Theorem If X_1, X_2, ..., X_n are independent and identically distributed standard uniform random variables, then $X_{(r)}$ has the beta(β, γ) distribution where $\beta = r$ and $\gamma = n - r + 1$.

Proof Let X_1, X_2, ..., X_n be standard uniform random variables with probability density function

$$f(x) = 1 \qquad 0 < x < 1$$

and cumulative distribution function

$$F(x) = x \qquad 0 < x < 1.$$

Using the order statistic result

$$f_{X_{(k)}}(x_{(k)}) = \frac{n!}{(k-1)!(n-k)!}[F(x_{(k)})]^{k-1}[1-F(x_{(k)})]^{n-k} f(x_{(k)}) \qquad a < x_{(k)} < b; k = 1, 2, \ldots, n,$$

where $f(\cdot)$ and $F(\cdot)$ denote the population probability density function and cumulative distribution function, and a and b are the minimum and maximum of the population support, the probability density function of $X_{(r)}$ is

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} x^{r-1}(1-x)^{n-r} \qquad 0 < x < 1.$$

This is the same as

$$f(x) = \frac{\Gamma(n+1)}{\Gamma(r)\Gamma(n-r+1)} x^{r-1}(1-x)^{n-r} \qquad 0 < x < 1,$$

which is the probability density function of a beta($r, n - r + 1$) random variable.

APPL verification: The APPL statements

```
X := UniformRV(0, 1);
Y := OrderStat(X, r, n);
PDF(Y);
```

yield the probability density function of a beta($r, n - r + 1$) random variable.