Example 7.34 A random sample of $n = 2$ observations is drawn from a $U(0, 1)$ population. Find the joint probability density function of the order statistics.

Let X_1 and X_2 denote the observations and let

$$X_{(1)} = \min\{X_1, X_2\} \qquad \text{and} \qquad X_{(2)} = \max\{X_1, X_2\}$$

denote the associated order statistics. Although the ordering of the data might not look like a transformation at first glance, the transformation technique can be applied to this bivariate transformation from X_1 and X_2 to $X_{(1)}$ and $X_{(2)}$. Figure 7.14 shows the support of X_1 and X_2 on the left-hand graph, which is the unit square, and the support of $X_{(1)}$ and $X_{(2)}$ on the right-hand graph. More specifically, consider the mapping of the two points, $(X_1, X_2) = (0.8, 0.9)$ and $(X_1, X_2) = (0.9, 0.8)$, which are plotted on the left-hand graph. The first point, $(X_1, X_2) = (0.8, 0.9)$, which has the random variables drawn in increasing order, is mapped to $(X_{(1)}, X_{(2)}) = (0.8, 0.9)$. In fact, all points that fall above the dashed line in the left-hand graph in Figure 7.14 are mapped directly to identical coordinates on the right-hand graph. The second point, $(X_1, X_2) = (0.9, 0.8)$, which has the random variables drawn in decreasing order, is also mapped to $(X_{(1)}, X_{(2)}) = (0.8, 0.9)$. In fact, all points that fall below the dashed line in the left-hand graph in Figure 7.14 are reflected across the dashed line by the mapping. Using the terminology from earlier in this section, this is a two-to-one bivariate transformation from $\mathcal{A}$ to $\mathcal{B}$, that can be viewed as a folding of $\mathcal{A}$ over the dashed line. We can apply Theorem 7.7 to determine the joint probability density function of the order statistics. The only awkward part of this transformation is the notation; we are working with $X_{(1)}$ and $X_{(2)}$ rather than the usual Y_1 and Y_2.

First, consider the transformation from $\mathcal{A}_1$ to $\mathcal{B}$. The transformation

$$x_{(1)} = g_1(x_1, x_2) = \min\{x_1, x_2\} = x_1 \qquad \text{and} \qquad x_{(2)} = g_2(x_1, x_2) = \max\{x_1, x_2\} = x_2$$

is a bivariate one-to-one transformation from

$$\mathcal{A}_1 = \{(x_1, x_2) \,|\, 0 < x_1 < x_2 < 1\}$$

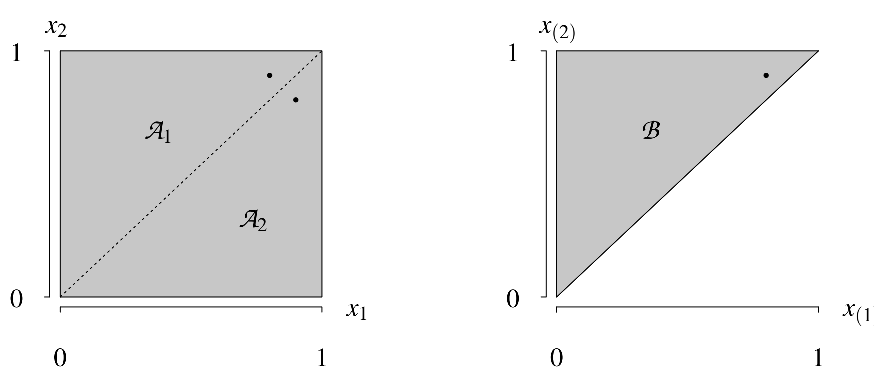

Figure 7.14: Support of X_1 and X_2 and support of $X_{(1)}$ and $X_{(2)}$.

to the support of $X_{(1)}$ and $X_{(2)}$, which is

$$\mathcal{B} = \{(x_{(1)}, x_{(2)}) \,|\, 0 < x_{(1)} < x_{(2)} < 1\}.$$

These functions can be solved in closed form for x_1 and x_2 as

$$x_1 = g_{11}^{-1}(x_{(1)}, x_{(2)}) = x_{(1)} \qquad \text{and} \qquad x_2 = g_{12}^{-1}(x_{(1)}, x_{(2)}) = x_{(2)}$$

with associated Jacobian

$$J_1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Next, consider the transformation from $\mathcal{A}_2$ to $\mathcal{B}$. The transformation

$$x_{(1)} = g_1(x_1, x_2) = \min\{x_1, x_2\} = x_2 \qquad \text{and} \qquad x_{(2)} = g_2(x_1, x_2) = \max\{x_1, x_2\} = x_1$$

is a bivariate one-to-one transformation from

$$\mathcal{A}_2 = \{(x_1, x_2) \,|\, 0 < x_2 < x_1 < 1\}$$

to the support of $X_{(1)}$ and $X_{(2)}$, which is

$$\mathcal{B} = \{(x_{(1)}, x_{(2)}) \,|\, 0 < x_{(1)} < x_{(2)} < 1\}.$$

These functions can be solved in closed form for x_1 and x_2 as

$$x_1 = g_{21}^{-1}(x_{(1)}, x_{(2)}) = x_{(2)} \qquad \text{and} \qquad x_2 = g_{22}^{-1}(x_{(1)}, x_{(2)}) = x_{(1)}$$

with associated Jacobian

$$J_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Theorem 7.7 with $n = 2$ and $k = 2$ can now be used to find the joint probability density function of $X_{(1)}$ and $X_{(2)}$. First, because X_1 and X_2 are independent, their joint probability density function is the product of their marginal probability density functions:

$$f_{X_1, X_2}(x_1, x_2) = 1 \qquad 0 < x_1 < 1, 0 < x_2 < 1.$$

The joint probability density function of $X_{(1)}$ and $X_{(2)}$ is

$$f_{X_{(1)}, X_{(2)}}(x_{(1)}, x_{(2)}) = 1 \cdot |1| + 1 \cdot |-1| = 2 \qquad 0 < x_{(1)} < x_{(2)} < 1.$$

The joint probability density function of $X_{(1)}$ and $X_{(2)}$ is uniformly distributed over the support $\mathcal{B}$ shown in Figure 7.14. So while X_1 and X_2 are independent random variables defined on a product space, the order statistics $X_{(1)}$ and $X_{(2)}$ are dependent random variables. The marginal probability density functions of $X_{(1)}$ and $X_{(2)}$ are easily calculated as

$$f_{X_{(1)}}(x_{(1)}) = \int_{x_{(1)}}^{1} 2\,dx_{(2)} = 2(1 - x_{(1)}) \qquad 0 < x_{(1)} < 1$$

and

$$f_{X_{(2)}}(x_{(2)}) = \int_{0}^{x_{(2)}} 2\,dx_{(1)} = 2x_{(2)} \qquad 0 < x_{(2)} < 1.$$