Theorem The limiting distribution of $n(1 - \max\{X_1, X_2, \ldots, X_n\})$, where $X_1, X_2, \ldots, X_n$ are mutually independent and identically distributed $U(0, 1)$ random variables, is exponential with mean 1.

Proof Let $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$. We want to show that the limiting distribution of $Y_n = n(1 - X_{(n)})$ is exponential with mean 1. Using the order statistic result

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1 - F(x)]^{n-k}f(x) \qquad a < x < b; k = 1, 2, \ldots, n,$$

where $f(\cdot)$ and $F(\cdot)$ denote the population probability density function and cumulative distribution function, and a and b are the minimum and maximum of the population support. For a population of $U(0, 1)$ random variables, the probability density function of $X_{(n)}$ is

$$f_{X_{(n)}}(x) = nx^{n-1} \qquad 0 < x < 1.$$

The transformation $Y_n = n(1 - X_{(n)})$ is a 1–1 transformation from $\mathcal{X} = \{x_{(n)} \mid 0 < x_{(n)} < 1\}$ to $\mathcal{Y} = \{y_n \mid 0 < y_n < n\}$ with inverse $X_{(n)} = 1 - Y_n/n$ and Jacobian

$$\frac{dX_{(n)}}{dY_n} = -\frac{1}{n}$$

So by the transformation technique, the probability density function of Y_n is

$$f_{Y_n}(y_n) = n\left(1 - \frac{y_n}{n}\right)^{n-1}\left|-\frac{1}{n}\right| = \left(1 - \frac{y_n}{n}\right)^{n-1} \qquad 0 < y_n < n.$$

The associated cumulative distribution function is

$$\begin{aligned} F_{Y_n}(y_n) &= \int_0^{y_n}\left(1 - \frac{w}{n}\right)^{n-1} dw \\ &= \left[-\left(1 - \frac{w}{n}\right)^n\right]_0^{y_n} \\ &= 1 - \left(1 - \frac{y_n}{n}\right)^n \qquad 0 < y_n < n. \end{aligned}$$

So the limiting distribution of Y_n is exponential with a mean of 1 because

$$\lim_{n\to\infty} F_{Y_n}(y_n) = \begin{cases} 0 & y_n < 0 \\ 1 - e^{-y_n} & y_n \geq 0. \end{cases}$$

APPL illustration: The APPL statements

```
X := UniformRV(0, 1);
n := 10;
T := OrderStat(X, n, n);
g := [[x -> n * (1 - x)], [0, 1]];
Y := Transform(T, g);
PlotDist(Y);
```

yield a probability density function that resembles an exponential probability density function with a mean of 1.