Theorem If $X \sim \text{Pascal}(n, p)$ and $p \sim \text{beta}(\alpha, \beta)$ then the probability mass function of X is

$$f_X(x) = \binom{n-1+x}{x} \frac{B(n+\alpha, x+\beta)}{B(\alpha, \beta)} \qquad\qquad x = 0,\, 1,\, 2,\, \ldots,$$

which is known as the beta–Pascal distribution.

Proof Let X be a Pascal random variable with parameters n and p, where $p \sim \text{beta}(\alpha, \beta)$. The conditional probability mass function of X for a positive integer n and a fixed $p \in (0, 1)$ is

$$f_{X|p}(x) = \binom{n-1+x}{x} p^n (1-p)^x \qquad\qquad x = 0,\, 1,\, 2,\, \ldots$$

The random variable p has probability density function

$$f_P(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \qquad\qquad 0 < p < 1,$$

where β and γ are fixed parameters. The goal is to find the unconditional probability mass function of X. We can find this by integrating the product of $f_{X|p}$ and $f_P(p)$ over the support of p for a fixed x in the support of X:

$$\begin{aligned}
f_X(x) &= \int_0^1 f_{X|p}(x) f_P(p)\, dp \\
&= \int_0^1 \binom{n-1+x}{x} p^n (1-p)^x \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] p^{\alpha-1}(1-p)^{\beta-1}\, dp \\
&= \binom{n-1+x}{x} \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \int_0^1 p^n (1-p)^x p^{\alpha-1}(1-p)^{\beta-1}\, dp \\
&= \binom{n-1+x}{x} \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \int_0^1 p^{n+\alpha-1}(1-p)^{x+\beta-1}\, dp \\
&= \binom{n-1+x}{x} \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \frac{\Gamma(n+\alpha)\Gamma(x+\beta)}{\Gamma(n+x+\alpha+\beta)} \\
&= \binom{n-1+x}{x} \frac{1}{B(\alpha, \beta)} B(n+\alpha, x+\beta) \\
&= \binom{n-1+x}{x} \frac{B(n+\alpha, x+\beta)}{B(\alpha, \beta)} \qquad\qquad x = 0,\, 1,\, 2,\, \ldots,
\end{aligned}$$

by definition of the beta function. This is the probability mass function of a beta–Pascal random variable with parameters α and β.

APPL verification: The APPL statements

```
Y := BetaRV(a, b);
X := NegativeBinomialRV(n, p);
int(Y[1][1](p) * X[1][1](x), p = 0 .. 1);
```

yield the probability mass function of the beta–Pascal distribution from the theorem.