Theorem If $X \sim N(0,1)$, then $Y = \mu + \sigma X$ has the normal distribution with mean μ and variance σ^2.

Proof Let the random variable $X \sim N(0,1)$. The probability density function of X is

$$f_X(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \qquad -\infty < x < \infty.$$

The transformation $Y = \mu + \sigma X$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, -\infty < x < \infty\}$ to $\mathcal{Y} = \{y \,|\, -\infty < y < \infty\}$, with inverse $X = (Y - \mu)/\sigma$ and Jacobian

$$\frac{dX}{dY} = \frac{1}{\sigma}.$$

Therefore, by the transformation technique, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left|\frac{dx}{dy}\right| \\ &= \frac{e^{-(y-\mu)^2/2\sigma^2}}{\sqrt{2\pi}} \left|\frac{1}{\sigma}\right| \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty, \end{aligned}$$

which is the probability density function of a $N(\mu, \sigma^2)$ random variable.

APPL verification: The APPL statements

```
X := StandardNormalRV():
assume(sigma > 0);
g := [[x -> mu + sigma * x], [-infinity,infinity]]:
Y := Transform(X, g);
```

yield the probability density function of a $N(\mu, \sigma^2)$ random variable.