Theorem The limiting distribution of a Zipf(α, n) random variable as $n \to \infty$ is the zeta(α) distribution.

Proof Let $X \sim$ Zipf(α). Then, X has probability mass function

$$f(x) = \frac{1}{x^\alpha \sum_{i=1}^{n} (1/i)^\alpha} \qquad x = 1, 2, \ldots, n$$

and cumulative distribution function

$$F(x) = \frac{\sum_{i=1}^{x} (1/i)^\alpha}{\sum_{i=1}^{n} (1/i)^\alpha} \qquad x = 1, 2, \ldots, n.$$

Taking the limit of the cumulative distribution function as $n \to \infty$ yields

$$\lim_{n \to \infty} F(x) = \frac{\sum_{i=1}^{x} (1/i)^\alpha}{\sum_{i=1}^{\infty} (1/i)^\alpha} \qquad x = 1, 2, \ldots, \infty,$$

which is the cumulative distribution function of a zeta(α) random variable.