2.3 Computing Probabilities

A variety of examples are presented here illustrating various applications of the Kolmogorov axioms and subsequent results from the previous section. Many of these examples include the flipping of coins, the rolling of dice, and the random sampling of items from an urn or bag. These settings are used to allow for exact results in the case of the analytic solution of the simple problems presented here. These will lead to more practical probability problems encountered later in the book.

Example 2.12 Three men and two women sit in a row of chairs in a random order. Let the event A be that men and women alternate (that is, MWMWM). Find $P(A)$.

Using the multiplication rule, there are $5! = 120$ equally likely outcomes to the random ordering, and, again by the multiplication rule, there are 12 orderings that correspond to men and women alternating (see Examples 1.14 and 1.15). Therefore,

$$P(A) = \frac{3 \cdot 2 \cdot 2 \cdot 1 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{12}{120} = \frac{1}{10}.$$

This analytic solution is exact and correct. It can be checked by Monte Carlo simulation. The R function `sample` can be used to generate a random ordering of the five people, who will be numbered 1, 2, 3, 4, 5 (the women are even and the men are odd, an agreeable convention for the women). The event A associated with men and women alternating is equivalent to chairs 1, 3, and 5 being occupied by the men, so the product of their indices will be 15. Increasing the number of replications from 1000 (in the coin flipping experiment in Example 2.7) to 100,000, the R code for the Monte Carlo simulation experiment is shown below.

```
nrep = 100000
count = 0
for (i in 1:nrep) {
  x = sample(5)
  if (x[1] * x[3] * x[5] == 15) count = count + 1
}
print(count / nrep)
```

Using indices 1, 3, and 5 to denote both the chairs and the men is coincidental. We could have used 1, 2, and 3, for example, to denote the men. After a call to `set.seed(3)` to initialize the random number seed, the code segment is run five times yielding

0.09948 0.10076 0.10100 0.10105 0.09859.

The fact that the five probability estimates are closer to the analytic value than in the Monte Carlo simulation experiment in Example 2.7 is due to the larger number of replications. Two of these estimates of the probability that men and women alternate are less than the true value $(1/10)$ and three of these estimates are greater than the true value, so the analytic solution is considered "verified." Although we could use the terms like *verified* and *confirmed* when Monte Carlo simulation results hover around the analytic value, these terms are a bit misleading. The Monte Carlo simulation results provide *supporting* evidence, but they do not provide a mathematical verification of an analytic solution. Monte Carlo can be helpful to see when an analytic solution is incorrect, however, because the estimates will not hover around the analytic solution.

Example 2.13 Karen collects n hats and returns them at random. Let the event A be the proper return of the hats to their owners. Find $P(A)$.

This is a variant of a classic problem in probability known as the *hatcheck girl problem*. In the early 20th century, many restaurants and theaters would employ young women to check coats, hats, and other items for safekeeping in a cloakroom. By the multiplication rule, the hats can be returned in $n!$ different orders. Of these orders, only one of the orders is correct. Thus, the probability of returning all hats to the correct owners is

$$P(A) = \frac{1 \cdot 1 \cdot \ldots \cdot 1}{n \cdot (n-1) \cdot \ldots \cdot 1} = \frac{1}{n!}.$$

Example 2.14 Roll a pair of fair dice 24 times. Let the event A be rolling double aces (that is, double ones) at least once. Find $P(A)$.

Using complementary probabilities, the probability of rolling double aces is one minus the probability of not rolling double aces. There are 36^{24} different possible outcomes for 24 rolls of the dice by the multiplication rule. Of these possible outcomes, 35^{24} correspond to not rolling double aces. Therefore, the probability of rolling double aces somewhere in the 24 rolls is

$$P(A) = 1 - P(A') = 1 - \frac{35 \cdot 35 \cdot \ldots \cdot 35}{36 \cdot 36 \cdot \ldots \cdot 36} = 1 - \frac{35^{24}}{36^{24}} \cong 0.4914.$$

The next four examples concern five-card poker hands, so it is worthwhile reviewing the contents of a *standard deck* of playing cards. Each of the 52 cards in a standard deck belongs to one of four suits: spades, denoted by ♠, hearts, denoted by ♡, diamonds, denoted by ♢, and clubs, denoted by ♣. Each suit contains 13 cards, known as ranks: the numbered cards $2, 3, \ldots, 10$, the jack (J), the queen (Q), the king (K), and the ace (A). The queen of hearts, for example, is denoted by $Q\heartsuit$.

Example 2.15 A five-card poker hand is dealt from a well-shuffled deck. Let the event A be that there are exactly 2 kings in the hand. Find $P(A)$.

There are $\binom{52}{5} = 2,598,960$ equally-likely five-card poker hands, which goes in the denominator of the expression for $P(A)$. There is the assumption that the *order* in which the cards are dealt into the poker hand is not relevant, and that fact will be reflected in the numerator. Determining how many of these hands contain exactly two kings requires the use of both the multiplication rule and combinations. First, choose the two kings (for example, the king of hearts and the king of diamonds) out of the four kings, which can be done in $\binom{4}{2} = 6$ different ways, where order is not relevant. Likewise, choose the three non-kings out of the 48 non-kings in $\binom{48}{3} = 17,296$ different ways, where order is again not relevant. Finally, the total number of hands with exactly two kings can be found by the multiplication rule by taking the product of the two binomial coefficients, yielding

$$P(A) = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}} = \frac{103,776}{2,598,960} = \frac{2162}{54,145} \cong 0.0399.$$

This example illustrates that it is helpful to express the solution as an exact fraction, but also to express the solution as a decimal. Approximately 4% of the random deals of a five-card poker hand will contain exactly 2 kings.