Theorem The limiting distribution of a Poisson(λ) distribution as $\lambda \to \infty$ is normal.

Proof Let $X_n \sim$ Poisson(n), for $n = 1, 2, \ldots$. The probability mass function of X_n is

$$f_{X_n}(x) = \frac{n^x e^{-n}}{x!} \qquad\qquad x = 0, 1, 2, \ldots.$$

The moment generating function of X_n is

$$M_{X_n}(t) = E\left[e^{tX_n}\right] = e^{n(e^t - 1)}$$

for $-\infty < t < \infty$. Taking the limit gets us nowhere because

$$\lim_{n\to\infty} M_{X_n}(t) = \lim_{n\to\infty} e^{n(e^t-1)} = \infty.$$

So now consider a "standardized" Poisson random variable

$$\frac{X_n - n}{\sqrt{n}}$$

which has limiting moment generating function

$$\begin{aligned}
\lim_{n\to\infty} M_{(X_n-n)/\sqrt{n}}(t) &= \lim_{n\to\infty} E\left[\exp\left(t \cdot \frac{X_n - n}{\sqrt{n}}\right)\right] \\
&= \lim_{n\to\infty} \exp\left(-t\sqrt{n}\right) E\left[\exp\left(\frac{tX_n}{\sqrt{n}}\right)\right] \\
&= \lim_{n\to\infty} \exp\left(-t\sqrt{n}\right) \exp\left(n(e^{t/\sqrt{n}} - 1)\right) \\
&= \lim_{n\to\infty} \exp\left(-t\sqrt{n} + n\left(tn^{-1/2} + t^2n^{-1}/2 + t^3n^{-3/2}/6 + \cdots\right)\right) \\
&= \lim_{n\to\infty} \exp\left(t^2/2 + t^3n^{-1/2}/6 + \cdots\right) \\
&= \exp\left(t^2/2\right) \qquad\qquad -\infty < t < \infty
\end{aligned}$$

by using the moment generating function of a Poisson random variable and expanding the exponential function as a series. This can be recognized as the moment generating function of a standard normal random variable. This implies that the associated unstandardized random variable X_n has a limiting distribution that is normal with mean n and variance n. This result is the basis for the "normal approximation to the Poisson distribution."