Theorem 6.5 If X and Y are random variables with finite population variances and covariance, then
$$V[X+Y] = V[X] + V[Y] + 2\text{Cov}(X, Y).$$

Proof Using the definition of the population variance, Theorem 6.2, and the fact that expected value is a linear operator,
$$\begin{aligned} V[X+Y] &= E\left[\left((X+Y) - E[X+Y]\right)^2\right] \\ &= E\left[\left((X-\mu_X) + (Y-\mu_Y)\right)^2\right] \\ &= E\left[(X-\mu_X)^2 + (Y-\mu_Y)^2 + 2(X-\mu_X)(Y-\mu_Y)\right] \\ &= E\left[(X-\mu_X)^2\right] + E\left[(Y-\mu_Y)^2\right] + 2E\left[(X-\mu_X)(Y-\mu_Y)\right] \\ &= V[X] + V[Y] + 2\text{Cov}(X, Y), \end{aligned}$$
which proves the result. □

Example 6.34 A fair coin is tossed twice. Let X be the number of heads that appear and Y be the number of tails that appear. Find the population variance of $X+Y$.

Since $X \sim$ binomial$(2, 1/2)$ and $Y \sim$ binomial$(2, 1/2)$, the population variances of the two random variables are
$$V[X] = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$
and
$$V[Y] = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$
because the population variance of a binomial(n, p) random variable is $np(1-p)$. Using Theorem 6.5, the population variance of the sum of X and Y is
$$V[X+Y] = V[X] + V[Y] + 2\text{Cov}(X, Y) = \frac{1}{2} + \frac{1}{2} - 2 \cdot \frac{1}{2} = 0,$$
where the population covariance was calculated as $\text{Cov}(X, Y) = -1/2$ in Examples 6.31 and 6.32. The population variance of the sum of X and Y must be zero because the sum of the heads and tails tossed in the random experiment is always 2, and the population variance of a constant is zero.

The next result concerns the population covariance between independent random variables.

Theorem 6.6 If X and Y are independent random variables, then $\text{Cov}(X, Y) = 0$.

Proof Since X and Y are independent random variables, $E[XY] = E[X]E[Y]$ by Theorem 6.3. Using the shortcut formula for the population covariance from Theorem 6.4, $\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 0$, which proves the result. □

One question that arises from Theorem 6.6 is whether the converse is true. That is, does a joint distribution of random variables X and Y with a population covariance of 0 imply that the random variables are independent? The converse is not true in general, and this is best established with a counterexample. The particular counterexample given next is not unique. There are many similar examples for both discrete and continuous probability distributions for X and Y.