**Theorem** If $X_1$ and $X_2$ are independent standard normal random variables, then $Y = X_1/X_2$ has the standard Cauchy distribution.

**Proof** Let $X_1$ and $X_2$ be independent standard normal random variables. We can write their probability density functions as

$$f_{X_1}(x_1) = \frac{e^{-x_1^2/2}}{\sqrt{2\pi}} \quad -\infty < x_1 < \infty$$

and

$$f_{X_2}(x_2) = \frac{e^{-x_2^2/2}}{\sqrt{2\pi}} \quad -\infty < x_2 < \infty.$$ 

Since $X_1$ and $X_2$ are independent, the joint probability density function of $X_1$ and $X_2$ is

$$f_{X_1,X_2}(x_1, x_2) = \frac{e^{-(x_1^2+x_2^2)/2}}{2\pi} \quad -\infty < x_1 < \infty, -\infty < x_2 < \infty.$$ 

Consider the $2 \times 2$ transformation

$$Y_1 = g_1(X_1, X_2) = \frac{X_1}{X_2} \quad \text{and} \quad Y_2 = g_2(X_1, X_2) = X_2$$

which is a 1–1 transformation from $\mathcal{X} = \{(x_1, x_2) \mid -\infty < x_1 < \infty, -\infty < x_2 < \infty\}$ to $\mathcal{Y} = \{(y_1, y_2) \mid -\infty < y_1 < \infty, -\infty < y_2 < \infty\}$ with inverses

$$X_1 = g_1^{-1}(Y_1, Y_2) = Y_1 Y_2 \quad \text{and} \quad X_2 = g_2^{-1}(Y_1, Y_2) = Y_2$$

and Jacobian

$$J = \begin{vmatrix} Y_2 & Y_1 \\ 0 & 1 \end{vmatrix} = Y_2.$$

Therefore, by the transformation technique, the joint probability density function of $Y_1$ and $Y_2$ is

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)) \left| J \right|$$

$$= \frac{e^{-(y_1^2+y_2^2)/2}}{2\pi} \left| y_2 \right| \quad -\infty < y_1 < \infty, -\infty < y_2 < \infty.$$ 

The probability density function of $Y_1$ is

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2) \, dy_2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| y_2 \right| e^{-y_2^2(y_1^2+1)/2} \, dy_2$$

$$= \frac{1}{\pi(y_1^2+1)} \quad -\infty < y_1 < \infty,$$

which is the probability density function of a standard Cauchy random variable.
APPL verification: The APPL statements

\[
\begin{align*}
X1 & := \text{NormalRV}(0, 1); \\
X2 & := \text{NormalRV}(0, 1); \\
g & := [[[x \rightarrow 1 / x, x \rightarrow 1 / x], [-\infty, 0, \infty]]]; \\
Y & := \text{Transform}(X2, g); \\
\text{Product}(X1, Y);
\end{align*}
\]

produce the probability density function of a Cauchy random variable.