Theorem  The limiting distribution of \( n(1 - \max\{X_1, X_2, \ldots, X_n\}) \), where \( X_1, X_2, \ldots, X_n \) are mutually independent and identically distributed \( U(0, 1) \) random variables, is exponential with mean 1.

Proof  Let \( X_{(n)} = \max\{X_1, X_2, \ldots, X_n\} \). We want to show that the limiting distribution of \( Y_n = n(1 - X_{(n)}) \) is exponential with mean 1. Using the order statistic result

\[
f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k}f(x) \quad a < x < b; \ k = 1, 2, \ldots, n,
\]

where \( f(\cdot) \) and \( F(\cdot) \) denote the population probability density function and cumulative distribution function, and \( a \) and \( b \) are the minimum and maximum of the population support.

For a population of \( U(0, 1) \) random variables, the probability density function of \( X_{(n)} \) is

\[
f_{X_{(n)}}(x) = nx^{n-1} \quad 0 < x < 1.
\]

The transformation \( Y_n = n(1 - X_{(n)}) \) is a 1–1 transformation from \( X = \{x_{(n)} | 0 < x_{(n)} < 1\} \) to \( Y = \{y_n | 0 < y_n < n\} \) with inverse \( X_{(n)} = 1 - Y_n/n \) and Jacobian

\[
\frac{dX_{(n)}}{dY_n} = -\frac{1}{n}
\]

So by the transformation technique, the probability density function of \( Y_n \) is

\[
f_{Y_n}(y_n) = n \left(1 - \frac{y_n}{n}\right)^{n-1} \left| -\frac{1}{n} \right| = \left(1 - \frac{y_n}{n}\right)^{n-1} \quad 0 < y_n < n.
\]

The associated cumulative distribution function is

\[
F_{Y_n}(y_n) = \int_0^{y_n} \left(1 - \frac{w}{n}\right)^{n-1} dw = \left[- \left(1 - \frac{w}{n}\right)^n\right]_0^{y_n} = 1 - \left(1 - \frac{y_n}{n}\right)^n \quad 0 < y_n < n.
\]

So the limiting distribution of \( Y_n \) is exponential with a mean of 1 because

\[
\lim_{n \to \infty} F_{Y_n}(y_n) = \begin{cases} 
0 & y_n < 0 \\
1 - e^{-y_n} & y_n \geq 0.
\end{cases}
\]

APPL illustration: The APPL statements

\[
X := \text{UniformRV}(0, 1); \\
n := 10; \\
T := \text{OrderStat}(X, n, n); \\
g := [[x -> n \times (1 - x)], [0, 1]]; \\
Y := \text{Transform}(T, g); \\
\text{PlotDist}(Y);
\]

yield a probability density function that resembles an exponential probability density function with a mean of 1.