Theorem If X has the standard Cauchy distribution, then $Y = a + \alpha X$ has the Cauchy distribution for $\alpha > 0$ and $-\infty < a < \infty$.

Proof Let the random variable X have the standard Cauchy distribution. The probability density function of X is

$$f_X(x) = \frac{1}{\pi\left[1 + x^2\right]} \qquad\qquad -\infty < x < \infty.$$

Using the transformation technique, the transformation $Y = g(X) = a + \alpha X$ is a 1–1 transformation from $\mathcal{X} = \{x \,|\, -\infty < x < \infty\}$ to $\mathcal{Y} = \{y \,|\, \infty < y < \infty\}$ with inverse $X = g^{-1}(y) = \frac{Y-a}{\alpha}$, and Jacobian $\frac{dX}{dY} = \frac{1}{\alpha}$. Therefore, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right)\left|\frac{dx}{dy}\right| \\ &= \frac{1}{\pi\left[1 + \left(\frac{y-a}{\alpha}\right)^2\right]} \cdot \frac{1}{\alpha} \\ &= \frac{1}{\alpha\pi\left(1 + \left((y-a)/\alpha\right)^2\right)} \qquad\qquad -\infty < y < \infty, \end{aligned}$$

which is recognized as the Cauchy distribution probability density function.

APPL Verification: The following APPL statements

```
assume(alpha > 0);
X := StandardCauchyRV();
g := [[x -> a + alpha * x], [-infinity, infinity]];
Transform(X, g);
```

yield the probability density function of the Cauchy distribution.