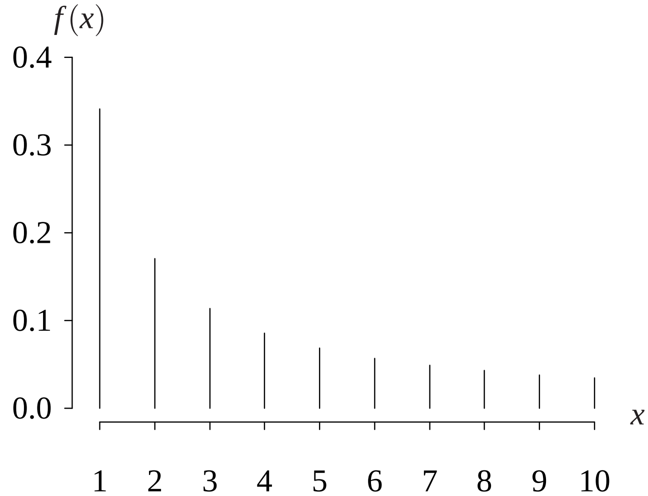
Zipf distribution (from `http://www.math.wm.edu/~leemis/chart/UDR/UDR.html`)
The shorthand $X \sim \text{Zipf}(\alpha, n)$ is used to indicate that the random variable X has the Zipf distribution with parameters α and n. A Zipf random variable X with parameters α and n has probability mass function

$$f(x) = \frac{1}{x^{\alpha} \sum_{i=1}^{n} (1/i)^{\alpha}} \qquad x = 1, 2, \ldots, n,$$

for all positive integers n and all $\alpha \geq 0$. The Zipf distribution can be used to account for the relative popularity of a few members of a population and the relative obscurity of other members of a population. Examples include the following.

- There are a few websites that get lots of hits, a greater number of websites that get a moderate number of hits, and a vast number of websites that hardly get any hits at all (like this one).
- A library has a few books that everyone wants to borrow (best sellers), a greater number of books that get borrowed occasionally (classics), and a vast number of books that hardly ever get borrowed.
- A natural language has a few words that are used with high frequency ("the" and "of" rank first and second in English), a greater number of words that get used with lower frequency (like "butter" and "joke"), and a vast number of words that hardly ever get used at all (like "defenestrate" which means to throw out of a window, "lucubration" which means a study or composition lasting late into the night, or "mascaron" which means a grotesque face on a door-knocker).
- The world population lives in several large cities, a greater number of medium-sized cities, and a vast number of small towns.

The probability mass function for $\alpha = 1$ and $n = 10$ is illustrated below.



Many times, the summation in the denominator is represented as

$$H_{n,\alpha} = \sum_{i=1}^{n} \left(\frac{1}{i}\right)^{\alpha}$$

An alternative representation is

$$f(x) = \frac{1}{x^{\alpha} H_{n,\alpha}} \qquad x = 1, 2, \ldots, n.$$

Using this shorthand notation, the cumulative distribution function on the support of X is

$$F(x) = P(X \le x) = \frac{H_{x,\alpha}}{H_{n,\alpha}} \qquad x = 1, 2, \ldots, n.$$

The survivor function on the support of X is

$$S(x) = P(X \ge x) = \frac{x^{\alpha} H_{n,\alpha} - x^{\alpha} H_{x,\alpha} + 1}{x^{\alpha} H_{n,\alpha}} \qquad x = 1, 2, \ldots, n.$$

The hazard function on the support of X is

$$h(x) = \frac{f(x)}{S(x)} = \frac{1}{x^{\alpha} H_{n,\alpha} - x^{\alpha} H_{x,\alpha} + 1} \qquad x = 1, 2, \ldots, n.$$

The cumulative hazard function on the support of X is

$$H(x) = -\ln S(x) = \ln\left(x^{\alpha} H_{n,\alpha}\right) - \ln\left(x^{\alpha} H_{n,\alpha} - x^{\alpha} H_{x,\alpha} + 1\right) \qquad x = 1, 2, \ldots, n.$$

The moment generating function of X is

$$M(t) = E\left[e^{tX}\right] = \frac{1}{H_{n,\alpha}} \sum_{j=1}^{n} \frac{e^{jt}}{j^{\alpha}} \qquad -\infty < t < \infty.$$

The characteristic function of X is

$$\phi(t) = E\left[e^{itX}\right] = \frac{1}{H_{n,\alpha}} \sum_{j=1}^{n} \frac{e^{ijt}}{j^{\alpha}} \qquad -\infty < t < \infty.$$

The population mean, variance, skewness, and kurtosis of X are

$$E[X] = \frac{H_{n,\alpha-1}}{H_{n,\alpha}} \qquad V[X] = \frac{H_{n,\alpha-2} H_{n,\alpha} - H_{n,\alpha-1}^{2}}{H_{n,\alpha}^{2}}$$

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^{3}\right] = \frac{\frac{H_{n,\alpha-3}}{H_{n,\alpha}} - 3\frac{H_{n,\alpha-1} H_{n,\alpha-2}}{H_{n,\alpha}^{2}} + 2\frac{H_{n,\alpha-1}^{3}}{H_{n,\alpha}^{3}}}{\left(\frac{H_{n,\alpha-2} H_{n,\alpha} - H_{n,\alpha-1}^{2}}{H_{n,\alpha}^{2}}\right)^{3/2}}$$

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^{4}\right] = \frac{H_{n,\alpha}^{3} H_{n,\alpha-4} - 4H_{n,\alpha}^{2} H_{n,\alpha-1} H_{n,\alpha-3} + 6H_{n,\alpha} H_{n,\alpha-1}^{2} H_{n,\alpha-2} - 3H_{n,\alpha-1}^{4}}{\left(H_{n,\alpha-2} H_{n,\alpha} - H_{n,\alpha-1}^{2}\right)^{2}}.$$

APPL verification: The APPL statements

```
assume(alpha >= 0)
X:=[[x -> 1/(x^alpha * sum((1/i)^alpha,i=1..n))], [1..n], ["Discrete", "PDF"]];
CDF(X);
SF(X);
HF(X);
CHF(X);
MGF(X);
Mean(X);
Variance(X);
Skewness(X);
Kurtosis(X);
```

verify the cumulative distribution, survivor function, hazard function, cumulative hazard function, moment generating function, population mean, variance, skewness, kurtosis.