Theorem If X_1 and X_2 are independent standard normal random variables, then $Y = X_1/X_2$ has the standard Cauchy distribution.

Proof Let X_1 and X_2 be independent standard normal random variables. We can write their probability density functions as

$$f_{X_1}(x_1) = \frac{e^{-x_1^2/2}}{\sqrt{2\pi}} \qquad -\infty < x_1 < \infty$$

and

$$f_{X_2}(x_2) = \frac{e^{-x_2^2/2}}{\sqrt{2\pi}} \qquad -\infty < x_2 < \infty.$$

Since X_1 and X_2 are independent, the joint probability density function of X_1 and X_2 is

$$f_{X_1,X_2}(x_1,x_2) = \frac{e^{-(x_1^2+x_2^2)/2}}{2\pi} \qquad -\infty < x_1 < \infty, -\infty < x_2 < \infty.$$

Consider the 2×2 transformation

$$Y_1 = g_1(X_1,X_2) = \frac{X_1}{X_2} \qquad \text{and} \qquad Y_2 = g_2(X_1,X_2) = X_2$$

which is a 1–1 transformation from $\mathcal{X} = \{(x_1,x_2) \mid -\infty < x_1 < \infty, -\infty < x_2 < \infty\}$ to $\mathcal{Y} = \{(y_1,y_2) \mid -\infty < y_1 < \infty, -\infty < y_2 < \infty\}$ with inverses

$$X_1 = g_1^{-1}(Y_1,Y_2) = Y_1\,Y_2 \qquad \text{and} \qquad X_2 = g_2^{-1}(Y_1,Y_2) = Y_2$$

and Jacobian

$$J = \begin{vmatrix} Y_2 & Y_1 \\ 0 & 1 \end{vmatrix} = Y_2.$$

Therefore, by the transformation technique, the joint probability density function of Y_1 and Y_2 is

$$\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}\left(g_1^{-1}(y_1,y_2), g_2^{-1}(y_1,y_2)\right)|J| \\
&= \frac{e^{-(y_1^2y_2^2+y_2^2)/2}}{2\pi}\,|y_2| \\
&= \frac{y_2 e^{-y_2^2(y_1^2+1)/2}}{2\pi} \qquad -\infty < y_1 < \infty, -\infty < y_2 < \infty.
\end{aligned}$$

Using integration by parts, the probability density function of Y_1 is

$$\begin{aligned}
f_{Y_1}(y_1) &= \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_2 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} y_2 e^{-y_2^2(y_1^2+1)/2}\,dy_2 \\
&= \frac{1}{\pi(y_1^2+1)} \qquad -\infty < y_1 < \infty,
\end{aligned}$$

which is the probability density function of a standard Cauchy random variable.

APPL verification: The APPL statements

```
X1 := NormalRV(0, 1);
X2 := NormalRV(0, 1);
g  := [[x -> 1 / x, x -> 1 / x], [-infinity, 0, infinity]];
Y  := Transform(X2, g);
Product(X1, Y);
```

produce the probability density function of a Cauchy random variable.