Theorem If $X_i \sim N(0,\,1)$, $i = 1,\,2,\,\ldots,\,n$ are mutually independent random variables, then $Y = \sum_{i=1}^{n} X_i^2$ has the chi-square distribution with n degrees of freedom.

Proof Let $V = X^2$, where $X \sim N(0,\,1)$. The cumulative distribution function of V is

$$\begin{aligned} F_V(v) &= P(V \le v) \\ &= P\left(X^2 \le v\right) \\ &= P(-\sqrt{v} \le X \le \sqrt{v}) \\ &= 2\int_0^{\sqrt{v}} \frac{1}{\sqrt{2\pi}}\, e^{-v^2/2} dv \qquad\qquad v > 0 \end{aligned}$$

by the symmetry of the standard normal distribution around 0. Letting $u = v^2$,

$$\begin{aligned} F_V(u) &= 2\int_0^{u} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2} \left(\frac{1}{2\sqrt{u}}\right) du \\ &= \int_0^{u} \frac{1}{\sqrt{\pi}\sqrt{2}}\, u^{1/2-1} e^{-u/2} du \qquad\qquad u > 0. \end{aligned}$$

Taking the derivative with respect to u,

$$f_V(u) = \frac{1}{\Gamma(1/2)\, 2^{1/2}}\, u^{1/2-1} e^{-u/2} \qquad\qquad u > 0,$$

which is the probability density function of the chi-square distribution with 1 degree of freedom. Since $X_i^2 \sim \chi^2(1)$, for $i = 1,\,2,\,\ldots,\,n$, the moment generating function of X_i is

$$M_{X_i}(t) = (1-2t)^{-1/2} \qquad\qquad t < 1/2.$$

Since $X_1,\, X_2, \ldots,\, X_n$ are mutually independent random variables, the moment generating function of $Y = \sum_{i=1}^{n} X_i^2$ is

$$\begin{aligned} M_Y(t) &= \prod_{i=1}^{n} M_{X_i}(t) \\ &= \prod_{i=1}^{n} (1-2t)^{-1/2} \\ &= (1-2t)^{-n/2} \qquad\qquad t < 1/2, \end{aligned}$$

which is the moment generating function of a chi-square random variable with n degrees of freedom.

APPL Verification: The APPL statements

```
X := NormalRV(0, 1);
g :=  [[x -> x ^ 2, x -> x ^ 2], [-infinity, 0, infinity]];
V := Transform(X, g);
n := 3;
Y := ConvolutionIID(V, n);
ChiSquareRV(n);
```

confirm the result for $n = 3$.