The smallest and largest of a group of random variables is oftentimes of interest. Examples include the time to failure of a series system of components (minimum), the time to failure of a parallel system of components (maximum), and the amount of rainfall associated with a 100-year flooding event (maximum). The example that follows considers the minimum of n independent exponential random variables.

Example 6.64 Let $X_1, X_2, \ldots, X_n$ be mutually independent random variables. Furthermore, let

$$X_i \sim \text{exponential}(\lambda_i)$$

for $i = 1, 2, \ldots, n$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive parameters. Find the distribution of $X = \min\{X_1, X_2, \ldots, X_n\}$.

The cumulative distribution function technique will be used to solve the problem. The cumulative distribution function of X_i on its support is

$$F_{X_i}(x_i) = 1 - e^{-\lambda_i x_i} \qquad x_i > 0,$$

for $i = 1, 2, \ldots, n$. Using complementary probability, equivalent events, and mutual independence, the cumulative function of X on its support is

$$\begin{aligned}
F_X(x) &= P(X \le x) \\
&= P(\min\{X_1, X_2, \ldots, X_n\} \le x) \\
&= 1 - P(\min\{X_1, X_2, \ldots, X_n\} > x) \\
&= 1 - P(X_1 > x, X_2 > x, \ldots, X_n > x) \\
&= 1 - P(X_1 > x)P(X_2 > x)\ldots P(X_n > x) \\
&= 1 - e^{-\lambda_1 x}e^{-\lambda_2 x}\ldots e^{-\lambda_n x} \\
&= 1 - e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)x} \qquad x > 0.
\end{aligned}$$

This cumulative distribution function can be recognized as the cumulative distribution function of an exponential$(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$ random variable. This result is well known to design engineers. If a series system consists of n electrical components with exponential times to failure with failure rates $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the system time to failure is also exponentially distributed with a failure rate that is the sum of the component failure rates.

The notion of an *expected value* of some function of n random variables generalizes directly from the two-dimensional case, as shown next.

Definition 6.14 Let $X_1,\ X_2,\ \ldots,\ X_n$ be random variables with joint probability mass function $f(x_1, x_2, \ldots, x_n)$ if the random variables are discrete or joint probability density function $f(x_1, x_2, \ldots, x_n)$ if the random variables are continuous. The *expected value* of $g(X_1, X_2, \ldots, X_n)$ is

$$E\left[g(X_1, X_2, \ldots, X_n)\right] = \sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} g(x_1, x_2, \ldots, x_n) f(x_1, x_2, \ldots, x_n)$$

when $X_1, X_2, \ldots, X_n$ are discrete and

$$E\left[g(X_1, X_2, \ldots, X_n)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1, x_2, \ldots, x_n) f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n$$

when $X_1, X_2, \ldots, X_n$ are continuous and when the sum or integral exist. When the sum or integral diverge, the expected value is undefined.