77}{2} = 38.5$ or \$38.50

3.111 (a) $\frac{r(n+m)}{m}$

(b) $E[X] = \sum_{x=r}^{n+r} x \cdot \frac{\binom{n}{x-r}\binom{m}{r-1}}{\binom{n+m}{x-1}} \cdot \frac{m-r+1}{n+m-x+1}$

3.113 $\frac{4(n+m-4)}{nm}$

3.115 (a) The (m, b) values on the line segment from $(-2, 2)$ to $(2, 1)$ in the (m, b) Cartesian coordinate system

(b) $\mu = \frac{m}{3} + \frac{b}{2}$, $\frac{1}{3} \le \mu \le \frac{2}{3}$

3.117 (a) $E[(X - \pi)^3] = -15\pi + 6\pi^2 - \pi^3 \cong -18.91$

(b) $V[17 - 4X] = 16$

(c) $V\left[X^2\right] = 5$

3.119 (a) $c = \frac{3}{2}$

(b) $F(x) = \begin{cases} 0 & x \le -1 \\ \frac{x^3}{2} + \frac{1}{2} & -1 < x < 1 \\ 1 & x \ge 1 \end{cases}$

(c) $\mu = 0, \sigma^2 = \frac{3}{5}$

(d) 0

(e) $\frac{8}{9} \cong 0.8889$

3.121 (a) $f(x) = \frac{5-x}{10} \qquad x = 1, 2, 3, 4$

(b) $E[X] = 2$

(c) $V[X] = 1$

3.123 (a) $f(x) = P(X = x) = P(X > x-1) - P(X > x)$, where $P(X > x)$ is $\sum_{i=1}^{n}(1-p_i)^x - \sum\sum_{i<j}(1-p_i-p_j)^x + \sum\sum\sum_{i<j<k}(1-p_i-p_j-p_k)^x - \cdots + (-1)^{n+1}(1-p_1-p_2-\cdots-p_n)^x$

(b) $E[X] = 14.7$

This is the expected number of rolls required to get all six outcomes in repeated rolls of a fair die

3.125 (a) $E[X] = 183$

(b) $E[1/X] \cong 0.01775$

3.127 (a) $f(x) = \frac{n!(x-1)}{n^x(n+1-x)!}$

for $x = 2, 3, \ldots, n+1$

(b) $E[X] = \sum_{x=2}^{n+1} x \cdot \frac{n!(x-1)}{n^x(n+1-x)!}$

(c) $E[X] = \frac{7{,}281{,}587}{1{,}562{,}500} \cong 4.6602$

3.129 $E[X] = 0.7, V[X] = 0.91$

3.131 (a) event

(b) undefined

(c) constant

(d) random variable

(e) constant

3.133 (a) $c = \frac{3}{4}$

(b) $E[X] = \frac{11}{16}$

3.135 $c = 1$

3.137 (a) Either stop when a 5 or 6 is rolled or stop when a 4, 5, or 6 is rolled

(b) Stop when a 6 is rolled

3.139 $E\left[\frac{1}{X}\right] = \frac{49}{120} \cong 0.4083$

3.141 $P(X \geq 4) = \frac{4}{27}, E[X]/4 = 1/4$

3.143 $P(4 < X < 12) = \frac{30{,}251}{32{,}768} \cong 0.9232$

$P(4 < X < 12) \geq \frac{3}{4}$

3.145 $P\left(X \geq \frac{3}{4}\right) = \frac{7}{16}, \frac{E[X]}{a} = \frac{8}{9}$

Chapter 4

4.1 $1 - 2p$

4.3 `pbinom(2, 9, 1 / 3)`

4.5 $P(\text{one or more fours}) = \frac{50700551}{60466176} \cong 0.8385$

4.7 $p^3 - 3p + 1 = 0$

4.9 $f(x) = 1 \qquad x = 0$

$f(x) = 1 \qquad x = n$

4.11 $f(x) = \begin{cases} 1/64 & x = 0 \\ 9/64 & x = 1 \\ 27/64 & x = 2 \\ 27/64 & x = 3 \end{cases}$

4.13 $f(x) = \binom{35}{x}\left(\frac{31}{365}\right)^x\left(\frac{334}{365}\right)^{35-x}$

for $x = 0, 1, \ldots, 35$

4.15 $\frac{41}{279{,}936} \cong 0.0001465$

4.17 $\sum_{x=3}^{5}\binom{5}{x}\left(\frac{244}{495}\right)^x\left(\frac{251}{495}\right)^{5-x} \cong 0.4867$

4.19 $P(\text{best team wins}) =$

$$\sum_{x=\frac{n+1}{2}}^{n}\binom{n}{x}[\max\{p, 1-p\}]^x[\min\{p, 1-p\}]^{n-x}$$

for $0 < p < 1$

4.21 $E\left[(X-c)^2\right] = c^2 - 3c + 3$

$$E\left[|X-c|\right] = \begin{cases} \frac{3-2c}{2} & c \leq 0 \\ \frac{6-3c}{4} & 0 < c \leq 1 \\ \frac{3}{4} & 1 < c \leq 2 \\ \frac{3c-3}{4} & 2 < c \leq 3 \\ \frac{2c-3}{2} & c > 3 \end{cases}$$

4.23 $P(X \leq 81) = \sum_{x=0}^{81}\binom{600}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{600-x}$ is calculated in R with

`pbinom(81, 600, 1 / 6)`

which returns approximately 0.0193. Assuming that the die is fair, the observed result (81 ones) or a more extreme result (fewer than 81 ones) will occur about only two times in 100.

4.25 $P(\text{Cubs win}) = \sum_{x=4}^{7}\binom{7}{x}p^x(1-p)^{7-x}$

$P(\text{Cubs win}) = \sum_{x=20}^{39}\binom{39}{x}p^x(1-p)^{39-x}$

4.27 (a) $P(\text{conviction}) =$
$r \cdot \sum_{x=n}^{m} \binom{m}{x}(1-p)^x p^{m-x} +$
$(1-r) \cdot \sum_{x=n}^{m} \binom{m}{x} q^x (1-q)^{m-x}$
(b) $P(\text{conviction}) \cong 0.5407$

4.29 (a) $f(x) = \binom{10}{x}\left(\frac{7}{8}\right)^x \left(\frac{1}{8}\right)^{10-x}$
for $x = 0, 1, 2, \ldots, 10$
(b) $E[X] = \frac{35}{4} = 8.75$
(c) The R code for a Monte Carlo experiment is

```
nrep = 10000
n = 10
count = 0
for (i in 1:nrep) {
  f1 = rbinom(n, 1, 1 / 2)
  f2 = rbinom(n, 1, 1 / 2)
  f3 = rbinom(n, 1, 1 / 2)
  f  = pmax(f1, f2, f3)
  nheads = sum(f)
  count = count + nheads
}
print(count / nrep)
```

4.31 $f_Y(y) = \frac{p(1-p)^y}{1-(1-p)^4} \qquad y = 0, 1, 2, 3$

4.33 3

4.35 12,476

4.37 $\frac{1{,}830{,}403{,}837{,}568}{9{,}906{,}146{,}353{,}125} \cong 0.1848$

4.39 $f(x) = \binom{x}{r-1} p^r (1-p)^{x-r+1}$
for $x = r-1, r, r+1, \ldots;\ 0 < p < 1;$
$r = 1, 2, \ldots$

4.41 (a) `pbinom(3, 8, 0.3)` yields 0.8059
(b) `pgeom(4, 1 / 3) - pgeom(1, 1 / 3)` yields 0.3128
(c) `dpois(3, 9)` yields 0.01499

4.43 $1 - \sum_{x=0}^{8} \frac{(147/31)^x e^{-147/31}}{x!} \cong 0.05254$

4.45 $\frac{13}{24e} \cong 0.1993$

4.47 $\displaystyle\sum_{y=0}^{23}\left[y \cdot \frac{18^y e^{-18}}{y!}\right] + 24\left[1 - \sum_{x=0}^{23} \frac{18^x e^{-18}}{x!}\right]$
$\cong 17.8183$

4.49 $\lambda = x$

4.51 $f_X(x) = \frac{(p\lambda)^x e^{-p\lambda}}{x!} \qquad x = 0, 1, 2, \ldots,$
that is, $X \sim \text{Poisson}(p\lambda)$

4.53 $E[X(X-1)] = \lambda^2$

4.55 $E\left[X^2\right] = 12$

4.57 $k = 20$

4.59 (a) $P(\text{passing}) = \frac{n(n-1)}{2} p^{n-2}(1-p)^2 + np^{n-1}(1-p) + p^n$
(b) $P(\text{passing}) = \frac{\binom{r}{n-2}\binom{100-r}{2}}{\binom{100}{n}} + \frac{\binom{r}{n-1}\binom{100-r}{1}}{\binom{100}{n}} + \frac{\binom{r}{n}\binom{100-r}{0}}{\binom{100}{n}}$ for $r = 2, 3, \ldots, 100$

4.61 (a) $1/9$
(b) $17/99$
(c) $E[Y] = 9k$

4.63 (a) true
(b) false
(c) true

Chapter 5

5.1 $13/10$

5.3 $F_Y(y) = \begin{cases} 0 & y \le 1 \\ 1 - 1/y & y > 1 \end{cases}$

5.5 $V\left[\sqrt{X}\right] = 1/18$

5.7 $\theta = 6$

5.9 (a) $3/4$
(b) $31\pi/3 \cong 32.4631$

5.11 (a) $x_0 = a/2$
(b) $\int_{x_0}^{\infty} f(x)\,dx = \frac{1}{2}$, indicating that the facility should be located at the *median* of the demand distribution
(c) The facility should be located at the *median* of the demand distribution, when the median exists

5.13 $f(x) = \frac{1}{2\sqrt{3}\sigma}$ for $\mu - \sqrt{3}\sigma < x < \mu + \sqrt{3}\sigma$

5.15 $2/3$

5.17 $1/5$

5.19 $f_Y(y) = \frac{1}{2\sqrt{3}} \qquad -\sqrt{3} < y < \sqrt{3}$

5.21 Use the cumulative distribution function technique

5.23 (a) $\displaystyle\sum_{y=0}^{10}\binom{30}{y}(0.3012)^y(0.6988)^{30-y}$

(b) `pbinom(10, 30, 1 - pexp(1200, 1 / 1000))` returns 0.7255

5.25 $E[X - x \mid X \geq x] = \frac{1}{\lambda}$

5.27 $\dfrac{1}{1 - \ln 2} \cong 3.2589$ years

5.29 $f_Y(y) = \frac{\lambda}{y^2} e^{-\lambda/y} \qquad y > 0$

5.31 (a) $f_Y(y) = \frac{1}{2\sqrt{3}} \qquad -\sqrt{3} < y < \sqrt{3}$

(b) $f_Y(y) = e^{-(y+1)} \qquad y > -1$

5.33 $F_Y(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-(y+1000)/3000} & 0 \leq y < 1000 \\ 1 - e^{-y/1500} & y \geq 1000 \end{cases}$

5.35 `pchisq(26.119, 14) - pchisq(5.629, 14)` returns 0.9499947

5.37 `qnorm(0.96)` returns 1.7507

5.39 3.8415

5.41 $\mu \cong 10.6$

5.43 $M_Y(t) = e^{(3\mu+4)t + \frac{9}{2}\sigma^2 t^2}$ for $-\infty < t < \infty$

5.45 (a) $P(X < 8) \cong 0.9772$

(b) $P(4 < X < 8) \cong 0.2297$

(c) $P(X > 10) \cong 0.003830$

(d) $a \cong 6.9346$

(e) $b \cong 6.9754$

(f) 0.007081

5.47 (a) 0.01475

(b) 73.2446 inches

5.49 Equate $f''(x)$ to zero and solve for x to find the values of the inflection points

5.51 (a) $P(4 \leq X \leq 6) = \frac{506660}{742729} \cong 0.6822$

(b) $P(4 \leq X \leq 6) = \frac{21}{32} = 0.65625$

(c) $P(4 \leq X \leq 6) = \frac{10625}{144} e^{-5} \cong 0.4972$

5.53 `qnorm(0.64, 50, 7)`

5.55 $\mu = 0.5551184$, $\sigma = 0.6595823$

5.57 $N(\mu, \sigma^2)$

5.59 $(\alpha - 1)/(\alpha + \beta - 2)$

5.61 A few seconds before 8:46 AM

5.63 \$45.26

5.65 (a) Cauchy

(b) Weibull

(c) beta

(d) Zipf

(e) triangular

5.67 (a) $E[X] = \beta/(\beta + 1)$

(b) $m = 2^{-1/\beta}$

5.69 Use the definition of the moment generating function

5.71 (a) $P(\mu - k\sigma < X < \mu + k\sigma) = P(-k < Z < k)$

(b) $P(\mu - k\sigma < X < \mu + k\sigma)$

$$= \begin{cases} e^{-(1-k)} - e^{-(1+k)} & 0 \leq k \leq 1 \\ 1 - e^{-(1+k)} & k > 1 \end{cases}$$

(c) $P(\mu - k\sigma < X < \mu + k\sigma)$

$$= \begin{cases} \displaystyle\sum_{x = \lfloor 5-2k \rfloor}^{\lceil 3+2k \rceil} \frac{4^x e^{-4}}{x!} & 0 < k \leq 2 \\ \displaystyle\sum_{x=0}^{\lceil 3+2k \rceil} \frac{4^x e^{-4}}{x!} & k > 2 \end{cases}$$

5.73 $P(a < X < b) = e^{-e^{-b}} - e^{-e^{-a}}$

Chapter 6

6.1 The pmf of $Y = (X_1 + X_2)/2$ is

$$f_Y(y) = \begin{cases} 1/9 & y = 1 \\ 2/9 & y = 1.5 \\ 3/9 & y = 2 \\ 2/9 & y = 2.5 \\ 1/9 & y = 3 \end{cases}$$

6.3 $P(2X + Y < 12) = 35/100$

6.5 $f_X(x) = 2\sqrt{1 - x^2}/\pi \qquad -1 < x < 1$

6.7 $P(a < X \leq b, c < Y \leq d) = F(b, d) - F(a, d) - F(b, c) + F(a, c)$

6.9 $\frac{2}{3} e^{-3} \cong 0.03319$

6.11 $f_Y(y) = \frac{\ln 10 - \ln(10-y)}{10} \qquad 0 < y < 10$

6.13 $P(X \leq Y) = \frac{6}{7} \cong 0.8571$

6.15 $$F(x,y)=\begin{cases} 0 & x \le 0 \text{ or } y \le 0 \\ 2xy & 0<x<1, \\ & 0<y<1, x+y<1 \\ 2y-y^2-(x-1)^2 & 0<x<1, \\ & 0<y<1, x+y \ge 1 \\ x(2-x) & 0<x<1, y \ge 1 \\ y(2-y) & 0<y<1, x \ge 1 \\ 1 & x \ge 1, y \ge 1 \end{cases}$$

6.17 (a) $f_{X_2}(x_2) = -\ln x_2 \qquad 0 < x_2 < 1$

(b) $E[X_2] = 1/4$

6.19 7/8, 3/4

6.21 (a) $f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{105}$
for $y_1 = 1, 2, \ldots, 14$;
$y_2 = 2, 3, \ldots, 15$; $y_1 < y_2$
$f_{Y_1}(y_1) = \frac{15-y_1}{105}$
for $y_1 = 1, 2, \ldots, 14$

(b) $$f_{Y_1,Y_2}(y_1,y_2)=\begin{cases} \frac{1}{225} & y_1 = y_2 = 1, 2, \ldots, 15 \\ \frac{2}{225} & y_1 = 1, 2, \ldots, 14; \\ & y_2 = 2, 3, \ldots, 15; \\ & y_1 < y_2 \end{cases}$$
$f_{Y_1}(y_1) = \frac{31-2y_1}{225}$ for $y_1 = 1, 2, \ldots, 15$

6.23 (a) $P(|X|+|Y|>1) = \frac{\pi-2}{\pi} \cong 0.3634$

(b) 11 regions

6.25 (a) $$P(X^2+Y^2 \le r^2) = \begin{cases} r^2 & 0 \le r < 1 \\ 1 & r \ge 1 \end{cases}$$

(b) $P(X+Y \le r) =$
$$\begin{cases} 0 & r < -\sqrt{2} \\ \int_{(r-\sqrt{2-r^2})/2}^{(r+\sqrt{2-r^2})/2} \int_{-\sqrt{1-x^2}}^{r-x} \frac{1}{\pi}\, dy\, dx & -\sqrt{2} \le r < -1 \\ \int_{-1}^{(r-\sqrt{2-r^2})/2} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy\, dx + & \\ \int_{(r-\sqrt{2-r^2})/2}^{(r+\sqrt{2-r^2})/2} \int_{-\sqrt{1-x^2}}^{r-x} \frac{1}{\pi}\, dy\, dx & -1 \le r < 1 \\ 1 - \int_{(r-\sqrt{2-r^2})/2}^{(r+\sqrt{2-r^2})/2} \int_{r-x}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy\, dx & 1 \le r < \sqrt{2} \\ 1 & r \ge \sqrt{2} \end{cases}$$

6.27 The median is $\sqrt{2/\pi} \cong 0.7979$

6.29 (a) $f_{X,Y}(x,y) = \frac{1}{10}$
for $x = 1, 2, 3, 4, 5$; $y = 2, 3, 4, 5$;
$x < y$. The joint pmf of X and Y is given below.

x \ y	1	2	3	4	5
1	0	1/10	1/10	1/10	1/10
2	0	0	1/10	1/10	1/10
3	0	0	0	1/10	1/10
4	0	0	0	0	1/10
5	0	0	0	0	0

(b) $f_Z(z) = (5-z)/10 \qquad z = 1, 2, 3, 4$

6.31 $F(1,1) = 7/12$

6.33 $(1/2, 1/2)$

6.35 $P(2<X<3) = 5e^{-4} - 7e^{-6} \cong 0.07423$

6.37 (a) $f_X(x) = \frac{3}{4}x(2-x)^2 \qquad 0 < x < 2$

(b) $P(2Y < X \,|\, X=1) = 1/4$

6.39 $P\left(X_1^2 + X_2^2 > 1\right) = 1 - \frac{\pi}{4} \cong 0.2146$

6.41 March 31

6.43 $P\left(X_1^2 + X_2^2 > 1\right) = 1/2$

6.45 $P\left(X_1^3 < X_2^4\right) = 3/7 \cong 0.4286$

6.47 (a) dependent random variables

(b) $c = 2/3$

(c) $f_X(x) = \frac{4x+1}{3} \qquad 0 < x < 1$

(d) $f_{Y\,|\,X=x}(y \,|\, X=x) = \frac{2x+y}{2x+1/2}$
for $0 < y < 1$ for any $0 < x < 1$

6.49 $P(Y=3) = 11/48 \cong 0.2292$

6.51 $$f_Y(y) = \begin{cases} p_0^2 & y=0 \\ 2p_0p_1 & y=1 \\ 2p_0p_2 + p_1^2 & y=2 \\ 2p_0p_3 + 2p_1p_2 & y=3 \\ 2p_0p_4 + 2p_1p_3 + p_2^2 & y=4 \\ 2p_1p_4 + 2p_2p_3 & y=5 \\ 2p_2p_4 + p_3^2 & y=6 \\ 2p_3p_4 & y=7 \\ p_4^2 & y=8 \end{cases}$$

6.53 $P(\min\{X,Y\}=3) = \frac{916{,}496}{1{,}953{,}125} \cong 0.4692$

6.55 (a) $f_X(x) = 2\lambda e^{-2\lambda x}$ for $x > 0$
$f_Y(y) = \lambda e^{-\lambda y}$ for $y > 0$

(b) $f_X(x) = (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)x}$
for $x > 0$
$f_Y(y) = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\left(e^{-\lambda_1 y} + e^{-\lambda_2 y}\right)$
for $y > 0$

(c) $f_X(x) = 2\kappa\lambda^\kappa x^{\kappa-1} e^{-2(\lambda x)^\kappa}$
for $x > 0$
$F_Y(y) = \kappa\lambda^\kappa \int_0^y t_1^{\kappa-1} e^{-(\lambda t_1)^\kappa}\left[1 - e^{-(\lambda(t_1+y))^\kappa}\right] dt_1 +$
$\kappa\lambda^\kappa \int_y^\infty t_1^{\kappa-1} e^{-(\lambda t_1)^\kappa}\left[e^{-(\lambda(t_1-y))^\kappa} - e^{-(\lambda(t_1+y))^\kappa}\right] dt_1$
for $y > 0$

6.57 $$f(x_1,x_2) = \begin{cases} 3/4 & x_1=0,\ x_2=1 \\ 1/4 & x_1=1,\ x_2=0 \end{cases}$$

6.59 $1 - \int_y^1 \int_{y/x_1}^1 f(x_1)f(x_2)\, dx_2\, dx_1 = y$
for $0 < y < 1$

6.61 $F_Z(z) = [F(z)]^2$ for z values on the support of X and Y

6.63 (a) $P(X=Y) = \frac{p}{2-p}$ for $0 < p < 1$

(b) $P(Y \bmod X = 0) = p^2 \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}}{1-(1-p)^{n+1}}$ for $0 < p < 1$

6.65 No

6.67 $\text{Cov}(X, Y) = p_1 p_4 - p_2 p_3$

6.69 $\text{Cov}(X, Y) = 2p(1-p)$. This is intuitive because large values of X tend to be with large values of Y and vice-versa.

6.71 $\text{Cov}(X, Y) = 0.02$

6.73 (a) 6

(b) 2/3

6.75 $\rho = -1$

6.77 $V[X+Y] = 3$

6.79 Statement (a) is true because the relationship is stronger between X_1 and X_2 than that between X_3 and X_4 because there are just two other suits to dilute the correlation rather than 11 other ranks to dilute the population correlation.

6.81 $l = \pi d/4 \cong 0.7854d$

6.83 $E[E[Y\,|\,X]] = p_2 + p_4$

6.85 $\rho = -1$

6.87 $\text{Cov}(X,Y) = -\frac{49}{272} \cong -0.1801$

6.89 $\rho = 1$

6.91 1

6.93 (a) Yes

(b) No

(c) $f_{X_1}(x_1) = \log_{10}\left(1 + \frac{1}{x_1}\right)$ for $x_1 = 0, 1, 2, \ldots, 9$

(d) $f_{X_2}(x_2) = \log_{10}\left[\left(\frac{11+x_2}{10+x_2}\right)\left(\frac{21+x_2}{20+x_2}\right)\cdots\left(\frac{91+x_2}{90+x_2}\right)\right]$ for $x_2 = 0, 1, 2, \ldots, 9$

(e) $E[X_2] = \sum_{x_2=0}^{9} x_2 \log_{10}\left[\prod_{x_1=1}^{9}\left(1 + \frac{1}{10x_1 + x_2}\right)\right] \cong 4.1874$

6.95 $V[E[Y\,|\,X]] = 0$

6.97 $V[E[V[E[X\,|\,Y]]]] = 0$

6.101 $\text{Cov}(X, Y) = p_4 - (p_2 + p_4)(p_3 + p_4)$

6.103 $E[\text{number of runs}] = 1 + 2(n-1)p(1-p)$

6.105 $\text{Cov}(X_1, X_2) = -mqr$

6.107 (a) $f_{X,Y}(x, y) = \frac{\binom{1}{x}\binom{4}{y}\binom{5}{3-x-y}}{\binom{10}{3}}$ for $x = 0, 1;\ y = 0, 1, 2, 3;\ x+y \le 3$

(b) $P(X=1, Y=1) = \frac{1}{6}$

(c) $E[X] = \frac{3}{10}$

6.109 $E[\,|X-Y|\,] = \frac{231}{442} \cong 0.5226$

6.111 (a) $\rho < 0$

(b) $\rho < 0$

(c) $\rho = 0$

(d) $\rho > 0$

(e) $\rho > 0$

6.113 (a) $k = 10$

(b) $\frac{100}{441} \le \frac{25}{98}$

6.115 (a) $f_Y(y) = \frac{y}{\sqrt{25-y^2}} \qquad 3 < y < 4$

(b) $\frac{25}{2}\arcsin\left(\frac{7}{25}\right) \cong 3.5474$

(c) $\frac{37}{3} - \frac{175}{4} \cdot \arcsin\frac{7}{25} \cong -0.08266$

(d)
```
nrep = 1000000
x = runif(nrep, 3, 4)
y = sqrt(25 - x ^ 2)
mean(y)
mean(x * y)
var(x, y)
```

6.117 (a) No

(b) $f_X(x) = \begin{cases} x+1 & -1 < x < 0 \\ 1-x & 0 \le x < 1 \end{cases}$

(c) $E[Y\,|\,X=x] = 0$ for $-1 < x < 1$

6.119 (a) $c = \frac{1}{9\pi}$

(b) Dependent

6.121 (a) $P\left(X^2 < Y^5\right) = 2/7$

(b) $E\left[X^2 Y^5\right] = 1/18$

(c) $E\left[\min\{X, Y\}\right] = 1/3$

6.123 $\text{Cov}(X, 1/X) = -\frac{2}{9}$

6.125 $f(x, y) =$
$$\begin{cases} \frac{1}{12}\left(\frac{1}{3}\right)^{y-1}\left(\frac{1}{2}\right)^{x-y-1} & x = 1, 2, \ldots;\ y = 1, 2, \ldots;\ x > y \\ \frac{1}{12}\left(\frac{1}{3}\right)^{x-1}\left(\frac{5}{6}\right)^{y-x-1} & x = 1, 2, \ldots;\ y = 1, 2, \ldots;\ x < y \end{cases}$$
$\rho = \text{Corr}(X, Y) = -\frac{3}{2\sqrt{60}} \cong -0.1936$

6.127 (a) $E[X] = 2/13 \cong 0.1538$
(b) $E[Y \mid X = 2] = 1/2$
(c) $E[V[E[Y \mid X]]] = 0$

6.129 $P(X > 0) = 2/3$

6.131 (a) $f(x, y) = \left(\frac{13}{15}\right)^{\min\{x,y\}-1}\left(\frac{1}{15}\right) \times \left(\frac{14}{15}\right)^{\max\{x,y\}-\min\{x,y\}-1}\left(\frac{1}{15}\right)$
$x = 1, 2, \ldots;\ y = 1, 2, \ldots;\ x \neq y$
(b) $E[Y \mid X = 2] = 111/7 \cong 15.8571$

6.133 -0.125

6.135 Counterexamples:

$$f(x, y) = \frac{1}{\sqrt{24}} \qquad x > 0, y > 0, x + y = 1$$

or

$$f(x, y) = \begin{cases} 1/2 & x = 0, y = 2 \\ 1/2 & x = 2, y = 0 \end{cases}$$

6.137 $\mathbf{\Sigma} = \begin{bmatrix} \sigma_X^2 & \sigma_X\sigma_Y\rho \\ \sigma_X\sigma_Y\rho & \sigma_Y^2 \end{bmatrix}$

6.139 (a) true
(b) false
(c) true
(d) true

6.141 $c = -\rho\sigma_X/\sigma_Y$

6.143 (a) Find the smallest n such that $P\left(\left|\bar{X} - \frac{1}{6}\right| < 0.005\right) > 0.9$
(b)
```
n = 100
while(pbinom(0.005 * n +
      n / 6, n, 1 / 6) -
      pbinom(-0.005 * n +
      n / 6, n, 1 / 6) <
      0.9) n = n + 1
print(n)
```
The die should be cast 14,901 times.

6.145 (a) 3.3
(b) 0.1525
(c) 0.18

6.147 Yes

6.149 0.006394

6.151 $E[\bar{X}] = 8.1$ and $V[\bar{X}] = \frac{22}{9} \cong 2.4444$

6.153 (a) $P(X_i > 6) = e^{-0.3^{2.5}} \cong 0.9519$
(b) $1000 \cdot e^{-0.3^{2.5}} \cong 951.9$
(c) $x_{0.25} = 20\left[-\ln(3/4)\right]^{2/5} \cong 12.1505$

6.155 $E[X + \cos Y \mid Z = z] =$
$$\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} (x + \cos y) \times \frac{f(x,y,z)}{\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dx}\,dy\,dx$$
for $-1 < z < 1$

6.157 $P\left(\bar{X} = 1\right) = p^4$

6.159 $V\left[X_1X_2^2X_3^3X_4^4\right] = p^4(1 - p^4)$

6.161 $M_Y(t) = 1 - p^4 + p^4e^t \qquad -\infty < t < \infty$

6.163 $M_{\bar{X}}(t) = \left(1 - p + pe^{t/4}\right)^4 \qquad -\infty < t < \infty$

6.165 (a) $P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$
(b) $P(X_1 < X_2 < X_3) = \frac{\lambda_1\lambda_2}{(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)}$

6.167 Expected loss is \$1.41

6.169 binomial$(n + m,\ p)$

6.171 chi-square(n)

6.173 $f_Y(y) = 10y^9 \qquad 0 < y < 1$

6.175 The joint probability distribution of X_2 and X_3 is bivariate normal with population mean vector and variance–covariance matrix
$$\begin{bmatrix} 8 & 3 \\ 3 & 9 \end{bmatrix}$$

6.177 The APPL statements

```
X := UniformDiscreteRV(1, 6);
Y := ConvolutionIID(X, 5);
PDF(Y, 17);
```

return $65/648 \cong 0.1003$. R code for the Monte Carlo simulation:

```
rollfive = function(n) {
  ceiling(runif(n, 0, 6)) +
  ceiling(runif(n, 0, 6)) +
  ceiling(runif(n, 0, 6)) +
  ceiling(runif(n, 0, 6)) +
  ceiling(runif(n, 0, 6))
}
sum(rollfive(10000) == 17) / 10000
```

6.179 $E[Y] = 9, V[Y] = 6$

```
nrep = 100000
y = rep(0, nrep)
for (i in 1:nrep) {
  x = sample(5, 3, replace = TRUE)
  y[i] = sum(x)
}
print(mean(y))
print(var(y))
```

6.181 $E[Y] = 9, V[Y] = 3$

```
nrep = 100000
y = rep(0, nrep)
for (i in 1:nrep) {
  x = sample(5, 3)
  y[i] = sum(x)
}
print(mean(y))
print(var(y))
```

6.183 $\frac{10}{81} \cong 0.1235$

6.185 (a) $f_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{1}{6}$
$0 < x_1 < 1, 0 < x_2 < 2, 0 < x_3 < 3$

(b) $f_{X_{(3)}}(x) = \begin{cases} x^2/2 & 0 < x \le 1 \\ x/3 & 1 < x \le 2 \\ 1/3 & 2 < x \le 3 \end{cases}$

6.187 $0.5 \le P(X_1 < X_2 < X_3) \le 0.7$

6.189 $P\left(X_{(1)} < m < X_{(4)}\right) = 7/8$

6.191
```
nrep = 1000000
x = matrix(runif(3 * nrep, 0, 10),
           nrep, 3)
y = apply(x, 1, max)
z = which(y < 8)
mean(x[z, 1])
```

6.193 $f(x) = \frac{1}{216} \qquad x = abc,$

where the hundreds digit a, the tens digit b, and the ones digit c, can each take on the values $1, 2, \ldots, 6$

6.195 $f(\mu_1, \mu_2, \ldots, \mu_n) = (2\pi)^{-n/2}$

Chapter 7

7.1

fcn?	1–1?	inv?	$g(9)$	$g^{-1}(9)$
yes	yes	yes	23	13/3
yes	no	no	81	$\{-3, 3\}$
no	no	yes	$\{-3, 3\}$	81

7.3 $f_Z(z) = \frac{\lambda_1\lambda_2\left(e^{-\lambda_2 z} - e^{-\lambda_1 z}\right)}{\lambda_1 - \lambda_2} \qquad z > 0$

7.5 $F_Z(z) = \begin{cases} 0 & z \le -\frac{1}{\sqrt{2}} \\ \dfrac{\pi + 4z^2 + 4z\sqrt{1-z^2} + 4\arcsin z}{4\pi} & -\frac{1}{\sqrt{2}} < z < \frac{1}{\sqrt{2}} \\ \dfrac{2z\sqrt{1-z^2} + \arcsin z}{\pi} & \frac{1}{\sqrt{2}} < z < 1 \\ 1 & z > 1 \end{cases}$

7.7 $\frac{65}{24} + \frac{1}{8\sqrt{5}} + \ln(2) - \ln\left(\sqrt{5} + 3\right) \cong 2.0254$

7.9 $f(y) = \frac{1}{n} y^{1/n-1} \qquad 0 < y < 1$

7.11
```
X := ExponentialRV(1);
Y := ExponentialRV(2);
V := Difference(X, Y);
g := [[x -> -x, x -> x],
      [-infinity, 0, infinity]];
W := Transform(V, g);
Variance(W);
```
$V\left[|X - Y|\right] = \frac{29}{36} \cong 0.8056$

7.13 $f_Y(y) = \begin{cases} 2y^3 & 0 < y < 1 \\ y/2 & \sqrt{2} < y < 2 \end{cases}$

7.15 \$864.66

7.17 $f_X(x) = \frac{1}{2} e^{-x/2} \qquad x > 0$

7.19 $f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2} \qquad y > 0$

7.21 (a) $f_Y(y) = e^{-1/y}/y^2 \qquad y > 0$

(b) $f_Y(y) = e^{-1/y}/y^2 \qquad y > 0$

7.23 (a) The probability mass function of $Z = X + Y$ is

$$f_Z(z) = \begin{cases} (1-p)e^{-\lambda} & z = 0 \\ \lambda^{z-1} e^{-\lambda}\left[\frac{(1-p)\lambda}{z!} + \frac{p}{(z-1)!}\right] & z = 1, 2, \ldots \end{cases}$$

(b) The probability mass function of $Z = X \cdot Y$ is

$$f_Z(z) = \begin{cases} (1-p) + pe^{-\lambda} & z = 0 \\ p\lambda^z e^{-\lambda}/z! & z = 1, 2, \ldots \end{cases}$$

7.25 (a) The probability mass function of $Y = X_1 + X_2 + X_3$ is

$$f_Y(y) = \begin{cases} (1-p)^3 & y = -3 \\ 3p(1-p)^2 & y = -1 \\ 3p^2(1-p) & y = 1 \\ p^3 & y = 3 \end{cases}$$

(b) The probability mass function of $Y = X_1/X_2$ is

$$f_Y(y) = \begin{cases} 2p(1-p) & y = -1 \\ p^2 + (1-p)^2 & y = 1 \end{cases}$$

7.27 The probability density function of $Y_1 = X_1^{X_2}$ is

$f_{Y_1}(y_1) = \int_0^\infty f_{X_1,X_2}\left(y_1^{1/y_2}, y_2\right) \frac{y_1^{1/y_2}}{y_1 y_2}\, dy_2$
for $y_1 > 0$

7.29 (a) The probability density function of $Z = X/Y$ is
$f_Z(z) = \frac{1}{z^2}\int_0^\infty x f_X(x) f_Y(x/z)\, dx$
for $z > 0$

(b) The probability density function of $Z = X/Y$ is
$f_Z(z) = \frac{1}{\pi\left(z^{3/2}+z^{1/2}\right)}$
for $z > 0$

7.31 (a) The probability density function of $Z = XY$ is

$$f_Z(z) = \int_0^\infty \frac{1}{x} f_X(x) f_Y(z/x)\, dx \qquad z > 0$$

(b) The probability density function of $Z = XY$ is

$$f_Z(z) = \frac{1}{2z\sqrt{\pi}} e^{-(\ln z)^2/4} \qquad z > 0$$

7.33 Check these:

(a) $P(\text{middle stick is the longest}) =$
$\int_0^{1/3} \int_{(1+y_1)/2}^1 [f(y_1, y_2) + f(y_2, y_1)]\, dy_2\, dy_1 +$
$\int_{1/3}^{1/2} \int_{2y_1}^1 [f(y_1, y_2) + f(y_2, y_1)]\, dy_2\, dy_1$

(b) $E(Y_2 - Y_1) =$
$\int_0^1 \int_{y_1}^1 (y_2 - y_1)[f(y_1, y_2) + f(y_2, y_1)]\, dy_2\, dy_1$

(c) $E[Y_1(1-Y_2)] =$
$\int_0^1 \int_{y_1}^1 y_1(1-y_2)[f(y_1, y_2) + f(y_2, y_1)]\, dy_2\, dy_1$

7.35 (a) $f\left(x_{(1)}, x_{(2)}\right) = 1/8$
for $0 < x_{(1)} < x_{(2)} < 4$

(b) $F_{X_{(1)},X_{(2)}}(1, 3) = 5/16$

7.37 $1 - n/2^{n-1}$

7.39 $f_{Y_1}(y_1) = \int_0^\infty f_{X_1,X_2}(y_1 y_2, y_2) y_2\, dy_2$
for $y_1 > 0$

7.41 (a) $f_Y(y) = 1/4 \qquad 0 < y < 4$

(b) $f_Y(y) = 1/8 \qquad y = 1, 2, \ldots, 8$

(c) $f_Y(y) = 1 \qquad y = 1$

(d) $f_Y(y) = \begin{cases} 1/4 & 0 < y < 2 \\ 1/8 & 2 \le y < 6 \end{cases}$

(e) $f_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & 1 < y < 2 \\ 1/8 & 2 \le y < 6 \end{cases}$

7.43 $f_Y(y) = \begin{cases} 1/9 & y = -8 \\ 1/9 & y = -5 \\ 1/9 & y = -3 \\ 3/9 & y = 0 \\ 1/9 & y = 3 \\ 1/9 & y = 5 \\ 1/9 & y = 8 \end{cases}$

7.45 (a)

```
p1 := 1 / 4;
U1 := UniformRV(0, 1);
U2 := UniformRV(0, 1);
T1 := Difference(U1, U2);
g1 := [[x -> -x, x -> x],
       [-1, 0, 1]];
D1 := Transform(T1, g1);

p2 := 1 / 2;
g2 := [[x -> x * x], [0, 1]];
T2 := Transform(U1, g2);
T3 := Convolution(T2, T2);
g3 := [[x -> sqrt(x)], [0, 2]];
D2 := Transform(T3, g3);

p3 := 1 / 4;
g4 := [[x -> 1 + x * x,
        x -> 1 + x * x],
       [-1, 0, 1]];
T4 := Transform(T1, g4);
g5 := [[x -> sqrt(x)], [1, 2]];
```

```
D3 := Transform(T4, g5);

DD := Mixture([p1, p2, p3],
              [D1, D2, D3]);
Mean(DD);
Variance(DD);
```

$$f_D(x) = \begin{cases} \frac{\pi x - 2x + 2}{4} & 0 < x < 1 \\ \frac{x}{2}\left(\frac{1}{\sqrt{x^2-1}} - \arcsin\left(1 - \frac{2}{x^2}\right) - 1\right) & 1 \le x < \sqrt{2} \end{cases}$$

$E[D] = \frac{3+\sqrt{2}+\ln(29\sqrt{2}+41)}{12} \cong 0.7350901248$

$V[D] = \frac{85}{144} - \frac{\sqrt{2}}{24} - \frac{(\sqrt{2}+3)\ln(29\sqrt{2}+41)}{72} - \frac{(\ln(29\sqrt{2}+41))^2}{144} \cong 0.1263091751$

(b) Monte Carlo simulation code in R

```
nrep = 100000
d = rep(nrep, 0)

for (i in 1:nrep) {
  side = sample(4, 1)
  if (side == 1) point1 = c(runif(1), 0)
  if (side == 2) point1 = c(1, runif(1))
  if (side == 3) point1 = c(runif(1), 1)
  if (side == 4) point1 = c(0, runif(1))

  side = sample(4, 1)
  if (side == 1) point2 = c(runif(1), 0)
  if (side == 2) point2 = c(1, runif(1))
  if (side == 3) point2 = c(runif(1), 1)
  if (side == 4) point2 = c(0, runif(1))
  d[i] = sqrt((point1[1] - point2[1]) ^ 2 +
              (point1[2] - point2[2]) ^ 2)
}
print(mean(d))
print(var(d))
```

7.47 $2 + \sqrt{2} \cong 3.4142$

7.49 $f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\pi(b-a)\sqrt{y_1^2+y_2^2}}$

for $a^2 < y_1^2 + y_2^2 < b^2$

7.51 $f_Z(z) = \begin{cases} 3z^2/2 & 0 \le z < 1 \\ 2 - z & 1 \le z < 2 \end{cases}$

7.53 $f(y_1, y_2) = \frac{1}{4}(y_1 + y_2)$

for $0 < y_1 < y_2 < 2$

$f_{Y_1}(y_1) = \frac{1}{2} + \frac{1}{2}y_1 - \frac{3}{8}y_1^2$ for $0 < y_1 < 2$

$f_{Y_2}(y_2) = \frac{3}{8}y_2^2$ for $0 < y_2 < 2$

7.55 $f_Y(y) = \dfrac{\theta y^{\theta/2-1}}{2} \qquad 0 < y < 1$

7.57 $f_Y(y) = \dfrac{1}{32} \qquad 0 < y < 32$

7.59 $f_{Y_1}(y_1) = \dfrac{\lambda_1\lambda_2}{(\lambda_1 y_1 + \lambda_2)^2} \qquad y_1 > 0$

7.61 $f_Y(y) = \begin{cases} 1 & 0 < y < 1/2 \\ 1/(4y^2) & y \ge 1/2 \end{cases}$

7.63 $f_Y(y) = \dfrac{1}{\pi\sqrt{1-y^2}} \qquad -1 < y < 1$

7.65 $f_{Y_1,Y_2}(y_1, y_2) = \lambda_1\lambda_2 e^{-\lambda_1(y_1-y_2)-\lambda_2 y_2}$

for $0 < y_2 < y_1$

7.67 The APPL code

```
R1 := ExponentialRV(100 / 6);
R2 := ExponentialRV(100 / 7);
R3 := ExponentialRV(100 / 8);
T1 := Convolution(R1, R2, R3);
g1 := [[x -> 1000000 * exp(x)],
       [0, infinity]];
A  := Transform(T1, g1);
Mean(A);
```

returns
$$\frac{125000000000}{100533}$$
or $1,243,372.82

7.69 The APPL statements

```
X := UniformRV(0, 1);
Y := ConvolutionIID(X, 4);
IDF(Y, 99 / 100) / 4;
```

return 0.8250182245

7.71 The APPL statements

```
A := [[1 / 6, 1 / 6, 1 / 6,
       1 / 6, 1 / 6, 1 / 6],
      [4, 8, 12, 16, 20, 24],
      ["Discrete", "PDF"]];
C := UniformDiscreteRV(1, 6);
R := [[1 / 6, 1 / 6, 1 / 6,
       1 / 6, 1 / 6, 1 / 6],
      [1, 4, 9, 16, 25, 36],
      ["Discrete", "PDF"]];
S := Product(A, C);
U := Difference(R, S);
V := [[CDF(U, -1), PDF(U, 0),
       SF(U, 1)], [0, 1, 2],
      ["Discrete", "PDF"]];
Mean(V);
```

return 3/8

7.73 $E\left[\dfrac{X_1/m}{X_2/n}\right] = \dfrac{n}{n-2}$ for $n > 2$

7.75 The APPL statements

```
A := UniformRV(0, 1);
B := UniformRV(0, 1);
g := [[x -> 1 / x ^ 2], [0, 1]];
R := Convolution(Transform(A, g),
                 Transform(B, g));
h := [[x -> sqrt(x)],
      [2, infinity]];
R := Transform(R, h);
Variance(R);
```

return an unevaluated integral

7.77 (a) The APPL statements

```
X := BinomialRV(5, 7 / 10);
Y := BinomialRV(4, 8 / 10);
Z := Product(X, Y);
Mean(Z);
```

return $\frac{56}{5}$ or \$11.20

(b) The Monte Carlo simulation code in R is

```
x = rbinom(10000, 5, 7 / 10)
y = rbinom(10000, 4, 8 / 10)
z = x * y
mean(z)
```

7.79 $P(0.052 < X < 9.55) \cong 0.9397$

7.81 0

7.83 $c \cong 31.8205$

7.85 $f_{X_{(2)}}(x) = 20\left(x^{-5} - x^{-6}\right) \qquad x > 1$

7.87 The APPL statements

```
X := NormalRV(2, 1);
Y := Truncate(X, 0, infinity);
Z := OrderStat(Y, 10, 10);
Mean(Z);
evalf(%);
```

return the mean as approximately 3.5511

7.89 $P(X_{(1)} < 1/3) = \dfrac{11}{27} \cong 0.4074$

7.91 The APPL statements

```
X  := WeibullRV(lambda, kappa);
Y3 := OrderStat(X, 10, 3);
CDF(Y3, 2);
```

give the probability

$P(X_{(3)} < 2) =$
$1 + 80e^{-9(2\lambda)^{\kappa}} - 36e^{-10(2\lambda)^{\kappa}} - 45e^{-8(2\lambda)^{\kappa}}$

for $\lambda > 0$ and $\kappa > 0$

7.93 (a) $\rho = 1/2$

(b) The R code for the Monte Carlo simulation is

```
n  = 500000
x1 = runif(n)
x2 = runif(n)
y1 = pmin(x1, x2)
y2 = pmax(x1, x2)
cor(y1, y2)
```

7.95 $2^{-1/n}$

7.97 The *approximate* shape of the probability distribution for the date of Easter Sunday can be found by taking the sum of independent $U(0, 29.53)$ and $U(0, 7)$ random variables, where 29.53 days is the length of a lunar cycle and 7 days is the length of a week.

7.99 The APPL statements

```
X := LogisticRV(1, 1);
Y := ConvolutionIID(X, 3);
Variance(Y);
```

return $V(Y) = \pi^2$

7.101 $P(X_1 + X_2 + \cdots + X_5 = 6) = \frac{128}{625} = 0.2048$

7.103 $Y = X_1 + X_2 + \cdots + X_n$
$f_Y(y) = \binom{y+nr-1}{nr-1} p^{nr}(1-p)^y$
for $y = 0, 1, 2, \ldots$

7.105 (a) 2.3263

(b) $2.3263n$

(c) $c = 2.3263\sqrt{n}$

7.107 (a) 0.5770

(b) 0.6421

(c) The R statements for part (b) are

```
n = 5
p = pnorm(0.07 / sqrt(0.13))
1 - pbinom(2, n, p)
```

7.109 $F(n_2, n_1)$

7.111 (a) The APPL statements below confirm that $f(x)$ is a legitimate probability density function

```
X := [[x -> x, x -> 1 / 4],
      [0, 1, 3],
      ["Continuous", "PDF"]];
VerifyPDF(X);
```

(b) The additional APPL statement

```
ExpectedValue(X, x -> cos(x));
```

returns
$\cos(1) + \frac{3}{4}\sin(1) - 1 + \frac{1}{4}\sin(3)$
or approximately 0.2067

(c) The additional APPL statements

```
Y := ConvolutionIID(X, 4);
IDF(Y, 95 / 100);
```

return 8.1087

(d) The additional APPL statements

```
U := UniformRV(0, 2);
Z := Product(X, U);
IDF(Z, 1 / 2);
```

return 0.9685

7.113 *F*

7.115 (a) $X \sim \text{Erlang}(n/10, k+1)$

(b) The APPL code

```
Z := ErlangRV(9 / 10, 8);
IDF(Z, 0.25);
IDF(Z, 0.50);
IDF(Z, 0.75);
```

returns the quartiles of the distribution as
$x_{0.25} \cong 6.6179$
$x_{0.50} \cong 8.5214$
$x_{0.75} \cong 10.7605$

7.117 (a) One inch and 0.000005 inches squared

(b) 1.0052 inches

7.119 1

7.121 $6p^2$

7.123 $X_1 \sim \text{binomial}(n, \lambda_1/(\lambda_1 + \lambda_2))$

7.125 9

7.127 0.1855

Chapter 8

8.1 (a) $f_{X_n}(x) = \binom{3n}{x}\left(\frac{1}{n}\right)^x\left(1 - \frac{1}{n}\right)^{3n-x}$
for $x = 0, 1, 2, \ldots, 3n$

(b) $f_X(x) = \frac{3^x e^{-3}}{x!}$ for $x = 0, 1, 2, \ldots$

8.3 (a) $M_{\bar{X}_n}(t) = \left[\frac{1}{1-\theta t/n}\right]^n$ for $t < n/\theta$

(b) $\lim_{n\to\infty} M_{\bar{X}_n}(t) = e^{\theta t}$ which is degenerate at θ

(c) $M_{\sqrt{n}(\bar{X}_n-\theta)/\theta}(t) = e^{-\sqrt{n}t}\left(1 - \frac{t}{\sqrt{n}}\right)^{-n}$
for $t < \sqrt{n}$

(d) $\lim_{n\to\infty} M_{\sqrt{n}(\bar{X}_n-\theta)/\theta}(t) = e^{t^2/2}$
for $-\infty < t < \infty$

8.5 (a) 2.14

(b)
```
nrep = 10000
ntrials = 5
x = rep(0, nrep)
p = rep(0, ntrials)
for (i in 1:ntrials) {
  for (j in 1:nrep) x[j] =
                    sum(runif(3))
  p[i] = sort(x)[0.90 * nrep]
}
t.test(p)
```
Five runs average to $\hat{x}_{0.90} = 2.16$

(c) $3 - \frac{\sqrt[3]{75}}{5} \cong 2.1566$

(d) $3 - \frac{\sqrt[3]{75}}{5} \cong 2.1566$

(e) The APPL code

```
X := UniformRV(0, 1);
Y := ConvolutionIID(X, 3);
IDF(Y, 0.90);
```

returns 2.156567334

8.7 (a) $P\left(-1 < \sum_{i=1}^{12} U_i < 1\right) \cong 0.6826$

(b) The APPL code

```
X := UniformRV(-1 / 2, 1 / 2);
Y := ConvolutionIID(X, 12);
CDF(Y, 1) - CDF(Y, -1);
```

returns $\frac{27085381}{39916800} \cong 0.6785$

8.9 $X_n \xrightarrow{D} 0$

8.11 (a) $P(\bar{X} > 1.6) \cong 0.03682$

(b) The APPL code

```
X := [[x -> x / 2], [0, 2],
      ["Continuous", "PDF"]];
Mean(X);
Variance(X);
Y := ConvolutionIID(X, 10);
g := [[x -> x / 10], [0, 20]];
Z := Transform(Y, g);
1 - CDF(Z, 8 / 5);
```

returns
$P(\bar{X} > 1.6) = \frac{36,411,338,728,543}{1,187,940,433,680,000} \cong$
0.03065

(c) Monte Carlo simulation code

```
nrep = 1000000
count = 0
for (i in 1:nrep) {
  if (mean(2 * sqrt(runif(10))) > 1.6)
    count = count + 1
}
print(count / nrep)
```

generates output that hover about the analytical solution

8.13 $P\left(\sum_{i=1}^{50} X_i < 60\right) = 0.1562$

8.15 (a) $P(X > 90) \cong 0.1459$

(b) $P(X > 90) \cong 0.1456$

8.17 0.2236