Theorem: The limiting distribution of a Poisson(\(\lambda\)) distribution as \(\lambda \to \infty\) is normal.

Proof: Let \(X_n \sim \text{Poisson}(n)\), for \(n = 1, 2, \ldots\). The probability mass function of \(X_n\) is

\[
f_{X_n}(x) = \frac{n^x e^{-n}}{x!} \quad x = 0, 1, 2, \ldots.
\]

The moment generating function of \(X_n\) is

\[
M_{X_n}(t) = E\left[ e^{tX_n} \right] = e^{n(e^t - 1)}
\]

for \(-\infty < t < \infty\). Taking the limit gets us nowhere because

\[
\lim_{n \to \infty} M_{X_n}(t) = \lim_{n \to \infty} e^{n(e^t - 1)} = \infty.
\]

So now consider a “standardized” Poisson random variable

\[
\frac{X_n - n}{\sqrt{n}}
\]

which has limiting moment generating function

\[
\lim_{n \to \infty} M_{(X_n - n)/\sqrt{n}}(t) = \lim_{n \to \infty} E\left[ \exp\left( t \cdot \frac{X_n - n}{\sqrt{n}} \right) \right]
\]

\[
= \lim_{n \to \infty} \exp\left( -t\sqrt{n} \right) E\left[ \exp\left( t\frac{X_n}{\sqrt{n}} \right) \right]
\]

\[
= \lim_{n \to \infty} \exp\left( -t\sqrt{n} \right) \exp\left( n(e^{t/\sqrt{n}} - 1) \right)
\]

\[
= \lim_{n \to \infty} \exp\left( -t\sqrt{n} + n\left( t n^{-1/2} + t^2 n^{-1/2} + t^3 n^{-3/2}/6 + \cdots \right) \right)
\]

\[
= \lim_{n \to \infty} \exp\left( t^2/2 + t^3 n^{-1/2}/6 + \cdots \right)
\]

\[
= \exp\left( t^2/2 \right) \quad -\infty < t < \infty
\]

by using the moment generating function of a Poisson random variable and expanding the exponential function as a series. This can be recognized as the moment generating function of a standard normal random variable. This implies that the associated unstandardized random variable \(X_n\) has a limiting distribution that is normal with mean \(n\) and variance \(n\). This result is the basis for the “normal approximation to the Poisson distribution.”