Theorem The limiting distribution of a $t(n)$ random variable is standard normal as $n \to \infty$.

Proof (Hogg, McKean, and Craig, 2005, pages 210–211) Let $X \sim t(n)$ have probability density function $f(x)$ and cumulative distribution function $F(x)$. The limiting cumulative distribution function can be found by integrating the probability density function:

$$\begin{aligned}
\lim_{n\to\infty} F(x) &= \lim_{n\to\infty} \int_{-\infty}^{x} f(y)\,dy \\
&= \lim_{n\to\infty} \int_{-\infty}^{x} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{(1+y^2/n)^{(n+1)/2}}\,dy \\
&= \int_{-\infty}^{x} \lim_{n\to\infty} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{(1+y^2/n)^{(n+1)/2}}\,dy \\
&= \int_{-\infty}^{x} \lim_{n\to\infty} \left[\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n/2}\,\Gamma\left(n/2\right)}\right] \cdot \lim_{n\to\infty} \left[\frac{1}{(1+y^2/n)^{1/2}}\right] \cdot \lim_{n\to\infty} \left[\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(1+y^2/n)^{n/2}}\right] dy \\
&= \int_{-\infty}^{x} 1 \cdot 1 \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-y^2/2}\,dy
\end{aligned}$$

The limit and integral can be interchanged by the Lebesgue Dominated Convergence Theorem because the absolute value of the integrand $|f(y)|$ is dominated by an integrable function. The first of the three limits can be shown to be 1 by using Stirling's approximation

$$\Gamma(n+1) \cong \sqrt{2\pi} n^{n+1/2} e^{-n}$$

for large values of n. (For integer values of n, Stirling's approximation is typically written as $n! = \sqrt{2\pi n}\, n^n e^{-n}$.) The second limit is easily seen to be 1. The third limit can be shown to be the probability density function of a standard normal random variable by a limit result from calculus.

APPL verification: The APPL statements

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X := TRV(n);
limit(X[1][1](x), n = infinity);
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yield the standard normal probability density function.