

5.2 Exponential Distribution

A continuous random variable with positive support $\mathcal{A} = \{x | x > 0\}$ is useful in a variety of applications. Examples include

- patient survival time after the diagnosis of a particular cancer,
- the lifetime of a light bulb,
- the sojourn time (waiting time plus service time) for a customer purchasing a ticket at a box office,
- the time between births at a hospital,
- the number of gallons purchased at a gas pump, or
- the time to construct an office building.

The common element associated with these random variables is that they can only assume positive values.

One fundamental distribution with positive support is the exponential distribution, which is the subject of this section. The gamma distribution also has positive support and is considered in the next section. The exponential distribution has a single scale parameter λ , as defined below.

Definition 5.2 A continuous random variable X with probability density function

$$f(x) = \lambda e^{-\lambda x} \quad x > 0$$

for some real constant $\lambda > 0$ is an exponential(λ) random variable.

In many applications, λ is referred to as a “rate,” for example the arrival rate or the service rate in queueing, the death rate in actuarial science, or the failure rate in reliability. The exponential distribution plays a pivotal role in modeling random processes that evolve over time that are known as “stochastic processes.”

The exponential distribution enjoys a particularly tractable cumulative distribution function:

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_0^x f(w) dw \\ &= \int_0^x \lambda e^{-\lambda w} dw \\ &= \left[-e^{-\lambda w} \right]_0^x \\ &= 1 - e^{-\lambda x} \end{aligned}$$

for $x > 0$. Thus, for all values of x , the cumulative distribution function is

$$F(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-\lambda x} & x > 0. \end{cases}$$

The geometric distribution, which was introduced in Section 4.3, is the only discrete distribution to possess the *memoryless property*. The only continuous distribution to possess this property is the exponential distribution.

Theorem 5.1 (memoryless property) For $X \sim \text{exponential}(\lambda)$ and any two positive real numbers x and y ,

$$P(X \geq x+y | X \geq x) = P(X \geq y).$$

Proof The conditional probability is

$$\begin{aligned} P(X \geq x+y | X \geq x) &= \frac{P(X \geq x+y, X \geq x)}{P(X \geq x)} \\ &= \frac{P(X \geq x+y)}{P(X \geq x)} \\ &= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} \\ &= e^{-\lambda y} \\ &= P(X \geq y), \end{aligned}$$

which proves the memoryless property. \square

The memoryless property also has a geometric interpretation. Consider the probability density function of an $\text{exponential}(\lambda)$ random variable from some value x to infinity. The area under the probability density function from x to infinity does not equal 1; rather it equals $e^{-\lambda x}$. The conditional probability density function given that the random variable X is greater than or equal to x is found by rescaling:

$$f_{X|X \geq x}(w) = \frac{\lambda e^{-\lambda w}}{e^{-\lambda x}} = \lambda e^{-\lambda(w-x)} \quad w > x.$$

This conditional distribution, if shifted x units to the left, is identical to the original $\text{exponential}(\lambda)$ distribution. The geometry associated with the memoryless property is shown in Figure 5.3. The left-hand curve is an $\text{exponential}(\lambda)$ probability density function; the right-hand curve is the conditional probability density function of an $\text{exponential}(\lambda)$ random variable that is greater than x .

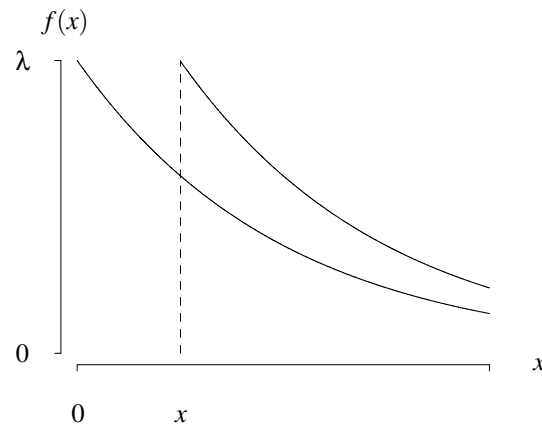


Figure 5.3: Memoryless property illustration for the exponential distribution.