Theorem If $X \sim t(n)$ then $Y = X^2 \sim F(1, n)$.

Proof The Student's t distribution with n degrees of freedom has probability density function

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)\left(\frac{x^2}{n}+1\right)^{(n+1)/2}} \qquad -\infty < x < \infty.$$

The transformation $Y = g(X) = X^2$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, -\infty < x < \infty\}$ to $\mathcal{Y} = \{y \,|\, y > 0\}$ with inverse $X = g^{-1}(Y) = \sqrt{Y}$ and Jacobian

$$\frac{dX}{dY} = \frac{1}{2\sqrt{Y}}.$$

Therefore, by the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\left|\frac{dx}{dy}\right| \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)\left(\frac{\sqrt{y}^2}{n}+1\right)^{(n+1)/2}}\left|\frac{1}{2\sqrt{y}}\right| \\ &= \frac{\Gamma((n+1)/2)(1/n)^{1/2}y^{1/2-1}}{\Gamma(1/2)\Gamma(n/2)[y/n+1]^{(n+1)/2}} \qquad y > 0, \end{aligned}$$

which is the probability density function of an $F(1, n)$ random variable.

APPL failure: The APPL statements

```
X := TRV(n);
g := [[x -> x ^ 2], [0, infinity]];
Y := Transform(X, g);
PDF(Y);
```

fail to produce the probability density function of an $F(1, n)$ random variable.