Theorem If $X_i \sim \text{Pascal}(n_i, p)$, for $i = 1, 2, \ldots, k$, and $X_1, X_2, \ldots, X_k$ are mutually independent random variables, then

$$\sum_{i=1}^{k} X_i \sim \text{Pascal}\left(\sum_{i=1}^{k} n_i, p\right).$$

Proof The moment generating function of X_i is

$$\begin{aligned}
M_{X_i}(t) &= E\left[e^{tX_i}\right] \\
&= \sum_{x=0}^{\infty} e^{tx} \binom{n_i - 1 + x}{x} p^{n_i} (1-p)^x \\
&= p^{n_i} \sum_{x=0}^{\infty} \binom{n_i - 1 + x}{x} \left[e^t (1-p)\right]^x \\
&= \frac{p^{n_i}}{(1 - (1-p)e^t)^{n_i}} \\
&= \left(\frac{p}{1 - (1-p)e^t}\right)^{n_i}
\end{aligned}$$

for $t < -\ln(1-p)$ and $i = 1, 2, \ldots, k$. Since the moment generating function of a sum of mutually independent random variables is the product of their moment generating functions,

$$\begin{aligned}
M_{X_1 + X_2 + \cdots + X_k}(t) &= \prod_{i=1}^{k} M_{X_i}(t) \\
&= \prod_{i=1}^{k} \left(\frac{p}{1 - (1-p)e^t}\right)^{n_i} \\
&= \left(\frac{p}{1 - (1-p)e^t}\right)^{\sum_{i=1}^{k} n_i}
\end{aligned}$$

for $t < -\ln(1-p)$. This moment generating function is recognized as that of a Pascal random variable with parameters $\sum_{i=1}^{k} n_i$ and p.

APPL illustration: The APPL statements

```
X1 := NegativeBinomialRV(n1, p);
X2 := NegativeBinomialRV(n2, p);
simplify(MGF(X1) * MGF(X2));
```

yield the appropriate moment generating function (could be further simplified)

$$M_{X_1+X_2}(t) = p^{n_1} e^{t(n_1+n_2)} (1 - e^t + pe^t)^{-n_1} p^{n_2} (1 - e^t + pe^t)^{-n_2}$$

for $t < -\ln(1-p)$ and $n = 2$. Notice the negative binomial (Pascal) distribution built in APPL is different from the one used here. The result holds for larger values of k by induction.