4.4 Negative Binomial Distribution

The geometric distribution models the number of failures before the first success in repeated, independent Bernoulli trials, each with probability of success p. The negative binomial distribution is a generalization of the geometric distribution. The negative binomial distribution models the number of failures before the rth success in a sequence of repeated, mutually independent, and identically distributed Bernoulli trials, each with probability of success p. Since there can be as few as zero failures before the rth success and no upper bound on the number of trials before the rth success, the support of a negative binomial random variable is the denumerable set $\mathcal{A} = \{x \,|\, x = 0, 1, \ldots\}$.

We now turn to the more difficult question of determining the probability mass function for a negative binomial random variable X. The probability of r successes and x failures *in a specified order*, for example

$$\text{FFSFFFSS}$$

associated with $r = 3$ and $x = 5$, is

$$p^r(1-p)^x.$$

The rth success occurs at position $x+r$ in the sequence. This means that there are $x+r-1$ positions prior to the rth success, where the prior $r-1$ successes must be distributed. Since the order is not important as to which positions are selected for the $r-1$ initial successes, there are

$$\binom{x+r-1}{r-1}$$

different sequences of failures and successes associated with x failures prior to the rth success. This leads to the definition of a negative binomial random variable.

Definition 4.6 A discrete random variable X with probability mass function

$$f(x) = \binom{x+r-1}{r-1} p^r(1-p)^x \qquad x = 0, 1, 2, \ldots$$

for some positive integer r and $0 < p < 1$ is a negative binomial(r, p) random variable.

The negative binomial is also known as the Pascal distribution. We denote a negative binomial distribution with parameters r and p by $X \sim$ negative binomial(r, p). The origins of this distribution are that values of $f(x)$ are successive terms in the expansion of $p^r(1-(1-p))^{-r}$. The geometric distribution is a special case of the negative binomial distribution when $r = 1$.

A negative binomial random variable can be thought of as the concatenation of r random experiments associated with the geometric distribution in the following fashion. A geometric random variable involves repeated Bernoulli(p) trials until the first success. A negative binomial random variable is r of these random experiments placed back-to-back. Stated another way, if $X_1, X_2, \ldots, X_r$ are mutually independent geometric(p) random variables then $X_1 + X_2 + \cdots + X_r$ is a negative binomial(r, p) random variable. The notion of independence for random variables will be defined in Chapter 6.

The moment generating function for a negative binomial(r, p) random variable is

$$M(t) = \left[\frac{p}{1-(1-p)e^t}\right]^r$$

for $(1-p)e^t < 1$ or $t < -\ln(1-p)$. Using $t = 0$ as an argument in $M'(t)$ yields the population mean

$$\mu = E[X] = \frac{r(1-p)}{p}.$$