

Answers to selected exercises

Chapter 1

- 1.1** Answers will vary. Example: How many sequences of “heads” and “tails” are possible in 32 flips of a coin?
- 1.3** (a) 1296
(b) 126
- 1.5** 120
- 1.7** (a) 676,000
(b) 626,000
- 1.9** (a) 720
(b) 240
(c) 480
(d) 72
(e) 144
(f) 96
(g) 48
- 1.11** 7200
- 1.13** (a) 350
(b) 230
(c) 14,850
- 1.15** 4200
- 1.17** 5040
- 1.19** Use the binomial theorem with $x = 1$ and $y = -1$
- 1.21** Use the binomial theorem with $x = 1$ and $y = 1$
- 1.23** Consider the two cases where the first six and last six digits are identical and non-identical
- 1.25** (a) 55
(b) 28

(c) 10

1.27 $\binom{n_1+n_2-1}{n_2-1} = \binom{n_1+n_2-1}{n_1}$

1.29 60

1.31 324,000

1.33 2730

1.35 28

1.37 $A = \{1, 2, 4, 7, 8, 11, 13, 14\}$

1.39 $A \cap B \neq \emptyset, B \cap C \neq \emptyset, A \cap C = \emptyset$

1.43 $(A \cup B)'$ or $A' \cap B'$

Chapter 2

2.1 (a) 81

(b) 3, 24, 18, 36

2.3 0.7

2.5 Use probability axioms and associated results

2.9 $\frac{4}{15} \leq P(A \cap B) \leq \frac{3}{5}$

2.11 $P(B) = \frac{5}{12}$

2.13 $\frac{840}{4199} \cong 0.20005$

2.15 $\frac{1}{10,000}$

2.17 $\frac{48}{95} \cong 0.5053$

2.19 $\frac{155}{323} \cong 0.4799$

2.21 $\frac{5}{54} \cong 0.09259$

2.23 $\frac{1}{81} \cong 0.01235$

2.25 $\frac{2197}{8330} \cong 0.2637$

2.27 $\frac{25}{108} \cong 0.2315$

2.29 (a) $\frac{n-1}{2n}$ for $n = 3, 4, \dots$

- (b) $P(\text{sum} \leq n) = \begin{cases} \frac{n-1}{2n} & n \text{ odd} \\ \frac{n-2}{2(n-1)} & n \text{ even} \end{cases}$
- (c) $\lim_{n \rightarrow \infty} P(\text{sum} \leq n) = \frac{1}{2}$
- 2.31** $\frac{29}{52} \cong 0.5577$
- 2.33** $P(\text{all faces are different}) = \frac{6!}{(6-n)!6^n}$
for $n = 1, 2, \dots, 6$
 $P(\text{all faces are different}) = 0$
for $n = 7, 8, \dots$
- 2.35** $\frac{5}{11} \cong 0.4545$
- 2.37** $\frac{126}{253} \cong 0.4980$
- 2.39** $\binom{x-1}{3} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^{x-4}$ for $x = 4, 5, \dots$
- 2.41** $\frac{2^n}{(2n)!}$
- 2.43** $\frac{\binom{18}{x} \binom{18-x}{8-2x} 2^{8-2x}}{\binom{36}{8}}$ for $x = 0, 1, 2, 3, 4$
- 2.45** (a) $\frac{352}{833} \cong 0.4226$
(b) $\frac{198}{4165} \cong 0.04754$
(c) $\frac{6}{4165} \cong 0.001441$
(d) $\frac{88}{4165} \cong 0.02113$
(e) $\frac{1}{4165} \cong 0.0002401$
- 2.47** (a) $\frac{25}{54} \cong 0.4630$
(b) $\frac{25}{108} \cong 0.2315$
(c) $\frac{25}{162} \cong 0.1543$
- 2.49** $\frac{324,701}{643,250} \cong 0.5048$
- 2.51** $\frac{1}{786,832,248,020} \cong 1.2709 \cdot 10^{-12}$
- 2.53** $\frac{1}{3243} \cong 0.0003084$
- 2.55** $\frac{1}{9}$
- 2.57** $\frac{14,350,336}{4,805,419,921,875} \cong 2.9863 \cdot 10^{-6}$
- 2.59** $x = 5$
- 2.61** $P(A|B) = \frac{1}{3}$
- 2.63** $\frac{n+1}{2n}$
- 2.65** $\frac{23}{40} = 0.575$
- 2.67** $\frac{11}{28} \cong 0.3929$
- 2.69** $\frac{8}{5525} \cong 0.001448$
- 2.71** (a) $\frac{2}{3}$
(b) $\frac{1}{2}$
- 2.73** $P(R|A_i) = \begin{cases} 1/5 & i = 6 \\ 1/6 & i = 7 \\ 1/5 & i = 8 \\ 1/4 & i = 9 \\ 1/3 & i = 10 \\ 1/2 & i = 11 \end{cases}$
- 2.75** $\frac{72}{5525} \cong 0.01303$
- 2.77** $\frac{2i}{n(n+1)}$ for $i = 1, 2, \dots, n$
- 2.79** $\frac{2}{3}$
- 2.81** 0.58
- 2.83** $\frac{26}{45} \cong 0.5778$
- 2.85** $\frac{1}{24} \cong 0.04167$
- 2.87** (a) $\frac{3m+2}{6m}$
(b) $\frac{2m}{3m+2}$
(c) $\left(\frac{3m+2}{6m}\right)^n$
(d) $\binom{n}{x} \left(\frac{3m+2}{6m}\right)^x \left(1 - \frac{3m+2}{6m}\right)^{n-x}$
for $x = 0, 1, 2, \dots, n$
- 2.89** $\left(\frac{1}{2}\right)^{n-1} \frac{1}{2} + \left(\frac{3}{4}\right)^{n-1} \frac{1}{4} + \left(\frac{7}{8}\right)^{n-1} \frac{1}{8} + \left(\frac{15}{16}\right)^{n-1} \frac{1}{16} + \left(\frac{31}{32}\right)^{n-1} \frac{1}{32} + \left(\frac{31}{32}\right)^{n-1} \frac{1}{32}$
- 2.91** Use an algebraic approach or induction
- 2.93** (a) $P(A|B) = 0$
(b) $P(A|B) = P(A)$
(c) $P(A|B) = P(A)/P(B)$
(d) $P(B|A) = 1$
(e) $P(B) > 0$
- 2.95** Use complementary probability
- 2.97** $560p_D^2 p_I^3 p_R^4$
- 2.99** 38
- 2.101** 17
- 2.103** $\frac{27}{64} \cong 0.4219$
- 2.105** $\left(\frac{23}{24}\right)^{11} \cong 0.6262$
- 2.107** $(1+m-mp)p^n$
- 2.109** $p = \frac{1}{3}$

- 2.111 (a) false
 (b) true
 (c) false
 (d) true
 (e) false

2.113 840

Chapter 3

$$3.1 f(x) = \frac{\binom{r}{x} \binom{w}{x-3}}{\binom{r+w}{x-1}} \cdot \frac{r-2}{r+w-(x-1)}$$

for $x = 3, 4, \dots, w+3$

$$3.3 P(X \leq 11) = \frac{457}{7776} \cong 0.05877$$

$$3.5 f(x) = \begin{cases} 1/6 & x = 0 \\ 1/3 & x = 1 \\ 1/3 & x = 2 \\ 1/6 & x = 3 \end{cases}$$

$$3.7 f(x) = \frac{5-x}{15} \text{ for } x = 0, 1, 2, 3, 4$$

$$3.9 \frac{1}{4}$$

$$3.11 e^{-0.0039} \cong 0.9961$$

3.13 When $b > 0$

$$F_Y(y) = F_X\left(\frac{y-a}{b}\right)$$

When $b = 0$

$$F_Y(y) = \begin{cases} 0 & y < a \\ 1 & y \geq a \end{cases}$$

When $b < 0$

$$F_Y(y) = 1 - F_X\left(\frac{y-a}{b}\right)$$

$$3.15 F(x) = \begin{cases} 0 & x < 3 \\ 3/10 & 3 \leq x < 7 \\ 1 & x \geq 7 \end{cases}$$

3.17 7

$$3.19 F(x) = \begin{cases} 0 & x \leq 0 \\ x^2/2 & 0 < x < 1 \\ -x^2/2 + 2x - 1 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

$$3.21 f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}} & 0 < y < 1 \\ \frac{1}{6\sqrt{y}} & 1 \leq y < 4 \end{cases}$$

$$3.23 F(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-\lambda x} & 0 < x < 1 \\ 1 - e^{-\lambda(x^2+1)/2} & x \geq 1 \end{cases}$$

$$3.25 2^{-1/\theta}$$

3.27 (a) yes

(b) no

(c) yes

(d) no

(e) yes

$$3.29 E[X(3-X)] = 0$$

$$3.31 E[X] = \frac{28}{3}$$

$$3.33 E[X] = 4$$

$$3.35 P(X = 7) = 0.7$$

$$3.37 -\frac{14}{495} \cong -0.02828$$

(a loss of approximately 2.8 cents)

$$3.39 \frac{41}{2}, \frac{297}{4}$$

3.41 2

3.43 Answers will vary. For example, the probability density function $f_X(x) = |x|$ for $-1 < x < 1$ has population median 0 and $f_X(0) = 0$.

3.45 (a) $\sqrt{2}$

(b) A Monte Carlo simulation in R for 10,001 replications:

```
t1 = runif(10001, 0, 2 * pi)
t2 = runif(10001, 0, 2 * pi)
x1 = cos(t1)
y1 = sin(t1)
x2 = cos(t2)
y2 = sin(t2)
chord = sqrt((x1 - x2) ^ 2 +
              (y1 - y2) ^ 2)
sort(chord)[5001]
```

$$3.47 E[X] = \frac{2\ln 2 - 4\ln 3 + 8\ln 5 - \ln 7}{\ln 2 + \ln 5} \cong 3.4402$$

$$3.49 m = \begin{cases} 0 & 0 < p < 1 - \frac{1}{\sqrt[3]{2}} \\ 1 & 1 - \frac{1}{\sqrt[3]{2}} < p < \frac{1}{2} \\ 2 & \frac{1}{2} < p < \frac{1}{\sqrt[3]{2}} \\ 3 & \frac{1}{\sqrt[3]{2}} < p < 1 \end{cases}$$

$$3.51 E[X] = 3$$

- 3.53** (a) $E[X] = 7$
 (b) $E[X] = \frac{213}{31} \cong 6.8710$
- 3.55** (a) $E[|X - m|] = \frac{b-a}{4}$
 (b) $E[|X - m|] = \begin{cases} 2p & 0 < p < 1 - \frac{1}{\sqrt{2}} \\ 2p^2 - 2p + 1 & 1 - \frac{1}{\sqrt{2}} < p < \frac{1}{\sqrt{2}} \\ 2 - 2p & \frac{1}{\sqrt{2}} < p < 1 \end{cases}$
 and undefined for $p = 1 - \frac{1}{\sqrt{2}}$ and $p = \frac{1}{\sqrt{2}}$
- 3.57** $\frac{1717}{18} \cong 95.3889$
- 3.59** $E[X] = \frac{\theta}{\theta+1}$
- 3.61** (a) $f(x) = \frac{\binom{80}{2} \binom{20}{x}}{\binom{100}{2+x}} \cdot \frac{78}{98-x}$
 for $x = 0, 1, 2, \dots, 20$
 (b) $E[X] = 20/27 \cong 0.7407$
- 3.63** (a) $E[X] = \frac{170}{13} \cong 13.0769$
 (b) 13, 18
- 3.65** $\frac{77}{2} = 38.5$ or \$38.50
- 3.67** $\frac{4(n+m-4)}{nm}$
- 3.69** (a) The (m, b) values on the line segment from $(-2, 2)$ to $(2, 1)$ in the (m, b) Cartesian coordinate system
 (b) $\mu = \frac{m}{3} + \frac{b}{2}$
 $\frac{1}{3} \leq \mu \leq \frac{2}{3}$
- 3.71** (a) 0
 (b) $2\sqrt[3]{1/2} \cong 1.5874$
 (c) $\frac{13}{27} \cong 0.4815$
- 3.73** $E[X] = 2$
- 3.75** (a) $f(x) = \begin{cases} 2/5 & x = 1 \\ 3/10 & x = 2 \\ 1/5 & x = 3 \\ 1/10 & x = 4 \end{cases}$
 or
 $f(x) = \frac{5-x}{10}$ for $x = 1, 2, 3, 4$
 (b) $E[X] = 2$
 (c) $V[X] = 1$
- 3.77** Use the definition of the survivor function
- 3.79** (a) $\frac{1}{\lambda} \Gamma\left(1 + \frac{1}{\kappa}\right), \frac{1}{\lambda} [\ln 2]^{1/\kappa}, \frac{1}{\lambda} \left(\frac{\kappa-1}{\kappa}\right)^{1/\kappa}$
 (mode expression valid for $\kappa \geq 1$)
- (b) $\kappa = \frac{1}{1 - \ln 2} \cong 3.2589$
 $\kappa \cong 3.4395$ (solved numerically)
 $\kappa \cong 3.3125$ (solved numerically)
- 3.81** (a) $f(x) = \frac{n!(x-1)}{n^x(n+1-x)!}$
 for $x = 2, 3, \dots, n+1$
 (b) $E[X] = \sum_{x=2}^{n+1} x \cdot \frac{n!(x-1)}{n^x(n+1-x)!}$
 (c) $E[X] = \frac{7,281,587}{1,562,500} \cong 4.6602$
- 3.83** $E[X] = 0.7, V[X] = 0.91$
- 3.85** (a) event
 (b) undefined
 (c) constant
 (d) random variable
 (e) constant
- 3.87** (a) $c = \frac{3}{4}$
 (b) $E[X] = \frac{11}{16}$
- 3.89** (a) $\frac{1}{2}$
 (b) Monte Carlo simulation code:
 $w = 3.14$
 $h = 1.62$
 $nrep = 100000$
 $area.big = rep(w * h, nrep)$
 $x = sqrt(runif(nrep))$
 $area.small = w * x * h * x$
 $mean(area.small / area.big)$
- 3.91** (a) $f(x) = \begin{cases} 1/3 & x = 0 \\ 1/2 & x = 1 \\ 1/6 & x = 3 \end{cases}$
 (b) $\mu = 1$
 (c) $\sigma^2 = 1$
- 3.93** $c = 1$
- 3.95** $P(4 < X < 12) = \frac{30,251}{32,768} \cong 0.9232$
 $P(4 < X < 12) \geq \frac{3}{4}$

Chapter 4

4.1 $1 - 2p$

4.3 $k = 12$

4.5 $1 - \left(\frac{364}{365}\right)^{35} \cong 0.09156$

4.7 $1 - (1 - e^{-\lambda c})^n - ne^{-\lambda c} (1 - e^{-\lambda c})^{n-1}$
 for $c > 0$

- 4.9 $E[X] = \frac{9}{10}n$
 $V[X] = \frac{9}{100}n$
- 4.11 (a) $P(10 \leq X \leq 20) = \sum_{x=10}^{20} \binom{100}{x} \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{100-x}$
 (b) The R statement

```
pbinom(20, 100, 1 / 6) -
pbinom(9, 100, 1 / 6)
```

 returns approximately 0.8268
- 4.13 $1 - \binom{120}{0} \left(\frac{1}{12}\right)^0 \left(\frac{11}{12}\right)^{120} \cong 0.99997$
- 4.15 (a) $2p^2(1-p) + p^3$
 $2p(1-p)^2 + (1-p)^3$
 $p(1-p)^2 + p^2(1-p)$
 (b) $E[X] = 260(1-p)^3 + 264p(1-p)^2 + \dots + 278p^3$
- 4.17 $P(X \geq 2) = \frac{16,867}{19,683} \cong 0.8569$
- 4.19 `pbinom(8, 100, 1 / 10)`
- 4.21 $P(X \bmod 5 = 2) = \frac{8}{31}$
- 4.23 $1 / (1 - F(c))$
- 4.25 $\frac{1,830,403,837,568}{9,906,146,353,125} \cong 0.1848$
- 4.27 $f(x) = \binom{x}{r-1} p^r (1-p)^{x-r+1}$
 for $x = r-1, r, r+1, \dots; 0 < p < 1;$
 $r = 1, 2, \dots$
- 4.29 (a) `pbinom(3, 8, 0.3)`
 yields 0.8059
 (b) `pgeom(4, 1 / 3) -`
`pgeom(1, 1 / 3)`
 yields 0.3128
 (c) `dpois(3, 9)`
 yields 0.01499
- 4.31 $1 - \sum_{x=0}^8 \frac{(147/31)^x e^{-147/31}}{x!} \cong 0.05254$
- 4.33 $\frac{13}{24e} \cong 0.1993$
- 4.35 $\sum_{y=0}^{23} \left[y \cdot \frac{18^y e^{-18}}{y!} \right] + 24 \cdot \left[1 - \sum_{x=0}^{23} \frac{18^x e^{-18}}{x!} \right] \cong 17.8183$
- 4.37 $\lambda = x$
- 4.39 $f_X(x) = \frac{(p\lambda)^x e^{-p\lambda}}{x!}$ for $x = 0, 1, 2, \dots$
- 4.41 (a) $P(\text{passing}) = \frac{n(n-1)}{2} p^{n-2} (1-p)^2 + n p^{n-1} (1-p) + p^n$
 (b) $P(\text{passing}) = \frac{\binom{r}{n-2} \binom{100-r}{2}}{\binom{100}{n}} + \frac{\binom{r}{n-1} \binom{100-r}{1}}{\binom{100}{n}} + \frac{\binom{r}{n} \binom{100-r}{0}}{\binom{100}{n}}$
 for $r = 2, 3, \dots, 100$
- 4.43 (a) $\frac{1}{9}$
 (b) $\frac{17}{99}$
 (c) $E[Y] = 9k$

Chapter 5

5.1 $\frac{13}{10}$

5.3 $\frac{1}{3}$

$$5.5 f_Y(y) = \begin{cases} 2/5 & y = 1 \\ 2/5 & y = 2 \\ 1/10 & y = 5 \\ 1/10 & y = 10 \end{cases}$$

5.7 (a) $x_0 = a/2$

(b) $\int_{x_0}^{\infty} f(x) dx = \frac{1}{2}$, indicating that the facility should be located at the *median* of the demand distribution

(c) The facility should be located at the *median* of the demand distribution, when the median exists

5.9 Approximately 0.05

5.11 $f(x) = \binom{10000}{x} \left(\frac{1}{8}\right)^x \left(\frac{7}{8}\right)^{10000-x}$
 for $x = 0, 1, 2, \dots, 10000$

5.13 (a) $\sum_{y=0}^{10} \binom{30}{y} (0.3012)^y (0.6988)^{30-y}$
 (b) The R statement

```
pbinom(10, 30,
1 - pexp(1200, 1 / 1000))
```

 returns 0.7255

5.15 $E[X - x | X \geq x] = \frac{1}{\lambda}$

5.17 $f_Y(y) = \binom{n}{y} (1 - e^{-\lambda c})^y e^{-\lambda c(n-y)}$
 for $y = 0, 1, \dots, n$

5.19 7

5.21 14.9790

5.23 0, 3

5.25 (a) `qunif(0.90, 0, 20)`
returns 18

(b) `qnorm(0.20)`
returns -0.8416

(c) `qnorm(0.75, 5, 3) -`
`qnorm(0.25, 5, 3)`
returns 4.0469

(d) `qbinom(0.75, 10, 0.4) -`
`qbinom(0.25, 10, 0.4)`
returns 2

5.27 `x = rnorm(1000, 70, 4)`
`y = rnorm(1000, 67, 3)`
`z = pmax(x, y)`
`z = sort(z)`
`z[10]`
`z[990]`

5.29 (a) $\mu = \frac{500}{3} = 166\frac{2}{3}$
(b) 0.9975, 0.9980
(c) 0.9992, 0.9993

5.31 0.003653

5.33 Use the definition of the moment generating function

5.35 exponential

5.37 (a) Cauchy

(b) Weibull

(c) beta

(d) Zipf

(e) triangular

5.39 $P(a < X < b) = e^{-e^{-b}} - e^{-e^{-a}}$

Chapter 6

6.1 $Y = (X_1 + X_2)/2$

$$f_Y(y) = \begin{cases} 1/9 & y = 1 \\ 2/9 & y = 1.5 \\ 3/9 & y = 2 \\ 2/9 & y = 2.5 \\ 1/9 & y = 3 \end{cases}$$

6.3 $P(2X + Y < 12) = \frac{35}{100}$

6.5 $f_X(x) = \frac{2\sqrt{1-x^2}}{\pi}$ for $-1 < x < 1$

6.7 $P(a < X \leq b, c < Y \leq d) =$
 $F(b, d) - F(a, d) - F(b, c) + F(a, c)$

6.9 $\frac{2}{3}e^{-3} \cong 0.03319$

6.11 $f_Y(y) = \frac{\ln 10 - \ln(10-y)}{10}$ for $0 < y < 10$

6.13 $P(X \leq Y) = \frac{6}{7} \cong 0.8571$

6.15 $F(x, y) = \begin{cases} 0 & x \leq 0 \text{ or } y \leq 0 \\ 2xy & 0 < x < 1, 0 < y < 1, x+y < 1 \\ 2y-y^2-(x-1)^2 & 0 < x < 1, 0 < y < 1, x+y \geq 1 \\ x(2-x) & 0 < x < 1, y \geq 1 \\ y(2-y) & 0 < y < 1, x \geq 1 \\ 1 & x \geq 1, y \geq 1 \end{cases}$

6.17 $f_X(x) = \begin{cases} \int_{-x}^{\infty} f(x, y) dy & x \leq 0 \\ \int_x^{\infty} f(x, y) dy & x > 0 \end{cases}$
 $f_Y(y) = \int_{-y}^y f(x, y) dx \quad y > 0$

6.19 $\frac{7}{8}, \frac{3}{4}$

6.21 (a) $f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{105}$
for $y_1 = 1, 2, \dots, 14;$
 $y_2 = 2, 3, \dots, 15; y_1 < y_2$
 $f_{Y_1}(y_1) = \frac{15-y_1}{105}$
for $y_1 = 1, 2, \dots, 14$

(b) $f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} \frac{1}{225} & y_1 = y_2 = 1, 2, \dots, 15 \\ \frac{2}{225} & y_1 = 1, 2, \dots, 14; \\ & y_2 = 2, 3, \dots, 15; \\ & y_1 < y_2 \end{cases}$
 $f_{Y_1}(y_1) = \frac{31-2y_1}{225}$ for $y_1 = 1, 2, \dots, 15$

6.23 (a) $P(|X| + |Y| > 1) = \frac{\pi-2}{\pi} \cong 0.3634$

(b) 11 regions

6.27 The median is approximately 0.7979

6.29 (a) $f_{X, Y}(x, y) = \frac{1}{10}$
for $x = 1, 2, 3, 4, 5; y = 2, 3, 4, 5;$
 $x < y$

(b) $f(z) = \frac{5-z}{10}$ for $z = 1, 2, 3, 4$

6.31 (a) $f(x, y) = \frac{1}{\pi}$ for $x^2 + y^2 < 1$

(b) no

(c) $P(|X| + |Y| < 1) = \frac{2}{\pi}$

(d) $P(3X + Y < 0) = \frac{1}{2}$

(e) $P(X^2 + Y^2 < 1/4) = \frac{1}{4}$

6.33 $Y = (X_1 + X_2)/2$

$$f_Y(y) = \begin{cases} 1/9 & y = 2 \\ 4/9 & y = 5 \\ 4/9 & y = 8 \end{cases}$$

6.35 $P(X = Y) = \frac{1}{3}$

- 6.37** $f_{Y_2}(y_2) = p_1(1-p_1)^{y_2} + p_2(1-p_2)^{y_2} - (p_1 + p_2 - p_1p_2)(1-p_1)^{y_2}(1-p_2)^{y_2}$
for $y_2 = 0, 1, 2, \dots$
- 6.39** (a) $f_V(v) = \begin{cases} 1/3 & -1 < v \leq 0 \\ 1-2v/3 & 0 < v \leq 1 \end{cases}$
(b) $f_W(w) = \begin{cases} 1/3 & w = -1 \\ 1/2 & w = 0 \\ 1/6 & w = 1 \end{cases}$
- 6.41** $F_Z(z) = [F(z)]^2$ for z values on the support of X and Y
- 6.43** $F_Z(z) = \begin{cases} 0 & z \leq 1 \\ \frac{\lambda_1\lambda_2(z^2-1)}{\lambda_1\lambda_2z^2+(\lambda_1^2+\lambda_2^2)z+\lambda_1\lambda_2} & z > 1 \end{cases}$
- 6.45** $f_Z(z) = \sum_{w=0}^z f_X(z-w)f_Y(w)$
for $z = 0, 1, 2, \dots$
- 6.47** (a) $\frac{1}{5} + \frac{1}{8} + \frac{1}{18} + \frac{1}{48} + \frac{1}{120} = \frac{59}{144} \cong 0.4097$
(b) 95th percentile of T is approximately 0.8776
- 6.49** $\text{Cov}(X, Y) = -\frac{100}{8619} \cong -0.01160$
The negative population covariance is not surprising because the more jacks drawn, the fewer queens drawn. In other words, when the number of jacks tends to be above its population mean the number of queens tends to be below its population mean and vice-versa.
- 6.51** (a) $\text{Cov}(X, Y) = \frac{4}{225}k^2$
(b) $E[Y | X = x] = \frac{2(k^3-x^3)}{3(k^2-x^2)}$
for $0 < x < k$
(c) $V[Y | X = x] = \frac{(x-k)^2(k^2+4kx+x^2)}{18(k+x)^2}$
for $0 < x < k$
- 6.53** (a) $f_{X,Y}(x, y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{13-x-y}}{\binom{52}{13}}$
for $x = 0, 1, 2, \dots, 13$;
 $y = 0, 1, 2, \dots, 13$; $x + y \leq 13$
(b) $\text{Cov}(X, Y) = -\frac{169}{272} \cong -0.6213$
- 6.55** (a) $E[X(n-X)] = n(n-1)p(1-p)$
(b) -1
(c) $E[X(17-X)] = \frac{544}{9} \cong 60.44$
R code with 1000 replications is
- ```
n = 17
p = 1 / 3
nrep = 1000
numheads = rbinom(nrep, n, p)
numtails = n - numheads
mean(numheads * numtails)
cor(numheads, numtails)
```
- 6.57**  $\rho = -1$
- 6.59** (a)  $P(X > 2/3) = \frac{5}{18}$   
(b)  $P(Y > 2/3) = \frac{1}{9}$   
(c)  $E[XY^2] = 0$   
(d)  $E[\arccos X] = \frac{\pi}{2}$   
(e)  $f_Y(y) = 2 - 2y$  for  $0 < y < 1$
- 6.61**  $P(X < \frac{1}{2} \cos \Theta) = \frac{2l+2d \arccos(d/l)-2\sqrt{l^2-d^2}}{\pi d}$
- 6.63**  $\text{Cov}(X, Y) = -1$
- 6.65**  $E[X^5 | Y = 1] = \frac{p^4}{p_2+p_4}$
- 6.67**  $\rho = \frac{1}{2}$
- 6.69**  $E[X | Y = 5] = 3$
- 6.71**  $\rho = -\sqrt{\frac{13}{405}} \cong -0.1792$
- 6.73**  $E[Y | X = 1] = \frac{5}{2}$   
 $V[Y | X = 1] = \frac{3}{4}$
- 6.75** (a)  $Y_1 \sim N(0, 2)$   
(b)  $\text{Cov}(Y_1, Y_2) = 1$
- 6.77**  $V[E[Y | X]] = 0$
- 6.79**  $V[E[V[E[X | Y]]]] = 0$
- 6.81** \$28.50
- 6.83**  $E[\text{number of runs}] = 1 + 2(n-1)p(1-p)$
- 6.85**  $\text{Cov}(X, Y) = 0$
- 6.87**  $E[X] = 1 + \frac{p}{1-p} + \frac{1-p}{p}$   
 $V[X] = \frac{-2p^4+4p^3+p^2-3p+1}{p^2(1-p)^2}$
- 6.89**  $\text{Cov}(X, Y) = -\frac{n}{12}$
- 6.91**  $\text{Cov}(X, Y) = 0$
- 6.93**  $-3 < r < 3$

- 6.95** (a) false  
 (b) true  
 (c) false  
 (d) false  
 (e) true  
 (f) false  
 (g) false  
 (h) false  
 (i) true  
 (j) true

**6.97**  $150, \frac{1075}{3} \cong 358.3333$

- 6.99** (a)  $f(x_1, x_2, x_3) = \frac{1}{24}$   
 for  $x_i = 1, 2, 3, 4$ , for  $i = 1, 2, 3$ ;  
 $x_1 \neq x_2, x_1 \neq x_3, x_2 \neq x_3$   
 (b)  $f(x_1, x_2, x_3) = \frac{1}{64}$   
 for  $x_i = 1, 2, 3, 4$ ; for  $i = 1, 2, 3$

$$\mathbf{6.101} \quad f_X(x) = \begin{cases} 14336/78125 & x = 0 \\ 15616/78125 & x = 1 \\ 15808/78125 & x = 2 \\ 532/3125 & x = 3 \\ 1896/15625 & x = 4 \\ 5697/78125 & x = 5 \\ 5589/156250 & x = 6 \\ 2187/156250 & x = 7 \end{cases}$$

**6.103**  $E[X_1 X_2 X_3] = 120$

**6.105**  $V[X_1 + X_2 + X_3] = 45$

**6.107**  $\sum_{i=1}^4 F(k, \mu_i, \sigma_i^2) \prod_{j=1, j \neq i}^4 (1 - F(k, \mu_j, \sigma_j^2))$

**6.109** (a)  $V[X_1] = 2$

(b)  $P(X_1 + X_2 + X_3 = 4) = \frac{1}{4}$

**6.111**  $P(X_1 = X_2 = X_3 = X_4) = (1 - p)^4 + p^4$

**6.113**  $E[X_1 X_2 X_3 X_4] = p^4$

**6.115**  $P(X_1 \leq X_2 \leq X_3 \leq X_4) = (1 - p)^4 + p(1 - p)^3 + p^2(1 - p)^2 + p^3(1 - p) + p^4$

**6.117** (a)  $P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

(b)  $P(X_1 < X_2 < X_3) = \frac{\lambda_1 \lambda_2}{(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)}$

**6.119** \$1.41

**6.121** binomial( $n + m, p$ )

**6.123** chi-square( $n$ )

**6.125**  $f_Y(y) = 10y^9$  for  $0 < y < 1$

**6.127** The joint probability distribution of  $X_2$  and  $X_3$  is bivariate normal with population mean vector  $\mu = (5, 6)'$  and variance-covariance matrix

$$\Sigma = \begin{bmatrix} 8 & 3 \\ 3 & 9 \end{bmatrix}$$

**6.129** The APPL statements

```
X := UniformDiscreteRV(1, 6);
Y := ConvolutionIID(X, 5);
PDF(Y, 17);
```

return 65/648  $\cong$  0.1003

**6.131**  $E[Y] = 9, V[Y] = 6$

```
nrep = 100000
y = rep(0, nrep)
for (i in 1:nrep) {
 x = sample(5, 3, replace = T)
 y[i] = sum(x)
}
print(mean(y))
print(var(y))
```

**6.133**  $E[(X_1 - X_2)^2 + (Y_1 - Y_2)^2] = 1/3$

```
nrep = 100000
x1 = runif(nrep)
x2 = runif(nrep)
y1 = runif(nrep)
y2 = runif(nrep)
mean((x1 - x2) ^ 2 + (y1 - y2) ^ 2)
```

## Chapter 7

| fcn? | 1-1? | inv? | $g(9)$  | $g^{-1}(9)$ |
|------|------|------|---------|-------------|
| yes  | yes  | yes  | 23      | 13/3        |
| yes  | no   | no   | 81      | {-3, 3}     |
| no   | no   | yes  | {-3, 3} | 81          |

**7.3**  $f_Z(z) = \frac{\lambda_1 \lambda_2 (e^{-\lambda_2 z} - e^{-\lambda_1 z})}{\lambda_1 - \lambda_2}$  for  $z > 0$

**7.5**  $F_Z(z) = \begin{cases} 0 & z \leq -\frac{1}{\sqrt{2}} \\ \frac{2z^2 + 4z\sqrt{1-z^2} + \arcsin z + \arcsin(\sqrt{1-z^2})}{2\pi} & -\frac{1}{\sqrt{2}} < z < 0 \\ \frac{\pi + 4z^2 + 4z\sqrt{1-z^2} + 4\arcsin z}{4\pi} & 0 < z < \frac{1}{\sqrt{2}} \\ \frac{2z\sqrt{1-z^2} + \arcsin z}{\pi} & \frac{1}{\sqrt{2}} < z < 1 \\ 1 & z > 1 \end{cases}$



$$7.7 \quad \frac{65}{24} + \frac{1}{8\sqrt{5}} + \ln(2) - \ln(\sqrt{5} + 3) \cong 2.0254$$

$$7.9 \quad E[D] = \frac{2}{15} + \frac{\sqrt{2}}{15} - \frac{\ln(\sqrt{2}-1)}{3} \cong 0.5214$$

$$V[D] = \frac{23}{75} - \frac{4\sqrt{2}}{225} + \frac{(4+2\sqrt{2})}{45} \ln(\sqrt{2}-1) - \frac{1}{9} \left( \ln(\sqrt{2}-1) \right)^2 \cong 0.06147$$

$$x_{0.95} \cong 0.9372$$

$$7.11 \quad V[|X - Y|] = \frac{29}{36} \cong 0.8056$$

$$7.13 \quad f_Y(y) = \begin{cases} 2y^3 & 0 < y < 1 \\ y/2 & \sqrt{2} < y < 2 \end{cases}$$

$$7.15 \quad f_{Y_1}(y_1) = \int_0^\infty f_{X_1, X_2}(y_1 y_2, y_2) y_2 dy_2$$

for  $y_1 > 0$

$$7.17 \quad \begin{aligned} \text{(a)} \quad & f_Y(y) = 1/4 \text{ for } 0 < y < 4 \\ \text{(b)} \quad & f_Y(y) = 1/8 \text{ for } y = 1, 2, \dots, 8 \\ \text{(c)} \quad & f_Y(y) = 1 \text{ for } y = 1 \end{aligned}$$

$$\text{(d)} \quad f_Y(y) = \begin{cases} 1/4 & 0 < y < 2 \\ 1/8 & 2 \leq y < 6 \end{cases}$$

$$\text{(e)} \quad f_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & 1 < y < 2 \\ 1/8 & 2 \leq y < 6 \end{cases}$$

$$7.19 \quad f_Y(y) = \begin{cases} 1/9 & y = -8 \\ 1/9 & y = -5 \\ 1/9 & y = -3 \\ 3/9 & y = 0 \\ 1/9 & y = 3 \\ 1/9 & y = 5 \\ 1/9 & y = 8 \end{cases}$$

$$7.23 \quad 2 + \sqrt{2} = 3.4142$$

$$7.25 \quad f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{2\pi(b-a)\sqrt{y_1^2 + y_2^2}}$$

for  $a^2 < y_1^2 + y_2^2 < b^2$

$$7.27 \quad f_Z(z) = \begin{cases} 3z^2/2 & 0 \leq z < 1 \\ 2-z & 1 \leq z < 2 \end{cases}$$

$$7.29 \quad f_{Y_1}(y_1) = \begin{cases} 1/17 & y = 0 \\ 13/102 & y = 1 \\ 16/51 & y = 2 \\ 13/51 & y = 3 \\ 25/102 & y = 8 \end{cases}$$

$$7.31 \quad f_Y(y) = \frac{\theta y^{\theta/2-1}}{2} \text{ for } 0 < y < 1$$

$$7.33 \quad f_Y(y) = \frac{1}{32} \text{ for } 0 < y < 32$$

$$7.35 \quad f_{Y_1}(y_1) = \frac{\lambda_1 \lambda_2}{(\lambda_1 y_1 + \lambda_2)^2} \text{ for } y_1 > 0$$

$$7.37 \quad f_Y(y) = \begin{cases} 1 & 0 < y < 1/2 \\ 1/(4y^2) & y \geq 1/2 \end{cases}$$

$$7.39 \quad f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}} \text{ for } -1 < y < 1$$

$$7.41 \quad f_{Y_1, Y_2}(y_1, y_2) = \lambda_1 \lambda_2 e^{-\lambda_1(y_1 - y_2) - \lambda_2 y_2}$$

for  $0 < y_2 < y_1$

#### 7.43 The APPL code

```
R1 := ExponentialRV(100 / 6);
R2 := ExponentialRV(100 / 7);
R3 := ExponentialRV(100 / 8);
T1 := Convolution(R1, R2, R3);
g1 := [[x -> 1000000 * exp(x)],
 [0, infinity]];
A := Transform(T1, g1);
Mean(A);
```

returns  $\frac{125000000000}{100533}$

or \$1,243,372.82

#### 7.45 The APPL statements

```
X := UniformRV(0, 1);
Y := ConvolutionIID(X, 4);
IDF(Y, 99 / 100) / 4;
```

return 0.8250182245

#### 7.47 The APPL statements

```
A := [[1 / 6, 1 / 6, 1 / 6, 1 / 6, 1 / 6, 1 / 6],
 [4, 8, 12, 16, 20, 24], ["Discrete", "PDF"]];
C := UniformDiscreteRV(1, 6);
R := [[1 / 6, 1 / 6, 1 / 6, 1 / 6, 1 / 6, 1 / 6],
 [1, 4, 9, 16, 25, 36],
 ["Discrete", "PDF"]];
S := Product(A, C);
U := Difference(R, S);
V := [[CDF(U, -1), PDF(U, 0), SF(U, 1)], [0, 1, 2],
 ["Discrete", "PDF"]];
Mean(V);
```

return 3/8

$$7.49 \quad E\left[\frac{X_1/m}{X_2/n}\right] = \frac{n}{n-2} \text{ for } n > 2$$

#### 7.51 The APPL statements

```

A := UniformRV(0, 1);
B := UniformRV(0, 1);
g := [[x -> 1 / x ^ 2], [0, 1]];
R := Convolution(Transform(A, g),
 Transform(B, g));
h := [[x -> sqrt(x)], [2, infinity]];
R := Transform(R, h);
Variance(R);

```

return an unevaluated integral

**7.53** (a) The APPL statements

```

X := BinomialRV(5, 7 / 10);
Y := BinomialRV(4, 8 / 10);
Z := Product(X, Y);
Mean(Z);

```

return  $\frac{56}{5}$  or \$11.20

(b) The Monte Carlo simulation code in R is

```

x = rbinom(10000, 5, 7 / 10)
y = rbinom(10000, 4, 8 / 10)
z = x * y
mean(z)

```

**7.55**  $P(0.052 < X < 9.55) \cong 0.9397$

**7.57** 0

**7.59**  $c \cong 31.8205$

**7.61**  $f_{X_{(2)}}(x) = 20(x^{-5} - x^{-6})$  for  $x > 1$

**7.63** The APPL statements

```

X := NormalRV(2, 1);
Y := Truncate(X, 0, infinity);
Z := OrderStat(Y, 10, 10);
Mean(Z);
evalf(%);

```

return the mean as approximately 3.5511

**7.65**  $P(X_{(1)} < 1/3) = \frac{11}{27} \cong 0.4074$

**7.67** The APPL statements

```

X := WeibullRV(lambda, kappa);
Y3 := OrderStat(X, 10, 3);
CDF(Y3, 2);

```

give the probability  $P(X_{(3)} < 2) = 1 + 80e^{-9(2\lambda)^\kappa} - 36e^{-10(2\lambda)^\kappa} - 45e^{-8(2\lambda)^\kappa}$  for  $\lambda > 0$  and  $\kappa > 0$

**7.69** (a)  $\rho = \frac{1}{2}$

(b) The R code is given below

```

n = 500000
x1 = runif(n)
x2 = runif(n)
y1 = pmin(x1, x2)
y2 = pmax(x1, x2)
cor(y1, y2)

```

**7.71**  $2^{-1/n}$

**7.73**  $P(Z_1^2 + Z_2^2 + \dots + Z_8^2 < 10) \cong 0.7350$

**7.75**  $Y \sim N(0, 1)$

**7.77**  $P(\bar{X} \leq c) = \sum_{x=0}^{\lfloor nc \rfloor} \frac{(n\lambda)^x e^{-n\lambda}}{x!}$

**7.79**  $n = 6$

**7.81** 0.1517

**7.83** 0.2007

**7.85** gamma

**7.87**  $F$

**7.89**  $P(|X - Y| > 3) \cong 0.0339$

**7.91** The APPL statements:

```

X := UniformRV(0, 1);
Y := ConvolutionIID(X, 5);
g := [[x -> x / 5], [0, 5]];
Xbar := Transform(Y, g);

```

yield the probability density function

$$f_{\bar{X}}(y) = \begin{cases} \frac{3125}{24} y^4 & 0 < y \leq 1/5 \\ \frac{125}{6} y - \frac{625}{4} y^2 + \frac{3125}{6} y^3 - \frac{3125}{6} y^4 - \frac{25}{24} & 1/5 < y \leq 2/5 \\ \frac{775}{24} - \frac{625}{2} y + \frac{4375}{4} y^2 - \frac{3125}{2} y^3 + \frac{3125}{4} y^4 & 2/5 < y \leq 3/5 \\ -\frac{3275}{24} + \frac{1625}{2} y - \frac{6875}{4} y^2 + \frac{3125}{2} y^3 - \frac{3125}{6} y^4 & 3/5 < y \leq 4/5 \\ \frac{3125}{24} - \frac{3125}{6} y + \frac{3125}{4} y^2 - \frac{3125}{6} y^3 + \frac{3125}{24} y^4 & 4/5 < y \leq 1 \end{cases}$$

**7.93**  $M_Y(t) = \left(\frac{\lambda_1}{\lambda_1 - a_1 t}\right)^{\kappa_1} \cdot \left(\frac{\lambda_2}{\lambda_2 - a_2 t}\right)^{\kappa_2} \cdot \left(\frac{\lambda_3}{\lambda_3 - a_3 t}\right)^{\kappa_3}$   
for  $t < \min\left\{\frac{\lambda_1}{a_1}, \frac{\lambda_2}{a_2}, \frac{\lambda_3}{a_3}\right\}$

**7.95**  $6p^2$

**7.97**  $c = \frac{43460\sqrt{29} + 3480\sqrt{265}}{1060\sqrt{29} + 87\sqrt{265}} \cong 40.8012$

## Chapter 8

- 8.1** (a)  $f_{X_n}(x) = \binom{3n}{x} \left(\frac{1}{n}\right)^x \left(1 - \frac{1}{n}\right)^{3n-x}$   
for  $x = 0, 1, 2, \dots, 3n$   
(b)  $f_X(x) = \frac{3^x e^{-3}}{x!}$  for  $x = 0, 1, 2, \dots$

- 8.3** (a)  $M_{\bar{X}_n}(t) = \left[\frac{1}{1-\theta t/n}\right]^n$   
for  $t < n/\theta$   
(b)  $\lim_{n \rightarrow \infty} M_{\bar{X}_n}(t) = e^{\theta t}$   
which is degenerate at  $\theta$   
(c)  $M_{\sqrt{n}(\bar{X}_n - \theta)/\theta}(t) = e^{-\sqrt{nt} \left(1 - \frac{t}{\sqrt{n}}\right)^{-n}}$   
for  $t < \sqrt{n}$   
(d)  $\lim_{n \rightarrow \infty} M_{\sqrt{n}(\bar{X}_n - \theta)/\theta}(t) = e^{t^2/2}$   
for  $-\infty < t < \infty$

- 8.5** (a) 2.14  
(b) 

```
nrep = 10000
ntrials = 5
x = rep(0, nrep)
p = rep(0, ntrials)
for (i in 1:ntrials) {
 for (j in 1:nrep) x[j] =
 sum(runif(3))
 p[i] = sort(x)[0.90 * nrep]
}
t.test(p)
```

  
(c)  $3 - \frac{\sqrt[3]{75}}{5} \cong 2.1566$   
(d)  $3 - \frac{\sqrt[3]{75}}{5} \cong 2.1566$   
(e) The APPL code  

```
X := UniformRV(0, 1);
Y := ConvolutionIID(X, 3);
IDF(Y, 0.90);
returns 2.156567334
```

**8.7** The limiting distribution is  $N(0, 1)$

**8.9** The APPL statements

```
Y := UniformRV(0, 1);
for n from 1 to 20 do
 T := MaximumIID(Y, n);
 g := [[x -> n * (1 - x)],
 [0, 1]];
 X := Transform(T, g);
 m := Mean(X);
 v := Variance(X);
 s := Skewness(X);
 k := Kurtosis(X);
 print(n, m, v, s, k);
od;
```

compute the values needed for the plot

- 8.11** (a)  $P(\bar{X} > 1.6) \cong 0.03682$   
(b) The APPL code  

```
X := [[x -> x / 2], [0, 2],
 ["Continuous", "PDF"]];
Mean(X);
Variance(X);
Y := ConvolutionIID(X, 10);
g := [[x -> x / 10], [0, 20]];
Z := Transform(Y, g);
1 - CDF(Z, 8 / 5);
returns P(\bar{X} > 1.6) = \frac{36,411,338,728,543}{1,187,940,433,680,000} \cong 0.03065
```

  
(c) Monte Carlo simulation code:  

```
nrep = 1000000
count = 0
for (i in 1:nrep) {
 if (mean(2 * sqrt(runif(10))) > 1.6)
 count = count + 1
}
print(count / nrep)
```

  
generates output that hover about the analytical solution  
**8.13** (a)  $P(\bar{X} > 4) \cong 0.2871$   
(b)  $P\{\bar{X} > 4\} \cong 0.2681$   
**8.15** (a)  $\chi_p^2(n) \stackrel{a}{=} n + z_{1-p}\sqrt{2n}$   
(b) The relative error of the approximation decreases with  $n$