Theorem Let X have the standard uniform distribution. Then $Y = \frac{\ln[1-(\ln X)(\ln \kappa)/\delta]}{\ln \kappa}$ has the Gompertz(κ, δ) distribution, where κ and δ are positive parameters with $\kappa > 1$.

Proof Let the random variable X have the standard uniform distribution with probability density function

$$f_X(x) = 1 \qquad\qquad 0 < x < 1.$$

The transformation $Y = g(X) = \frac{\ln[1-(\ln X)(\ln \kappa)/\delta]}{\ln \kappa}$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, 0 < x < 1\}$ to $\mathcal{Y} = \{y \,|\, y > 0\}$ with inverse $X = g^{-1}(Y) = e^{\delta(1-\kappa^Y)/\ln \kappa}$ and Jacobian

$$\frac{dX}{dY} = -\delta \kappa^Y e^{\delta(1-\kappa^Y)/\ln \kappa}.$$

Therefore, by the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right| \\ &= 1 \left| -\delta \kappa^y e^{\delta(1-\kappa^y)/\ln \kappa} \right| \\ &= \delta \kappa^y e^{\delta(1-\kappa^y)/\ln \kappa} \\ &= \delta \kappa^y e^{-\delta(\kappa^y-1)/\ln \kappa} \qquad\qquad y > 0, \end{aligned}$$

which is the probability density function of the Gompertz(κ, δ) distribution.

APPL verification: The APPL statements

```
assume(delta > 0);
assume(kappa > 1);
X := StandardUniformRV();
g := [[x -> log(1 - ((log(x) * log(kappa)) / delta)) / log(kappa)], [0, 1]];
Y := Transform(X, g);
```

yield the probability density function of a Gompertz(κ, δ) random variable

$$f_Y(y) = \delta \kappa^y e^{-\delta(\kappa^y-1)/\ln \kappa} \qquad\qquad y > 0.$$