Theorem If $X \sim \text{Poisson}(\mu)$ and $\mu \sim \text{gamma}(\alpha, \beta)$ then the probability mass function of X is

$$f_X(x) = \frac{\Gamma(x+\beta)\alpha^x}{\Gamma(\beta)(1+\alpha)^{\beta+x} x!} \qquad x = 0, 1, 2, \ldots,$$

which is known as the gamma–Poisson distribution.

Proof The unconditional distribution of X (also known as the *compound distribution*) is

$$\begin{aligned}
f_X(x) &= \int_0^\infty f_\mu(\mu) f_{X\,|\,\mu}(x\,|\,\mu)\,d\mu \\
&= \int_0^\infty \left[\frac{1}{\Gamma(\beta)\alpha^\beta}\,\mu^{\beta-1}e^{-\mu/\alpha}\right]\left[\frac{\mu^x e^{-\mu}}{x!}\right] d\mu \\
&= \frac{1}{\Gamma(\beta)\alpha^\beta x!}\int_0^\infty \mu^{\beta+x-1}e^{-\mu(1+\alpha)/\alpha}d\mu \\
&= \frac{1}{\Gamma(\beta)\alpha^\beta x!}\int_0^\infty \left(\frac{\alpha t}{1+\alpha}\right)^{\beta+x-1} e^{-t}\left(\frac{\alpha}{1+\alpha}\right)dt \\
&= \frac{1}{\Gamma(\beta)\alpha^\beta x!}\cdot\left(\frac{\alpha}{1+\alpha}\right)^{\beta+x}\int_0^\infty t^{\beta+x-1}e^{-t}dt \\
&= \frac{\Gamma(x+\beta)\alpha^x}{\Gamma(\beta)(1+\alpha)^{x+\beta}x!} \qquad x = 0, 1, 2, \ldots
\end{aligned}$$

by using the change of variable $t = \mu(1+\alpha)/\alpha$.

APPL verification: The APPL statements

```
assume(alpha > 0);
assume(beta > 0);
M := [[x -> x ^ (beta - 1) * exp(-x / alpha) / (GAMMA(beta) * alpha ^ beta)],
      [0, infinity], ["Continuous", "PDF"]];
X := PoissonRV(mu);
int(M[1][1](mu) * X[1][1](x), mu = 0 .. infinity);
```

yield the probability mass function of the gamma–Poisson distribution indicated in the theorem.